Canad. Math. Bull. Vol. **46** (4), 2003 pp. 559–574

On Density Conditions for Interpolation in the Ball

Nicolas Marco and Xavier Massaneda

Abstract. In this paper we study interpolating sequences for two related spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, $n > 1$. We first give density conditions for a sequence to be interpolating for the class $A^{-\infty}$ of holomorphic functions with polynomial growth. The sufficient condition is formally identical to the characterizing condition in dimension 1, whereas the necessary one goes along the lines of the results given by Li and Taylor for some spaces of entire functions. In the second part of the paper we show that a density condition, which for $n = 1$ coincides with the characterizing condition given by Seip, is sufficient for interpolation in the (weighted) Bergman space.

1 Introduction

Let $A^{-\infty}$ denote the space of holomorphic functions in $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ (denoted $\mathbb{B}$ if no confusion can arise) satisfying the growth condition

$$\log|f(z)| \leq C_f \log\left(\frac{e}{1-|z|}\right) \quad C_f > 0.$$

$A^{-\infty}$ is the smallest algebra of holomorphic functions that contains the class H^∞ of bounded holomorphic functions and is closed by differentiation. $A^{-\infty}$ can be thought of as the union of the spaces

$$A^{-p} = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{A^{-p}} =: \sup_{z\in\mathbb{B}_n} (1-|z|)^p |f(z)| < \infty \right\} \quad p > 0.$$

It can also be thought as the union of weighted Bergman spaces

$$B^2_\alpha := \left\{ f \in H(\mathbb{B}_n) : \|f\|^2_\alpha := \int_{\mathbb{B}_n} |f(z)|^2 (1-|z|^2)^{2\alpha-1}\, dm(z) < \infty \right\} \quad \alpha > 0,$$

where dm denotes the Lebesgue measure.

The class $A^{-\infty}$ in the unit disk $\mathbb{D} \subset \mathbb{C}$ was intensively studied by Korenblum in two essential papers [Kor75, Kor77]. The first of them contains a characterization of the zero sequences and a factorization theory for $A^{-\infty}$, whereas the second one provides a complete description of its closed ideals.

Received by the editors October 10, 2001; revised January 6, 2003.

The authors were partially supported by a PICS program of Generalitat de Catalunya and CNRS. Second author also supported by the DGICYT grant PB98-1242-C02-01 and the CIRIT grant 2000-SGR00059.

AMS subject classification: Primary: 32A36, 32A38; secondary: 30E05.

Definition Let $\Gamma := \{a_k\}_k \subset \mathbb{B}_n$ be a discrete sequence. Γ is $A^{-\infty}$*-interpolating*, denoted $\Gamma \in \operatorname{Int} A^{-\infty}$, if for every sequence of values $\{v_k\}_{k\in\mathbb{N}}$, such that $\exists C > 0$ with

$$\log|v_k| \le C \log\left(\frac{e}{1-|a_k|}\right) \quad \text{for all } k \in \mathbb{N} \tag{1}$$

there exists $f \in A^{-\infty}$ with $f(a_k) = v_k$ for all $k \in \mathbb{N}$.

Bruna and Pascuas characterized $A^{-\infty}$-interpolation in the disk by a condition which is essentially Korenblum's condition for the zero sequences made invariant by automorphisms (see [BP89]). A different characterization in terms of a suitable density was given in [Mas99, Theorem 1.3]. We first prove that the formal analogue of such density is also sufficient in higher dimension.

Let ϕ_z denote an automorphism of the unit ball exchanging z and 0.

Theorem 1 *$\Gamma = \{a_k\}_k$ is $A^{-\infty}$-interpolating if there exists a constant $C > 0$ such that*

(a) $\sum_{k:\, 0<|\phi_{a_j}(a_k)|\le 1/2} \log \frac{1}{|\phi_{a_j}(a_k)|} \le C \log(\frac{e}{1-|a_j|})$ $j \in \mathbb{N}$,

(b) $\sum_{k:\, 1/2<|\phi_z(a_k)|\le |z|} \log \frac{1}{|\phi_z(a_k)|} \le C \log(\frac{e}{1-|z|})$ $|z| > 1/2$.

The constant $1/2$ can be replaced by any other $\delta \in (0,1)$, as will be clearly seen in the proof.

This result is proved using L^2-estimates for the $\bar\partial$, following the ideas in [BC95]. The main difficulty in the proof is, as usual, the choice of an appropriate subharmonic function with singularities on the sequence.

This sufficient condition cannot be improved, in the sense that no condition of type

$$\sum_{k:\, 1/2<|\phi_z(a_k)|\le|z|} \log\frac{1}{|\phi_z(a_k)|} \le C\Lambda(|z|),$$

where $\Lambda\colon [0,1) \to \mathbb{R}$ is increasing and $\lim_{r\to 1} \frac{\Lambda(r)}{|\log(1-r)|} = +\infty$, can be sufficient.

This is easily seen by considering sequences $\{a_k\}_k \subset \mathbb{B}_n$ with $a_k = (\alpha_k, 0) \in \mathbb{D} \times \{0\}^{n-1}$, and applying the result in the disk, since $\{a_k\}_k \in \operatorname{Int} A^{-\infty}(\mathbb{B}_n)$ if and only if $\{\alpha_k\}_k \in \operatorname{Int} A^{-\infty}(\mathbb{D})$.

When $n > 1$ the conditions above are far from being necessary. However, adapting to the ball a result by Li and Taylor [LT96] for some algebras of entire functions, we obtain a necessary density condition for $A^{-\infty}$-interpolation, which we include for the sake of completeness.

Given $z, \zeta \in \mathbb{B}$ consider the hyperbolic pseudodistance

$$d(z,\zeta) = |\phi_z(\zeta)|.$$

Also, given $\delta \in (0,1)$ and a sequence Γ, consider also the pseudoball

$$\mathcal{K}(z,\delta) = \{\zeta \in \mathbb{D} : d(z,\zeta) < \delta\}$$

and the associated counting function

$$n(z,\delta) := \#\Gamma \cap \mathcal{K}(z,\delta).$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

Theorem 2 *If $\Gamma = \{a_k\}_k$ is $A^{-\infty}$-interpolating then*

(a) *Γ is weakly separated, that is, there exist constants $\varepsilon, q > 0$ such that*

$$d(a_j, a_k) \geq 2\varepsilon \max[(1-|a_k|)^q, (1-|a_j|)^q] \quad \text{for all } j \neq k.$$

(b) *There exists a constant $C > 0$ such that for all $z \in \mathbb{B}_n$ and all $r \in (0,1)$*

$$n(z,r) \leq \frac{C}{(1-r)^n}\left[\log\left(\frac{e}{1-r}\right) + \log\left(\frac{e}{1-|z|}\right)\right]^n.$$

Since

$$\sum_{k:\, 1/2<|\phi_z(a_k)|\leq|z|} \log\frac{1}{|\phi_z(a_k)|} = \int_{1/2}^{|z|} n(z,t)\frac{dt}{t},$$

from (b) we obtain the following condition.

Corollary 3 *If $n > 1$ and $\Gamma = \{a_k\}_k$ is $A^{-\infty}$-interpolating then*

$$\sum_{k:\, 1/2<|\phi_z(a_k)|\leq|z|} \log\frac{1}{|\phi_z(a_k)|} \leq \frac{C}{(1-|z|)^{n-1}}\left[\log\left(\frac{e}{1-|z|}\right)\right]^n \quad |z| > 1/2.$$

Remark In the disk there is a gap between the necessary condition we obtain here and the necessary condition (b) of Theorem 1.

In order to see this let $d\lambda := (1-|z|^2)^{-(n+1)}\, dm$ denote the invariant measure in $\mathbb{B}_n$. Then Theorem 2 (b) can be rewritten as

$$\frac{n(z,r)}{\lambda\big(\mathcal{K}(z,r)\big)} \leq C\left[\log\left(\frac{e}{1-r}\right) + \log\left(\frac{e}{1-|z|}\right)\right]^n.$$

Similarly, the condition in Corollary 3 is equivalent to

$$\frac{\sum_{k:\, 1/2<|\phi_z(a_k)|\leq|z|} \log\frac{1}{|\phi_z(a_k)|}}{\int_{1/2}^{|z|} \lambda\big(\mathcal{K}(z,t)\big)\, dt/t} \leq C\left[\log\left(\frac{e}{1-|z|}\right)\right]^n \quad |z| > 1/2.$$

When $n = 1$ this differs by a logarithmic factor from the necessary condition (b) of Theorem 1, which is equivalent to

$$\sup_{z:\, |z|>1/2} \frac{\sum_{k:\, 1/2<|\phi_z(a_k)|\leq|z|} \log\frac{1}{|\phi_z(a_k)|}}{\int_{1/2}^{|z|} \lambda\big(\mathcal{K}(z,t)\big)\, dt/t} < \infty.$$

In the second part of the paper we show that the techniques used in the previous results can be adapted to give a sufficient condition for interpolation in the space B^2_α in the ball, following [BC95, Theorem 3].

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

Definition A sequence $\Gamma = \{a_k\}_k \subset \mathbb{B}_n$ is B^2_α*-interpolating* if for all sequence of values $\{v_k\}_{k\in\mathbb{N}}$ such that

$$\|v\|^2_\alpha := \sum_{k=1}^{\infty}(1-|a_k|^2)^{2\alpha+n}|v_k|^2 < \infty, \tag{2}$$

there exists $f \in B^2_\alpha$ such that $f(a_k) = v_k$.

The sufficient condition is given in terms of the following density

$$D(\Gamma) = \lim_{r\to 1}\sup\sup_{z\in\mathbb{B}} \frac{\sum_{k:\, 1/2<|\phi_z(a_k)|<r} \log \frac{1}{|\phi_z(a_k)|}}{\log\frac{1}{1-r}},$$

which is the analogue of the 1-dimensional upper density used by Seip to characterize B^2_α-interpolation in the unit disk [Sei94].

Theorem 4 *Let Γ be a separated sequence $\mathbb{B}_n$. If $D(\Gamma) < \alpha/n$ then Γ is B^2_α-interpolating.*

With this and [JMT96, Theorem 3.3] one can also give sufficient conditions for interpolation in the more general Bergman spaces $B^p_\alpha = H \cap L^p[(1-|z|)^{\alpha-1}\, dm]$, $p, \alpha > 0$.

A sequence $\{a_k\}_k$ is called *separated* when $\inf_{j\neq k} d(a_j, a_k) > 0$. The separation is a necessary condition for B^2_α-interpolation, and it implies $n(z,r) \le C(1-r)^{-n}$, whence by integration

$$\sup_{r>1/2}\sup_{z\in\mathbb{B}_n} \frac{\sum_{k:\, 1/2<|\phi_z(a_k)|<r} \log\frac{1}{|\phi_z(a_k)|}}{(1-r)^{1-n}} < \infty.$$

The order of this estimate cannot be improved, since $\{a_k\}_k$ is B^2_α-interpolating if it is separated enough (*i.e.* $\inf_{j\neq k} d(a_j, a_k)$ is big enough). We do not know of a more precise estimate (in terms of α and n).

The paper is organized as follows. In Section 2 give the proof of Theorem 1, except for some unpleasant computations that are left for an appendix. Section 3 contains the proof of Theorem 4. In Section 4 we outline the proof of Theorem 2 and show the minor adjustments to make the proof of Li and Taylor work in the ball.

A final remark about notation. C will always denote a positive constant and its actual value may change from one occurrence to the next. $A \preceq B$ means that $A \le cB$ for some $c > 0$, and $A \simeq B$ is $A \preceq B \preceq A$.

2 Proof of the Theorem 1

Let $\{v_k\}_k$ satisfy condition (1).

We first see that it is possible to construct a smooth interpolating function having the characteristic growth of $A^{-\infty}$.

Condition (a) implies that Γ is weakly separated (see the definition in Theorem 2(a)). Thus, the hyperbolic pseudoballs $\mathcal{K}_k := \mathcal{K}(a_k, \delta_k)$, where $\delta_k = \varepsilon(1-|a_k|)^q$, are pairwise disjoint.

Let $\mathcal{X}$ be a smooth cut-off function of one real variable with uniformly bounded derivative, $\mathcal{X}(t) \equiv 1$ for $t < 1/2$ and $\mathcal{X}(t) \equiv 0$ for $t > 1$.

Consider the smooth interpolating function

$$F(z) = \sum_{k=1}^{\infty} v_k \mathcal{X}\left(\frac{|\phi_z(a_k)|^2}{\delta_k^2}\right).$$

The support of F is contained in $\cup_k \mathcal{K}_k$ and for $z \in \mathcal{K}_k$

$$|F(z)| \leq |v_k|,$$

$$|\bar{\partial}F(z)| \preceq |v_k| \frac{|\bar{\partial}|\phi_{a_k}(z)|^2|}{\delta_k^2} \leq |v_k| \frac{1}{\delta_k} \frac{1}{1-|a_k|}.$$

Thus, for some $p > 0$

$$\sup_{z\in\mathbb{B}} |F(z)|(1-|z|)^p \preceq \sup_{k\in\mathbb{N}} |v_k|(1-|a_k|)^p < \infty,$$

and similarly, $\sup_{z\in\mathbb{B}} |\bar{\partial}F(z)|(1-|z|)^{p+q+1} < \infty$. In particular, for some $\alpha > 0$ big enough

$$\int_{\mathbb{B}} |F(z)|^2(1-|z|^2)^{\alpha} < \infty; \quad \int_{\mathbb{B}} |\bar{\partial}F(z)|^2(1-|z|^2)^{\alpha} < \infty. \tag{3}$$

Now, when looking for a holomorphic interpolating function of the form $f := F - u$ we are led to the $\bar{\partial}$-problem

$$\bar{\partial}u = \bar{\partial}F,$$

which we solve by Hörmander's theorem [Hör90]: given a pluirisubharmonic function ψ in $\mathbb{B}$, there exists a solution u to the above equation such that

$$\int_{\mathbb{B}} |u|^2 e^{-\psi} \leq \int_{\mathbb{B}} |\bar{\partial}F|^2_{\Omega} e^{-\psi},$$

where $|\cdot|_{\Omega}$ indicates the norm with respect to the metric $\Omega := i\partial\bar{\partial}\psi$ (see also [Del98, Theorem 1] for a more refined result).

In order to define a suitable weight ψ let

$$\rho_z = \begin{cases} 3/4 & \text{if } |z| \leq 3/4, \\ |z| & \text{if } |z| > 3/4, \end{cases}$$

and

$$K(r) = \int_{1/4}^{r} \frac{dt}{1-t} \simeq \log\frac{1}{1-r}.$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

Consider the negative function

$$F(\rho,x)=\begin{cases}0 & \text{if } \rho\le x,\\ \log x-\frac{1}{K(\rho)}\int_{1/4}^{\rho}\frac{\log\max(x,t)}{1-t}\,dt & \text{if } \rho>x,\end{cases}$$

which is $\mathcal{C}^1$ and vanishes at order two on $\{(x,\rho):x=\rho\}$, and let

$$v(z)=n\sum_{k=1}^{\infty}F\big(\rho_z^2,|\phi_z(a_k)|^2\big)\,.$$

Define the weights

$$\psi_\beta(z)=\beta\log\Big(\frac{1}{1-|z|^2}\Big)+v(z)\quad \beta>0.$$

Recall that the fundamental form of the Bergman metric is

$$\Psi(z)=i\partial\bar\partial\log\Big(\frac{1}{1-|z|^2}\Big)=\frac{(1-|z|^2)i\partial\bar\partial|z|^2+i\partial|z|^2\wedge\bar\partial|z|^2}{(1-|z|^2)^2}.$$

Lemma 5 *Under the hypotheses (a) and (b) of Theorem 1, there exists a constant $C>0$ such that $i\partial\bar\partial v\ge -C\Psi$.*

The proof of this lemma is a gloomy calculation, and it will be deferred till the end of the paper. Assuming this, for β big enough we have:

$$\Omega_\beta:=i\partial\bar\partial\psi_\beta=\beta\Psi+i\partial\bar\partial v\ge\Psi,$$

thus $|\bar\partial F|_{\Omega_\beta}\le|\bar\partial F|_\Psi\le|\bar\partial F|$ and

$$\int_{\mathbb{B}}|u|^2(1-|z|^2)^\beta\le\int_{\mathbb{B}}|u|^2e^{-\psi_\beta}\le\int_{\mathbb{B}}|\bar\partial F|^2(1-|z|^2)^{\beta+1}e^{-v}. \tag{4}$$

If z is in the support of $\bar\partial F$, there exists k such that $\delta_k/2<|\phi_z(a_k)|<\delta_k$. For such z, if $|z|>3/4$, the definition of v and the hypotheses yield:

$$\begin{aligned}-v(z)&\le n\sum_{j:\,|\phi_z(a_j)|\le|z|}\log\frac{1}{|\phi_z(a_j)|^2}\preceq\log\frac{1}{\delta_k}\\&\qquad+\sum_{0<|\phi_{a_k}(a_j)|\le 1/2}\log\frac{1}{|\phi_{a_k}(a_j)|}+\sum_{1/2<|\phi_z(a_j)|\le|z|}\log\frac{1}{|\phi_z(a_j)|}\\&\preceq\log\frac{1}{\delta_k}+\log\Big(\frac{1}{1-|a_k|}\Big)+\log\Big(\frac{1}{1-|z|}\Big)\preceq\log\Big(\frac{1}{1-|z|}\Big).\end{aligned}$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

If $|z| \leq 3/4$ it is clear that

$$-v(z) \preceq \sum_{j:\,|\phi_z(a_j)|\leq 3/4} \log \frac{1}{|\phi_z(a_j)|} \preceq \log\left(\frac{1}{1-|z|}\right).$$

Thus there is $c > 0$ such that $e^{-v} \preceq (1-|z|)^{-c}$ on $\operatorname{supp}(\bar\partial F)$. Taking β big enough, the above and (3) show that the last integral in (4) is finite.

This shows that $f \in A^{-\infty}$. Moreover $e^{-\psi_\beta} \simeq |\phi_z(a_k)|^{-2n}$ around each a_k, so the convergence of the second integral in (4) implies $u(a_k) = 0$ for all $k \in \mathbb{N}$, and therefore $f(a_k) = v_k$, as required. ∎

3 Proof of Theorem 4

This proof follows the same scheme as the previous one.

Take $\delta > 0$ such that the hyperbolic balls $\mathcal{K}_k := \mathcal{K}(a_k, \delta)$ are pairwise disjoint, and given $\{v_k\}_k$ satisfying (2), consider the smooth interpolating function

$$F(z) = \sum_{k=1}^{\infty} v_k \mathcal{X}\left(\frac{|\phi_z(a_k)|^2}{\delta^2}\right).$$

The support of F is contained in $\cup_k \mathcal{K}_k$, and for $z \in \mathcal{K}_k$ we have now $|F(z)| \leq |v_k|$, hence

$$\int_{\mathbb{B}} |F|^2(1-|z|^2)^{2\alpha-1} \preceq \sum_k \int_{\mathcal{K}_k} |F|^2(1-|z|^2)^{2\alpha-1} \preceq \|v\|_\alpha^2.$$

When solving the $\bar\partial$ equation we will need an estimate of the norm with respect to the Bergman metric

$$|\bar\partial F|_\Psi^2 = (1-|z|^2)^2|\bar\partial F|^2 + (1-|z|^2)\left|\bar\partial F \wedge \bar\partial |z|^2\right|^2.$$

Since

$$|\bar\partial F(z)| \preceq |v_k| \left|\bar\partial |\phi_z(a_k)|^2\right| \quad z \in \mathcal{K}_k$$

and

$$\bar\partial |\phi_{a_k}(z)|^2 = \frac{1-|a_k|^2}{|1-\bar a_k z|^2}\left[\bar\partial |z|^2 - \left(\frac{1-|z|^2}{1-a_k\bar z}\right)\left(\sum_j a_j d\bar z_j\right)\right],$$

we have

$$|\bar\partial F(z)|_\Psi \preceq |v_k| \quad \text{for } z \in \mathcal{K}_k.$$

From the above we get

$$\int_{\mathbb{B}} |\bar\partial F|_\Psi^2 (1-|z|^2)^{2\alpha-1} \preceq \sum_k \int_{\mathcal{K}_k} |v_k|^2(1-|z|)^{2\alpha-1}\, dm \simeq \|v\|_\alpha^2. \tag{5}$$

Now we solve the $\bar{\partial}$-equation $\bar{\partial}u = \bar{\partial}F$ by the following theorem.

***Theorem* [BC00, Theorem 2.3]** *Let w and ψ be plurisubharmonic functions such that $\psi \geq 0$ and $i\partial\psi \wedge \bar{\partial}\psi \leq si\partial\bar{\partial}\psi$ for some $s < 1$. Let f be a $\bar{\partial}$-closed $(0,1)$-form in $\mathbb{B}$ in $L^2(e^{\psi-w})$. There exists u solution to $\bar{\partial}u = f$ such that*

$$\int_{\mathbb{B}} |u|^2 e^{\psi-w} \leq C_s \int_{\mathbb{B}} |f|^2_{i\partial\bar{\partial}(\psi+w)} e^{\psi-w}.$$

Here $|\cdot|_{i\partial\bar{\partial}(\psi+w)}$ denotes the norm with respect to the metric $i\partial\bar{\partial}(\psi+w)$.

In order to define the weights ψ and w, let now

$$K(r) = \int_{1/4}^{r} \frac{\log 1/t}{(1-t)^2}\, dt,$$

and consider the corresponding negative function

$$F(\rho, x) = \begin{cases} 0 & \text{if } \rho \leq x, \\ \log x - \frac{1}{K(\rho)} \int_{1/4}^{\rho} \log \max(x,t) \frac{\log 1/t}{(1-t)^2}\, dt & \text{if } \rho > x. \end{cases}$$

Define

$$v(z) = n \sum_{k=1}^{\infty} F\big(r^2, |\phi_z(a_k)|^2\big).$$

We apply the theorem to

$$\psi_\epsilon(z) = (1-\epsilon) \log \frac{1}{1-|z|^2}, \tag{6}$$

$$w_\epsilon(z) = (2\alpha - \epsilon) \log \frac{1}{1-|z|^2} + v(z), \tag{7}$$

where $\epsilon > 0$ will be chosen later. Clearly $i\partial\psi_\epsilon \wedge \bar{\partial}\psi_\epsilon \leq (1-\epsilon) i\partial\bar{\partial}\psi_\epsilon = (1-\epsilon)^2\Psi$.

Lemma 6 *There exists $\epsilon > 0$ such that $i\partial\bar{\partial}w_\epsilon \geq \epsilon\Psi$.*

Assuming this we have $i\partial\bar{\partial}(\psi_\epsilon + w_\epsilon) \geq \Psi$, so the theorem provides a solution u to the $\bar{\partial}$-equation such that

$$\int_{\mathbb{B}} |u|^2 (1-|z|^2)^{2\alpha-1} \leq \int_{\mathbb{B}} |u|^2 e^{\psi_\epsilon - w_\epsilon} \leq C_\epsilon \int_{\mathbb{B}} |\bar{\partial}F|^2_\Psi (1-|z|^2)^{2\alpha-1} e^{-v}.$$

By construction $\operatorname{supp}(\bar{\partial}F) \subset \cup_k \mathcal{K}(a_k, \delta) \setminus \mathcal{K}(a_k, \delta/2)$ and for z in the region $\delta/2 < |\phi_z(a_k)| < \delta$ one has:

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

$$-v(z) \le \sum_{j:\,|\phi_z(a_j)|<r} \log \frac{1}{|\phi_z(a_j)|^2} \preceq C_{\delta,n} + \sum_{1/2<|\phi_z(a_j)|<r} \log \frac{1}{|\phi_z(a_j)|^2} \le C_{r,p,\delta,n}.$$

Thus, by (5), the last integral above is finite. Then $f = F - u \in B^2_\alpha$ and $f(a_k) = v_k$, as in the proof of Theorem 1.

Proof of Lemma 6 It is enough to see that $i\partial\bar\partial v \ge -2(\alpha - \epsilon)\Psi$.

Define the function

$$A(r) = \int_{1/4}^{r} \frac{-\log^2 t}{(1-t)^2}\,dt.$$

Performing the integral in the definition of v we have

$$\begin{aligned} v(z) = n &\sum_{k:\,|\phi_z(a_k)|<\frac{1}{2}} \left\{ \log|\phi_z(a_k)|^2 - \frac{A(r^2)}{K(r^2)} \right\} \\ &+ n \sum_{k:\,\frac{1}{2}<|\phi_z(a_k)|<r} \left\{ \left(1 - \frac{K\big(|\phi_z(a_k)|^2\big)}{K(r^2)}\right) \log|\phi_z(a_k)|^2 \right. \\ &\qquad\qquad \left. - \frac{A(r^2) - A\big(|\phi_z(a_k)|^2\big)}{K(r^2)} \right\}. \end{aligned}$$

Therefore, using that $i\partial\bar\partial \log|\phi_z(a_k)|^2 \ge 0$,

$$\begin{aligned} i\partial\bar\partial v(z) \ge n &\sum_{1/2<|\phi_z(a_k)|<r} i\partial\bar\partial \left[\left(1 - \frac{K\big(|\phi_z(a_k)|^2\big)}{K(r^2)}\right) \log|\phi_z(a_k)|^2 \right] \\ &+ \frac{i\partial\bar\partial A\big(|\phi_z(a_k)|^2\big)}{K(r^2)}. \end{aligned}$$

In order to simplify the expressions we use the shorthand notation

$$N\big(\phi_z(a_k)\big) = \frac{i\partial\bar\partial|\phi_z(a_k)|^2}{1 - |\phi_z(a_k)|^2}, \tag{8}$$

$$T\big(\phi_z(a_k)\big) = \frac{i\partial|\phi_z(a_k)|^2 \wedge \bar\partial|\phi_z(a_k)|^2}{\big(1 - |\phi_z(a_k)|^2\big)^2}. \tag{9}$$

Notice that by the invariance of the Bergman metric (or by a direct calculation):

$$N\big(\phi_z(a_k)\big) + T\big(\phi_z(a_k)\big) = \Psi(z). \tag{10}$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

A computation gives then

$$i\partial\bar\partial\left(\left(1-\frac{K\big(|\phi_z(a_k)|^2\big)}{K(r^2)}\right)\log|\phi_z(a_k)|^2\right)$$
$$=\left(1-\frac{K\big(|\phi_z(a_k)|^2\big)}{K(r^2)}\right)i\partial\bar\partial\log|\phi_z(a_k)|^2$$
$$+\left(3\frac{\log|\phi_z(a_k)|^2}{|\phi_z(a_k)|^2}+2\frac{\log^2|\phi_z(a_k)|^2}{1-|\phi_z(a_k)|^2}\right)\frac{T\big(\phi_z(a_k)\big)}{K(r^2)}$$
$$+\frac{\log^2|\phi_z(a_k)|^2}{1-|\phi_z(a_k)|^2}\frac{N\big(\phi_z(a_k)\big)}{K(r^2)}$$

$$i\partial\bar\partial A(|\phi_z(a_k)|^2)=-\frac{\log^2|\phi_z(a_k)|^2}{1-|\phi_z(a_k)|^2}N\big(\phi_z(a_k)\big)$$
$$-2\left(\frac{\log|\phi_z(a_k)|^2}{|\phi_z(a_k)|^2}+\frac{\log^2|\phi_z(a_k)|^2}{1-|\phi_z(a_k)|^2}\right)T\big(\phi_z(a_k)\big).$$

This and the fact that $1-\frac{K(|\phi_z(a_k)|^2)}{K(r^2)}\geq 0$ when $|\phi_z(a_k)|<r$ yield:

$$i\partial\bar\partial v(z)\geq -n\frac{1}{K(r^2)}\sum_{1/2<|\phi_z(a_k)|<r}\log\frac{1}{|\phi_z(a_k)|^2}\frac{T\big(\phi_z(a_k)\big)}{|\phi_z(a_k)|^2}.$$

Again from $i\partial\bar\partial\log|\phi_z(a_k)|^2\geq 0$, we deduce that:

$$\frac{T\big(\phi_z(a_k)\big)}{|\phi_z(a_k)|^2}\leq\frac{N\big(\phi_z(a_k)\big)}{1-|\phi_z(a_k)|^2},$$

or equivalently (by (10))

$$\frac{T\big(\phi_z(a_k)\big)}{|\phi_z(a_k)|^2}\leq\Psi(z). \tag{11}$$

Thus

$$i\partial\bar\partial v(z)\geq -2n\frac{\sum_{k:\,1/2<|\phi_z(a_k)|<r}\log\frac{1}{|\phi_z(a_k)|}}{K(r^2)}\Psi(z).$$

Take $\epsilon>0$ so that $D(\Gamma)<(\alpha-\epsilon)/n$. Since $\lim_{r\to 1}\frac{\log\frac{1}{1-r}}{K(r^2)}=1$, the density condition of Theorem 4 implies the existence of r big enough such that

$$i\partial\bar\partial v(z)\geq -2(\alpha-\epsilon)\Psi(z).$$ ∎

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

4 Proof of Theorem 2

(a) This is Theorem 2 in [Mas97].

(b) The proof we present is just an adaptation to the ball of the proof of Theorem 3.6 in [LT96]. We include an outline for the sake of completeness.

Theorem 2 (b) is a consequence of the following proposition and a certain invariance by automorphisms of interpolating sequences. Given a map $F = (F_1, \dots, F_n)$, let $\|F\|_{A^{-p}} = \sum_j \|F_j\|_{A^{-p}}$. Let $D_u F_j$ denote the derivative of F_j in the unitary direction u.

Proposition 7 *Let $\Gamma = \{a_k\}_k$. Assume $\gamma_0 > 0$, $p > 0$, and $F = (F_1, \dots, F_n)\colon \mathbb{B} \to \mathbb{C}^n$, $F_j \in A^{-p}$, are such that $F(a_k) = 0$ for all $k \in \mathbb{N}$ and*

$$\sum_{j=1}^{n} |D_u F_j(a_k)| \geq \gamma_0 \quad \forall k \in \mathbb{N}\, \forall u.$$

Then there exists $C = C(n, p) > 0$ such that

$$n(0, r) \leq \frac{C}{(1-r)^n} \Big[\log \|F\|_{A^{-p}} + \log \frac{1}{\gamma_0} + \log\Big(\frac{1}{1-r}\Big) \Big]^n.$$

Proof Fix $r \in (0, 1)$ and take a_k with $|a_k| \leq r$. Because of the estimate on $D_u F_j(a_k)$, there exist constants $c_1, c_2, A_1, A_2 > 0$ depending only on n, p and $\|F\|_{A^{-p}}$ such that:

$$d\big(a_k, F^{-1}(0) \setminus \{a_k\}\big) > d_r := \gamma_0 c_1 (1 - r^2)^{A_1}.$$

and

$$|F(w)| > \delta_r := \gamma_0 c_2 (1 - r^2)^{A_2} \quad \text{for } w \in \partial\mathcal{K}(a_k, d_r). \tag{12}$$

This is proved using the analog of [LT96, Lemma 3.9] for the ball.

By Sard's lemma there exists $\tau \in \mathbb{R}^n$ with

$$\frac{\delta_r}{4\sqrt{n}} < \tau_j < \frac{\delta_r}{2\sqrt{n}} \quad j = 1, \dots, n$$

and a zero measure set E in the unit torus $\mathbb{T}^n$ such that for all $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathbb{T}^n \setminus E$ the value $a_\varphi := \tau e^{i\varphi} = (\tau_1 e^{i\varphi_1}, \dots, \tau_n e^{i\varphi_n})$ is regular for F, hence $F^{-1}(a_\varphi)$ is a discrete variety in $\mathbb{C}^n$. Let

$$F_\varphi = F - a_\varphi.$$

By (12), on $\partial\mathcal{K}(a_k, d_r)$:

$$|F - F_\varphi|^2 = |a_\varphi|^2 = \sum_{j=1}^{n} |\tau_j|^2 < n \frac{\delta_r}{4n} < |F|^2.$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

From this and Rouché's lemma [BGVY93, Theorem 2.12] one deduces that $n_F\big(\mathcal{K}(a_k,d_r)\big) = n_{F_\varphi}\big(\mathcal{K}(a_k,d_r)\big)$ (denoting by $n_F(U)$ the number of zeros of a map F in U, counted with multiplicities). Thus

$$n(0,r) \le n_{F_\varphi}\big(B(0,r)\big) \quad \text{for all } \varphi \in \mathbb{T}^n \setminus E.$$

Now take $\beta > 1$ such that $\beta^n r = \frac{1+r}{2}$ and apply the average estimate [Gru77, Theorem 2.9]. If $d\varphi$ denotes the product measure in $\mathbb{T}^n$ and $M(F_j,t) = \sup_{|z|=t} |F_j(z)|$,

$$\begin{aligned} n(0,r) &\le \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} n_{F_\varphi}\big(B(0,R)\big)\, d\varphi \\ &\le \frac{c_n}{(\beta^2-1)^n} \prod_{j=1}^n \Big[\log^+ M(F_j,\beta^n R) + \log\frac{4\sqrt{n}}{\delta_r}\Big] \\ &\le \frac{c_n}{(\beta-1)^n} \Big[\log\|F\|_{A^{-p}} + \log\Big(\frac{1}{1-\beta^n R}\Big) + \log\Big(\frac{4\sqrt{n}}{c_2}\Big) \\ &\qquad + A_2\log\Big(\frac{1}{1-r}\Big) + 3\log\frac{1}{\gamma_0}\Big]^n. \end{aligned}$$

Since $\beta - 1 > \frac{1-r}{2n}$ and $1-\beta^n r = \frac{1-r}{2}$, there exists $C > 0$ depending on n, p and α such that $\log(\frac{1}{1-\beta^n r}) \preceq \log(\frac{1}{1-r})$, and the proof is finished. ■

Proof of Theorem 2 By [Mas98, proof of Main Theorem] there exist $p > 0$ and a map F satisfying the hypotheses of Proposition 7 with $\gamma_0 = 1$.

Fix $z \in \mathbb{B}$. Define $\Gamma_z = \{\phi_z(a_k)\}_k$ and $F^z = F \circ \phi_z$. Then $F^z \in A^{-p}$ and $\|F^z\|_{A^{-p}} \le \|F\|_{A^{-p}}(1-|z|)^{-p}$. Also $F^z \equiv 0$ on Γ_z, and letting $b_k = \phi_z(a_k)$:

$$\sum_{j=1}^n |D_u F^z_j(b_k)| \ge \sum_{j=1}^n |D_u F_j(a_k)|\, \big\| \mathcal{J}\phi_z\big(\phi_z(a_k)\big) \cdot \mathcal{J}\phi_z(a_k)\big\| \ge \frac{1}{\sqrt{n}}(1-|z|^2)^{2(n+1)}.$$

Then we get the desired result as an application of Proposition 7. ■

5 Appendix. Proof of Lemma 5

Before the proof we state some easy consequences of (a) and (b) in Theorem 1.

***Lemma 8* (See [Mas99], Lemma 3.5)** *Conditions (a) and (b) in Theorem 1 imply:*

(i) *For every $0 < r < 1$ there exists a constant $C(r) > 0$ such that $n(z,r) \le C(r)\log(\frac{e}{1-|z|})$.*
(ii) *There exists $C > 0$ such that $\log\frac{1}{|z|} n(z,|z|) \le C\log(\frac{e}{1-|z|})$.*

Proof of Lemma 5 Let

$$A(x) = \int_{1/4}^{x} \frac{\log t}{1-t}\, dt.$$

With this and the definition of $K(r)$ given in Section 2, v can be rewritten as:

$$v(z) = v_1(z) + v_2(z),$$

where

$$v_1(z) = n \sum_{k\colon |\phi_z(a_k)| \le 1/2} \left[\log |\phi_z(a_k)|^2 - \frac{A(\rho_z^2)}{K(\rho_z^2)} \right],$$

$$(13) \qquad v_2(z) = n \sum_{k\colon 1/2 < |\phi_z(a_k)| \le \rho_z} \left[\left(1 - \frac{K\big(|\phi_z(a_k)|^2\big)}{K(\rho_z^2)} \right) \log |\phi_z(a_k)|^2 - \frac{A(\rho_z^2) - A\big(|\phi_z(a_k)|^2\big)}{K(\rho_z^2)} \right].$$

Besides the notations (8), (9), we use:

$$\beta(z) = (1 - |z|^2) i\partial\bar\partial |z|^2, \quad \gamma(z) = i\partial |z|^2 \wedge \bar\partial |z|^2,$$

$$M\big(\phi_z(a_k)\big) = \frac{i\partial |\phi_z(a_k)|^2 \wedge \bar\partial |z|^2 + i\partial |z|^2 \wedge \bar\partial |\phi_z(a_k)|^2}{(1 - |z|^2)\big(1 - |\phi_z(a_k)|^2\big)}.$$

Notice that $\Psi(z) = (1 - |z|^2)^{-2}\big(\beta(z) + \gamma(z)\big)$.

We recap in the following lemma the computations we will use in the estimate of $i\partial\bar\partial v$ in the following lemma. The proof is straightforward but tedious, and we omit it.

Lemma 9 *The following equalities hold*

$$i\partial\bar\partial K\big(|\phi_z(a_k)|^2\big) = N\big(\phi_z(a_k)\big) + T\big(\phi_z(a_k)\big) = \Psi(z),$$

$$i\partial\bar\partial A\big(|\phi_z(a_k)|^2\big) = \frac{1 - |\phi_z(a_k)|^2}{|\phi_z(a_k)|^2} T\big(\phi_z(a_k)\big) + \big(\log |\phi_z(a_k)|^2\big) \Psi(z),$$

$$(1 - |z|^2)^2 i\partial\bar\partial \frac{A(|z|^2)}{K(|z|^2)} = \left(\frac{1 - |z|^2 - |z|^2 \log \frac{1}{|z|^2}}{|z|^2} + \frac{2 \log \frac{1}{|z|^2}}{K(|z|^2)} - \frac{A(|z|^2)}{K(|z|^2)} \left(1 - \frac{2}{K(|z|^2)} \right) \right) \frac{\gamma(z)}{K(|z|^2)} + \left(\log |z|^2 - \frac{A(|z|^2)}{K(|z|^2)} \right) \frac{\beta(z)}{K(|z|^2)},$$

$$i\partial\bar\partial \frac{A\big(|\phi_z(a_k)|^2\big)}{K(|z|^2)} = \frac{1 - |\phi_z(a_k)|^2}{|\phi_z(a_k)|^2} \frac{T\big(\phi_z(a_k)\big)}{K(|z|^2)} + \big(\log |\phi_z(a_k)|^2\big) \frac{\Psi(z)}{K(|z|^2)} - \big(\log |\phi_z(a_k)|^2\big) \frac{M\big(\phi_z(a_k)\big)}{K^2(|z|^2)} - \frac{A\big(|\phi_z(a_k)|^2\big)}{K^2(|z|^2)} \Psi(z) + \frac{2A\big(|\phi_z(a_k)|^2\big)}{K^3(|z|^2)} \frac{\gamma(z)}{(1 - |z|^2)^2},$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

$$i\partial\bar\partial \frac{K\big(|\phi_z(a_k)|^2\big)}{K(|z|^2)} = \left(1 - \frac{K\big(|\phi_z(a_k)|^2\big)}{K(|z|^2)}\right)\frac{\Psi(z)}{K(|z|^2)} - \frac{M\big(\phi_z(a_k)\big)}{K^2(|z|^2)} - 2\frac{K\big(|\phi_z(a_k)|^2\big)}{K^3(|z|^2)}\frac{\gamma(z)}{(1-|z|^2)^2}.$$

Now we can estimate $i\partial\bar\partial v$.

Case $|z| \le 3/4$ Here $\rho_z = 3/4$. Let $K = K(3/4)$ and $A = A(3/4)$. Using (11) and $\log|\phi_z(a_k)|^2 \ge 0$, from (13) we have:

$$\begin{aligned}
i\partial\bar\partial v(z) &\ge i\partial\bar\partial v_2(z)\\
&= n \sum_{k:\,1/2<|\phi_z(a_k)|\le 3/4} i\partial\bar\partial\left[\left(1 - \frac{K\big(|\phi_z(a_k)|^2\big)}{K}\right)\log|\phi_z(a_k)|^2 + \frac{A\big(|\phi_z(a_k)|^2\big)}{K}\right]\\
&= n \sum_{k:\,1/2<|\phi_z(a_k)|\le 3/4}\left[-\frac{\log|\phi_z(a_k)|^2}{K}\Psi(z) - \frac{2\big(1-|\phi_z(a_k)|^2\big)}{K\cdot|\phi_z(a_k)|^2}T\big(\phi_z(a_k)\big)\right.\\
&\qquad\left. + \frac{1-|\phi_z(a_k)|^2}{K\cdot|\phi_z(a_k)|^2}T\big(\phi_z(a_k)\big) + \frac{\log|\phi_z(a_k)|^2}{K}\Psi(z)\right]\\
&\ge -\frac{1}{K}\sum_{k:\,1/2<|\phi_z(a_k)|\le 3/4} \frac{1-|\phi_z(a_k)|^2}{|\phi_z(a_k)|^2}T\big(\phi_z(a_k)\big)\\
&\ge -\frac{1}{K}\sum_{k:\,1/2<|\phi_z(a_k)|\le 3/4} \big(1-|\phi_z(a_k)|^2\big)\Psi(z)\\
&\succeq -n(z,3/4)\Psi(z).
\end{aligned}$$

By Lemma 8, $n(z,3/4) \preceq \log(\frac{e}{1-|z|}) \preceq 1$, since $|z| \le 3/4$. Thus

$$i\partial\bar\partial v(z) \succeq -\Psi(z).$$

Case $|z| > 3/4$ Now $\rho_z = |z|$. We estimate separately $i\partial\bar\partial v_1$ and $i\partial\bar\partial v_2$.
Using Lemma 9, throwing away positive terms, and finally applying Lemma 8:

$$\begin{aligned}
i\partial\bar\partial v_1(z) &\succeq -n(z,1/2)\, i\partial\bar\partial\frac{A(|z|^2)}{K(|z|^2)}\\
&\ge -\frac{n(z,1/2)}{(1-|z|^2)^2}\left[\left(\frac{1-|z|^2}{|z|^2} + 2\frac{\log\frac{1}{|z|^2}}{K(|z|^2)} - \frac{A(|z|^2)}{K(|z|^2)}\right)\frac{\gamma(z)}{K(|z|^2)}\right.\\
&\qquad\left. - \frac{A(|z|^2)}{K(|z|^2)}\frac{\beta(z)}{K(|z|^2)}\right]\\
&\succeq -\frac{n(z,1/2)}{(1-|z|^2)^2}\left(\frac{\gamma(z)+\beta(z)}{K(|z|^2)}\right) \succeq -n(z,1/2)\frac{\Psi(z)}{K(|z|^2)} \succeq -\Psi(z).
\end{aligned}$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

Consider now the term (13).

$$
\begin{aligned}
i\partial\bar\partial v_2(z) \geq n \sum_{1/2<|\phi_z(a_k)|\leq|z|} & -\log|\phi_z(a_k)|^2 i\partial\bar\partial\Big(\frac{K\big(|\phi_z(a_k)|^2\big)}{K(|z|^2)}\Big) \\
& - 2\frac{1-|\phi_z(a_k)|^2}{|\phi_z(a_k)|^2}\frac{T\big(\phi_z(a_k)\big)}{K(|z|^2)} \\
& + \frac{1-|\phi_z(a_k)|^2}{|\phi_z(a_k)|^2}\frac{K\big(|\phi_z(a_k)|^2\big)\cdot M\big(\phi_z(a_k)\big)}{K^2(|z|^2)} \\
& - i\partial\bar\partial\frac{A(|z|^2)}{K(|z|^2)} + i\partial\bar\partial\frac{A\big(|\phi_z(a_k)|^2\big)}{K(|z|^2)}.
\end{aligned}
$$

Applying again Lemma 9 and (11), and throwing away positive terms

$$
\begin{aligned}
i\partial\bar\partial v_2(z) \geq n \sum_{1/2<|\phi_z(a_k)|\leq|z|} & -2\log\frac{1}{|\phi_z(a_k)|^2}\frac{K\big(|\phi_z(a_k)|^2\big)}{K^3(|z|^2)}\frac{\gamma(z)}{(1-|z|^2)^2} \\
& - \frac{1-|\phi_z(a_k)|^2}{|\phi_z(a_k)|^2}\frac{T\big(\phi_z(a_k)\big)}{K(|z|^2)} \\
& - \left[\frac{2\log\frac{1}{|z|^2}}{K(|z|^2)} - \frac{A(|z|^2)}{K(|z|^2)}\right]\frac{\gamma(z)}{(1-|z|^2)^2K(|z|^2)} \\
& + \frac{A(|z|^2)}{K(|z|^2)}\frac{\beta(z)}{(1-|z|^2)^2K(|z|^2)} \\
& - \log\frac{1}{|\phi_z(a_k)|^2}\frac{\Psi(z)}{K(|z|^2)} - \frac{A\big(|\phi_z(a_k)|^2\big)}{K(|z|^2)}\frac{\Psi(z)}{K(|z|^2)} \\
\succeq - \sum_{k\colon 1/2<|\phi_z(a_k)|\leq|z|} & \frac{\log\frac{1}{|\phi_z(a_k)|^2}}{K(|z|^2)}\Psi(z) - \frac{\log\frac{1}{|z|^2}n(z,|z|)}{K^2(|z|^2)}\Psi(z) \\
- \sum_{k\colon 1/2<|\phi_z(a_k)|\leq|z|} & \frac{A(|z|^2) - A\big(|\phi_z(a_k)|^2\big)}{K^2(|z|^2)}\Psi(z).
\end{aligned}
$$

We will be done as soon as we prove that each of the last three terms is bounded below by a constant times $-\Psi(z)$. Since $K(|z|^2) \simeq \log(\frac{1}{1-|z|})$, this is so for the first term (by condition (b) in Theorem 1) and the second term (by Lemma 8). For the third one just notice that it is implied by condition (b) in Theorem 1, since

$$
\begin{aligned}
\Big|-A(|z|^2) + A\big(|\phi_z(a_k)|^2\big)\Big| &\leq \sup_{1/4<t<1}|A'(t)|\,\Big||z|^2 - |\phi_z(a_k)|^2\Big| \\
&\leq (4/3\log 4)\big(1-|\phi_z(a_k)|^2\big). \qquad \blacksquare
\end{aligned}
$$

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press

References

[BC95] B. Berndtsson and J. Ortega Cerdà, *On interpolation and sampling in Hilbert spaces of analytic functions.* J. Reine Angew. Math. **464**(1995), 109–128.

[BC00] B. Berndtsson and P. Charpentier, *A Sobolev mapping property of the Bergman kernel.* Math. Z. (235) **1**(2000), 1–10.

[BGVY93] C. A. Berenstein, R. Gay, A. Vidras and A. Yger, *Residue currents and Bézout identities.* Progr. Math. **114**, Birkhäuser Boston, 1993.

[BP89] J. Bruna and D. Pascuas, *Interpolation in* $A^{-\infty}$. J. London. Math. Soc. (1989), 452–466.

[Del98] H. Delin, *Pointwise estimates for the weighted Bergman projection kernel in* $\mathbb{C}^n$*, using a weighted* L^2 *estimate for the* $\bar{\partial}$ *equation.* Ann. Inst. Fourier. (Grenoble) **48**(1998), 967–997.

[Gru77] L. Gruman, *The area of analytic varieties in* $\mathbb{C}^n$. Math. Scand. **41**(1977), 365–397.

[Hör90] L. Hörmander, *An introduction to complex analysis in several variables.* third ed., North-Holland, Amsterdam, 1990.

[JMT96] M. Jevtić, X. Massaneda, and P. J. Thomas, *Interpolating sequences for the weighted Bergman spaces of the ball.* Michigan Math. J. **43**(1996), 495–517.

[Kor75] B. Korenblum, *An extension of the Nevanlinna theory.* Acta. Math. **135**(1975), 187–219.

[Kor77] ———, *A Beurling type theorem.* Acta Math. **138**(1977), 265–293.

[LT96] B. Q. Li and B. A. Taylor, *On the Bézout problem and area of interpolating varieties in* $\mathbb{C}^n$. Amer. J. Math. **118**(1996), 989–1010.

[Mas97] X. Massaneda, *Interpolation by holomorphic functions in the unit ball with polynomial growth.* Ann. Fac. Sci. Toulouse Math. **VI**(1997), 277–296.

[Mas98] ———, $A^{-\infty}$*-interpolation in the ball*, Proc. Edinburgh Math. Soc. **41**(1998), 359–367.

[Mas99] ———, *Density conditions for interpolation in* $A^{-\infty}$. J. Analyse Math. **79**(1999), 299–314.

[Sei94] K. Seip, *Beurling type density theorems in the unit disk*, Invent. Math. **113**(1994), 21–29.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra
Spain
e-mail: nmarco@mat.uab.es

Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585
08071 Barcelona
Spain
e-mail: xavier@mat.ub.es

https://doi.org/10.4153/CMB-2003-053-x Published online by Cambridge University Press