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CLASSES OF SEMILATTICES ASSOCIATED WITH AN EQUATIONAL CLASS OF LATTICES

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In this paper we consider the question of whether there is a natural class of semilattices associated with a fixed equational class of lattices. We give four classes of semilattices which may be obtained from a given equational class of lattices and show that for the distributive case they coalesce into one class. An incomplete set of relations is given for the general case.

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1. For a general discussion of the basic material concerning semilattices, lattices, equational classes and direct limits we refer the reader to [**1**] and [**2**]. Any special notation will be discussed as it arises.

The concepts of distributive and modular semilattice have been defined by several authors [**2**; **4**]. Below we define four classes of semilattices which are associated with an equational class K of lattices. Of these, the first three seem to be particularly natural, and the fourth seems only mildly artificial. In this note we investigate the various inclusions holding between these four classes. Thus let K be fixed.

Definition 1. With K we associate classes of semilattices K_0, K_1, K_2, K_3 defined as follows:

(i) $\mathfrak{S} \in K_0$ if and only if $\mathfrak{J}(\mathfrak{S}) = \langle I(\mathfrak{S}); +; \cap \rangle$ is a lattice in K.

(ii) $\mathfrak{S} \in K_1$ if and only if every formula in Γ_K is valid in $\mathfrak{S}$, where $\Gamma_K = \{\Phi : \Phi$ is valid in each element of K and is a sentence of the form $\forall x_0 \ldots \forall x_{n-1} \exists y_0 \ldots \exists y_{m-1} \psi$ where ψ is open not involving $\cdot\}$.

(iii) $\mathfrak{S} \in K_2$ if and only if every map of the generators of $\mathscr{L}_n^K$ into $\mathfrak{S}$ extends to a join homomorphism with the property that the equivalence relation induced on $\mathscr{L}_n^K$ by this map is a congruence relation. Here $\mathscr{L}_n^K$ is the free K lattice on n generators.

(iv) $\mathfrak{S} \in K_3$ if and only if $\mathfrak{S}$ is a direct limit of lattices in K considered as semilattices.

For the general case we have the following theorems:

THEOREM 1. *$K_2 \subseteq K_1 \subseteq K_0$ and $K_3 \subseteq K_0$.*

THEOREM 2. *If for each $n \in \omega$, $|\mathscr{L}_n^K| < \aleph_0$, then $K_1 = K_2 \subseteq K_3$.*

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Nothing more is known at the present time. Rather than present proofs of these results, we will proceed directly to the distributive case which we consider in detail.

2. Grätzer has defined a *distributive semilattice* as a semilattice which satisfies

$$\forall x \forall y \forall z \exists u \exists v[(x + y \geqq z) \to (u \leqq x \,\&\, v \leqq w \,\&\, u + v = w)].$$

We will show that D_0, D_1, D_2 and D_3 are all equal to the class of distributive semilattices. It is easily seen that if $\mathfrak{S}$ is a distributive semilattice then $\mathfrak{J}(\mathfrak{S})$ is distributive. This result seems first to have been noticed by Green [**3**]. We state it as

LEMMA 1. *Let $\mathfrak{S}$ be any semilattice. Then $\mathfrak{S}$ is distributive if and only if $\mathfrak{J}(\mathfrak{S})$ is a distributive lattice.*

The remainder of this section we devote to showing that $D_0 = D_1 = D_2 = D_3$. We first prove the other three classes are contained in D_0.

LEMMA 2. *D_1, D_2 and $D_3 \subseteq D_0$.*

Proof. We first show that $D_1 \subseteq D_0$. Since the defining formula for distributive semilattices is valid in every distributive lattice, we are done.

To see that $D_2 \subseteq D_0$ we proceed as follows. Let a, b and $c \in S$ where $\langle S; +\rangle \in D_2$ and $a + b \geqq c$. Now if $\mathscr{L}_3^D$ is the free distributive lattice on three generators, x, y and z, let ϕ be any join homomorphism of $\mathscr{L}_3^D$ into $\mathfrak{S}$ such that $x\phi = a$, $y\phi = b$, $z\phi = c$ and $\equiv_\phi$ is a congruence relation on $\mathscr{L}_3^D$. Now by the distributive law we have

$$(xz)\phi + (yz)\phi = ((x + y)z)\phi.$$

Observe that we can define a second operation $\wedge$ on $(L_3^D)\phi$ so that $\mathscr{L}_3^D/\equiv_\phi \approx \langle L_3^D\phi; +, \wedge\rangle$. This is immediate from the fact that ϕ is a join homomorphism and $\equiv_\phi$ is a congruence relation. It follows that $((x + y)z)\phi \geqq c$. Since $(xz)\phi + (yz)\phi \leqq c$, we have $(xz)\phi + (yz)\phi = c$ and hence the defining formula for distributive semilattices is valid in $\mathfrak{S}$.

It remains to show that $D_3 \subseteq D_0$. Let

$$\mathscr{A} = \langle \{\mathfrak{S}_i : i \in I\}, \langle I; \leqq\rangle, \{\phi_i^j : j, i \in I \,\&\, i \leqq j\}\rangle$$

be a direct family of distributive semilattices. We show that $\lim_{\rightarrow}\mathscr{A}$ is a distributive semilattice. Let

$$\langle S; +\rangle = \lim_{\rightarrow} \mathscr{A},$$

and suppose we have a, b, and $c \in S$ such that $a + b \geqq c$. Now there exists $i \in I$ and x, y, and $z \in S_i$ such that $x\phi_i^\infty = a$, $y\phi_i^\infty = b$ and $z\phi_i^\infty = c$. Observe that since $((x + y) + z)\phi_i^\infty = (x + y)\phi_i^\infty$, there exists a $k \in I$ such that

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$i \leqq k$ and $((x + y) + z)\phi_i^k = (x + y)\phi_i^k$. It follows that $z\phi_i^k \leqq (x + y)\phi_i^k$, and hence since $\mathfrak{S}_k$ is a distributive semilattice we may choose x' and $y' \in S_k$ such that $x' \leqq x\phi_i^k$, $y' \leqq y\phi_i^k$ and $x' + y' = z\phi_i^k$. Now if $a' = x'\phi_k^\infty$ and $b' = y'\phi_k^\infty$ we have $a' \leqq a$, $b' \leqq b$ and $a' + b' = c$. Thus $\mathfrak{S}$ is a distributive semilattice. Now, since every distributive lattice is a distributive semilattice we have immediately that $D_3 \subseteq D_0$. This completes Lemma 2.

Observe that in the third part of Lemma 2 we actually proved more than was necessary. We state this as

COROLLARY 1. *The class of distributive semilattices is closed under direct limits.*

We wish to show that $D_0 \subseteq D_1$, D_2 and D_3. We claim that this would be straightforward given that in a distributive semilattice $\mathfrak{S}$, if $S_1 \subseteq S$ is finite then for some finite S_2, $S_1 \subseteq S_2 \subseteq S$ and $\langle S_2; +\rangle$ is a distributive semilattice. Let us see that this is the case. First observe that in any finite distributive semilattice a second operation, $\wedge$, can be defined so that $\langle S; +, \wedge\rangle$ is a distributive lattice. Thus every finite set S_1 is contained in a finite set S_2 for which $\langle S_2; +, \wedge\rangle$ is a distributive lattice. Here $\wedge$ is defined from $+$ in the only possible way. This clearly yields that $D_0 \subseteq D_2$. For $D_0 \subseteq D_1$, we let $\forall x_0 \ldots \forall x_{n-1} \exists y_0 \ldots \exists y_{m-1} \Psi \in \Gamma_D$. Now if $\mathfrak{S} \in D_0$ then we must show that $\forall x_0 \ldots \exists y_{m-1} \Psi$ is valid in $\mathfrak{S}$. Let $a_0, \ldots, a_{n-1}$ be any n-element sequence in S. Then there is a finite $S_2 \supseteq \{a_0, \ldots, a_{n-1}\}$ such that $\langle S_2; +\rangle$ is a distributive semilattice. Now as above, form $\langle S_2; +, \wedge\rangle$, a distributive lattice. Now $\exists y_0 \ldots \exists y_{m-1} \Psi[a_0, \ldots, a_{n-1}]$ is valid in $\langle S_2; +, \wedge\rangle$, hence in $\langle S_2; +\rangle$. It follows that $\exists y_0 \ldots \exists y_{m-1} \Psi[a_0, \ldots, a_{n-1}]$ is valid in $\mathfrak{S}$. Since the sequence was arbitrary, the result follows. For $D_0 \subseteq D_3$ we actually prove something stronger, which we state as Theorem 3.

THEOREM 3. *$\mathfrak{S}$ is a distributive semilattice if and only if it is a* $1 - 1$ *direct limit of finite distributive lattices considered as semilattices.*

Proof. Sufficiency is a consequence of $D_3 \subseteq D_0$. Now for the converse, let $\mathfrak{S}$ be any distributive semilattice. Recall that we assume that every finite subset of S is contained in a finite substructure which is distributive. It is well known that any algebra is a $1 - 1$ direct limit of its finitely generated subalgebra [**2**]. Observe that since finitely generated implies finite for semilattices, we are done since the finite subsemilattices which are distributive are cofinal in the collection of all finite subsemilattices, and each such subsemilattice is isomorphic to a distributive lattice.

THEOREM 4. $D_0 = D_1 = D_2 = D_3$.

To complete the proof of Theorem 4, we must show that for any distributive semilattice $\mathfrak{S}$ and any finite subset $S_1 \subseteq S$, there is a finite distributive subsemilattice $\mathfrak{S}_2$ with $S_1 \subseteq S_2 \subseteq S$. The idea behind the proof is to use the natural embedding of $\mathfrak{S}$ in $\mathfrak{J}(\mathfrak{S})$ in the following way. With S_1 as above, S_1

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generates a finite sublattice in $\mathfrak{J}(\mathfrak{S})$ via the embedding. Now if each point of this finite sublattice is principal we would be done. Unfortunately, some elements will be non-principal. We show, however, that principal replacements may be chosen so that a new finite semilattice is obtained which is join isomorphic to the old finite lattice. We then use the embedding to obtain images in $\mathfrak{S}$. With this in mind let ϕ_1 be the natural embedding of $\mathfrak{S}$ in $\mathfrak{J}(\mathfrak{S})$. Let

$$\mathfrak{S}_2^* = \langle [S_1\phi_1]; +; \cap \rangle$$

where $S_1 \subseteq S$ is finite, and $S_2^* = [S_1\phi_1]$ is the algebraic closure of $S_1\phi_1$ under $+$ and $\cap$. Now $[S_1\phi_1]$ is finite.

Definition 2. Let $S_2' \subseteq S_2^*$ consist of exactly those elements which are join irreducible in $\mathfrak{S}_2^*$. A map ϕ^* from S_2' into $S\phi_1$ is said to be *admissible* if

(i) for each $x \in S_2'$, $x\phi^* \leqq x$ and
(ii) ϕ^* is order preserving.

It is well-known that in a distributive lattice of finite length, each element has a unique representation as a join of a join-irredundant set of join irreducibles [**1**]. As a consequence, we have that each admissible map has a natural extension to a join homomorphism from $\mathfrak{S}_2^*$ into $\mathfrak{S}\phi_1$. Further, we observe that if ϕ^* and ψ^* are admissible maps, and we define $\phi^* \vee \psi^*$ by,

$$(x)(\phi^* \vee \psi^*) = (x\phi^*) + (x\psi^*),$$

then $\phi^* \vee \psi^*$ is an admissible map. For the remainder, let $S_2' = \{b_0, \ldots, b_{n-1}\}$, where this is a list in non-decreasing order.

LEMMA 3. *If $b \in S_2'$ and $a \in b$ then there is an admissible map ϕ^* such that $a \in b\phi^*$.*

Proof. Suppose that for each $i < j < n$ we have picked $a_i \leqq b_i$ such that (1) if $b = b_i$ then $a \in a_i$, and (2) if $i \leqq s < j$ and $b_i \leqq b_s$ then $a_i \leqq a_s$. We choose a_j in the following manner. If $b = b_0$ let $a_0 = a\phi_1$. Otherwise let a_0 be $c\phi_1$ for any $c \in b_0$. For $j > 0$, let a_j' be any element of b_j' and set

$$a_j = a_j'\phi_1 + \sum_{b_i \subseteq b_j} a_i + a\phi_1$$

where the last term, $a\phi_1$, is included only if $a \in b_j$. It is trivial that $a_j \leqq b_j$ and satisfies (1) and (2). We set $b_i\phi^* = a_i$ and it is clear that ϕ^* is an admissible map. This completes Lemma 3.

For each $J \subseteq n$ such that $\{b_i : i \in J\}$ is join irredundant and $a \in \sum_{i \in J} b_i$, let A_a^J be any set satisfying the following conditions:

(i) $A_a^J \subseteq S$,
(ii) $a \leqq \sum A_a^J$, and
(iii) $|A_a^J \cap b_i| = 1$ if $i \in J$ and $A_a^J \cap b_i = 0$ if $i \notin J$.

For any such J and a, A_a^J must clearly exist.

LEMMA 4. *Let $a \in b \in S_2^*$. Then there is an admissible map ϕ^* such that its extension ϕ to a homomorphism satisfies $a \in b\phi$.*

Proof. If $b \in S_2'$ we are done. Otherwise b is join reducible. Then there exists unique $J \subseteq n$ such that $b = \sum \{b_i : i \in J\}$ and J is minimal. We choose A_a^J and to each $i \in J$ an admissible ϕ_i^* which satisfies $A_a^J \cap b_i \subseteq b_i\phi_i^*$. Let $\phi^* = \vee_{i \in J} \phi_i^*$. Then if ϕ is the extension of ϕ^* to a join homomorphism, we have

$$a \leqq \sum A_a^J \in \sum_{i \in J} (b_i\phi^*) = b\phi,$$

as desired.

LEMMA 5. *There is a join isomorphism ψ of $\mathfrak{S}_2$ into $\mathfrak{S}$ such that if $x \in S_2$ is principal then $x\psi\phi_1 = x$.*

Proof. For each $b \in S_2^*$ we pick $a_b \in S$ as follows: if b is principal, let $a_b = b\phi_1^{-1}$, and if b is non-principal choose $a_b \in S$ such that for each $b^* \in S_2^*$, $a_b \in b^*$ if and only if $b \leqq b^*$. Such an a_b clearly exists in either case. Now by our preceding lemmas we choose ϕ^* such that its extension to a homomorphism ϕ satisfies $a_b \in b\phi$ for each $b \in S_2^*$. Suppose that $b \nleqq b'$. Then $a_b \in b - b'$, whence $b\phi \nsubseteq b'$. Since $b'\phi \subseteq b'$ from the admissibility of ϕ^*, we have $b\phi \nsubseteq b\phi'$. Thus ϕ is $1 - 1$. Now since $b\phi \leqq b$, we have that if b is principal then $b\phi = b$. Now $\psi = \phi\phi_1^{-1}$. Then ψ is an isomorphism and $x\psi\phi_1 = x$ as desired.

Now to complete the proof of Theorem 4, let S_1 be a finite subset of S. Obtain $\mathfrak{S}_2^*$ and let ψ satisfy the conclusion of Lemma 5. Then $S_1 \subseteq S_2^*\psi$ and $\langle S_2^*\psi; +\rangle$ is a distributive semilattice since it is join isomorphic to a distributive lattice.

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