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GENERALIZED FERMAT'S PROBLEM

Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

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ABSTRACT. The following problem is studied. Generalized Fermat's problem: in an n-dimensional Hadamard manifold M, locate a point whose distances from the given k vertices of M have the smallest possible sum.

1. **Introduction.** Let us recall an old problem in Euclidean plane geometry known as Fermat's problem (or also as Steiner's problem).

FERMAT'S PROBLEM. In a given triangle ABC, locate a point P whose distances from A, B, C have the smallest possible sum.

The answer to this problem is well-known and we refer the reader to H. S. M. Coxeter [4, p. 21] for example. The desired point uniquely exists and is given by

(1) the point P with $\angle APB = \angle BPC = \angle CPA = 2\pi / 3$ if any angle of the triangle is smaller than $2\pi / 3$, or

(2) the vertex with angle $\geq 2\pi / 3$ otherwise.

It may be interesting to ask the problem in a more general situation. Here we take the viewpoint of Riemannian geometry.

We mean by an *Hadamard manifold* a complete, simply connected, smooth Riemannian manifold with everywhere non-positive sectional curvature. Flat Euclidean spaces (which will be called simply *Euclidean spaces*), hyperbolic spaces of constant negative curvature and Riemannian symmetric spaces of non-compact type are examples of Hadamard manifolds. Let M be an n-dimensional Hadamard manifold and let $p_1, \ldots, p_k$ be k *mutually distinct* points in M, where k is an integer ≥ 3. We call these points $p_1, \ldots, p_k$ *vertices*. We say the vertices to be in *general position* if all the vertices together do not lie on any geodesic in M.

Our problem is:

GENERALIZED FERMAT'S PROBLEM. Locate a point in M whose distances from the given vertices $p_1, \ldots p_k$ have the smallest possible sum.

We call the desired points the *Fermat's points* (for the vertices $p_1, \ldots, p_k$), and call the set of the Fermat points the *Fermat set*. We will see in Proposition 2.4 that the Fermat set is a non-empty convex subset of the convex hull spanned by $p_1, \ldots, p_k$. For points p

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and q in M with $p \neq q$, we denote by $X(p,q)$ the unit tangent vector at p along the unique geodesic from p to q. We call a vertex p_h *singular* if we have

$$\|X_1 + \cdots + X_{h-1} + X_{h+1} + \cdots + X_k\| \leq 1, \tag{1.1}$$

where $X_i = X(p_h, p_i)$ for $i \neq h$, and $\| - \|$ denotes the norm of a tangent vector.

We can now state our main theorem.

THEOREM 1. *Let M be an n-dimensional Hadamard manifold and $p_1, \ldots, p_k$, $k \geq 3$, vertices on M. Then the following (I)–(IV) hold.*

(I) If the vertices $p_1, \ldots, p_k$ are in general position, then the Fermat set consists of exactly one point.

(II) In general a point p in $M - \{p_1, \ldots, p_k\}$ is the Fermat point if and only if

$$X_1 + X_2 + \cdots + X_k = 0, \tag{1.2}$$

where $X_i = X(p, p_i)$ for $i = 1, \ldots, k$.

(III) A vertex p_h is the Fermat point if and only if it is singular.

(IV) If all the vertices lie on a geodesic $\gamma\colon \mathbb{R} \longrightarrow M$ in such a way that $p_i = \gamma(t_i)$ with $t_1 < t_2 < \cdots < t_k$, then the Fermat set is $\{p_{(k+1)/2}\}$ or $\gamma\left([t_{k/2}, t_{k/2+1}]\right)$ according as k is odd or even.

The statement (IV) is obvious.

In Section 4 the Fermat point for four vertices of the 3-dimensional Euclidean space will be characterized in terms of solid angle.

We give another description of a singular vertex.

PROPOSITION 2. *The condition (1.1) is equivalent to the condition*

$$\sum_{i,j} \langle X_i, X_j \rangle \leq -(k-2)/2, \tag{1.1$'$}$$

where i and j range over the integers with $1 \leq i < j \leq k$, $i \neq h$ and $j \neq h$, and we put $X_i = X(p_h, p_i)$ for $i \neq h$.

Thus Theorem 1(I) and (III) imply

COROLLARY 3. *Let $A_1, \ldots, A_k$ be vertices in general position of the n-dimensional Euclidean space. Then the vertex A_h satisfying the following condition (1.3) is unique, if such a vertex exists:*

$$\sum_{i,j} \cos(\angle A_i A_h A_j) \leq -(k-2)/2, \tag{1.3}$$

where i and j range as in Proposition 2.

Next we give another description of the Fermat point in $M - \{p_1, \ldots, p_k\}$.

PROPOSITION 4. *The condition (1.2) is equivalent to the condition*

$$(1.2)' \qquad \text{for every } h = 1, \dots, k, \ \sum_{i,j \neq h} \langle X_i, X_j \rangle = -(k-2)/2,$$

where i and j run over the integers with $1 \leq i < j \leq k$, $i \neq h$ *and* $j \neq h$, *and we put* $X_i = X(p, p_i)$ *for* $i = 1, \dots, k$.

Theorem 1(I) and (II) imply

COROLLARY 5. *Let* $A_1, \dots, A_k$ *be vertices in general position of the n-dimensional Euclidean space* $\mathbb{R}^n$. *Suppose that there is no singular vertex. Then there exists (uniquely) a point P in* $\mathbb{R}^n$ *such that for every* $h = 1, \dots, k$, *one has*

$$(1.4) \qquad \sum_{i,j \neq h} \cos(\angle A_i P A_j) = -(k-2)/2,$$

where i and j run as in Proposition 4.

In the following sections some of Lemmas and Propositions may be obvious to specialists, but we give them proofs for the convenience of readers not specialized in the Riemannian geometry.

2. **Proof of Theorem 1.** We say a continuous function f on a Riemannian manifold N to be *convex*, if for every non-trivial geodesic $\gamma \colon [0, 1] \longrightarrow N$ and for every $t \in (0, 1)$, we have the inequality $f\big(\gamma(t)\big) \leq f\big(\gamma(0)\big) + t\Big(f\big(\gamma(1)\big) - f\big(\gamma(0)\big)\Big)$. We say the f to be *strictly convex*, if the inequality is strict. A subset C of N is defined to be *convex*, if for $p, q \in C$ there is (up to parametrization) a unique shortest geodesic from p to q in N and this geodesic is contained in C (see [2, p. 3]).

Let M be an n-dimensional Hadamard manifold. We denote by $d(-,-)$ the distance function of M. For an arbitrarily fixed point p of M, we obtain a continuous function $d(-,p) \colon M \longrightarrow \mathbb{R}\,; x \longmapsto d(x,p)$. This function is smooth except at the point p. Let q be a point in $M - \{p\}$. By the first variation formula, the gradient vector of $d(-,p)$ at q is equal to $-X(q,p)$. Let $\gamma \colon \mathbb{R} \longrightarrow M$ be a geodesic with $\gamma(0) = q$. Then by the second variation formula, we have

$$(2.1) \qquad \left(\frac{d}{dt}\right)^2 d\big(\gamma(t), p\big)|_{t=0} = \int_0^{\ell} \left\{-K(Y,X)\|Y \wedge X\|^2 + \|\nabla_X Y \wedge X\|^2\right\} dt \geq 0,$$

where $K(-,-)$ is the sectional curvature and $\|Y \wedge X\|^2 = \langle Y, Y\rangle\langle X, X\rangle - \langle Y, X\rangle^2 \geq 0$, (see [3, p. 158], [1, p. 85] or [7, p. 209]). Perhaps a word about X and Y is in order. Let $V \colon [0, \ell] \times (-\varepsilon, \varepsilon) \longrightarrow M$, $\ell = d(q,p)$, be the variation of the geodesic $\mu \colon [0, \ell] \longrightarrow M$ from q to p with arc-length parameter, such that $V(s, 0) = \mu(s)$ for $s \in [0, \ell]$ and $V(0, t) = \gamma(t)$ and $V(\ell, t) = p$ for $t \in (-\varepsilon, \varepsilon)$. Then $X = V_* \partial_s$ and $Y = V_* \partial_t$. By the inequality in (2.1), $d(-,p)|_{M - \{p\}}$ is convex. Since $d(p,p) = 0$ is the (absolute) minimum of $d(-,p)$, the continuous function $d(-,p)$ is convex on M (cf. Theorem 1.3 in p. 4 of [2]).

Let W be an open subset of M and $f\colon W \longrightarrow \mathbb{R}$ be a smooth function. For each tangent vector $Y_0 \in T_qM$, $q \in W$, there exists a unique geodesic $\gamma\colon \mathbb{R} \longrightarrow M$ with $\gamma(0) = q$ and $\dot\gamma(0) = Y_0$. The correspondence: $Y_0 \longmapsto \left(\frac{d}{dt}\right)^2 f\big(\gamma(t)\big)|_{t=0}$ gives a well-defined function $H(f)\colon T_qM \longrightarrow \mathbb{R}$. We call this $H(f)$ the *Hessian* of f at q, since $H(f)$ can be regarded as the quadratic form on T_qM associated with the bilinear form usually called the Hessian (cf. [1, p. 42]).

LEMMA 2.2. *The function $d(-,p)$ is proper and convex. At $q \in M - \{p\}$, the Hessian $H\big(d(-,p)\big)$ of $d(-,p)$ is positive semi-definite and its null space $H\big(d(-,p)\big)^{-1}(0)$ is the 1-dimensional subspace of T_qM spanned by $X(q,p)$.*

PROOF. It is sufficient to show that the null space $N = H\big(d(-,p)\big)^{-1}(0)$ is the subspace spanned by $X(q,p)$.

Suppose that Y_0 is in N. Then from (2.1) we have $\nabla_X Y \wedge X = 0$ at all points $\mu(s)$, $s \in [0, \ell\,]$. Since X is the vector field given by the geodesic μ, we have $\nabla_X X = 0$ and hence

$$\nabla_X(Y \wedge X) = \nabla_X Y \wedge X + Y \wedge \nabla_X X = \nabla_X Y \wedge X.$$

This implies $\nabla_X(Y \wedge X) = 0$ at all points $\mu(s)$, $s \in [0, \ell\,]$. The vector $Y \wedge X$ at $q = \mu(0)$, i.e., $Y_0 \wedge X(q,p)$, can be regarded as the parallel translation along the path μ of the vector $Y \wedge X$ at $p = \mu(\ell)$. The latter vector vanishes, since $Y = 0$ at p. Thus, the former vector $Y_0 \wedge X(q,p) = 0$. This shows that Y_0 lies in the subspace spanned by $X(q,p)$.

Next we show that N includes the subspace spanned by $X(q,p)$. It may suffice to prove $X(q,p) \in N$. The geodesic γ for $Y_0 = X(q,p)$ coincides with the geodesic μ, and hence $d\big(\gamma(t),p\big) = \ell - t$ for all $t \in [0, \ell\,]$. Immediately, we have

$$\left(\frac{d}{dt}\right)^2 d\big(\gamma(t),p\big)|_{t=0} = 0.$$

This completes the proof of Lemma 2.2.

DEFINITION 2.3. Let $p_1, \ldots, p_k$ be vertices in M. Hereafter we reserve the letter f for the continuous function $M \longrightarrow \mathbb{R}$ defined by $f(x) = \sum_{i=1}^k d(x,p_i)$ for $x \in M$.

The Fermat points are the points at which f takes the (absolute) minimum. Lemma 2.2 implies that f is proper and convex. Obviously f is non-negative.

PROPOSITION 2.4. *The Fermat set is non-empty, compact and convex. Furthermore it is included in the convex hull spanned by the vertices $p_1, \ldots, p_k$.*

Here the convex hull spanned by $p_1, \ldots, p_k$ is the smallest closed convex subset of M containing $p_1, \ldots, p_k$.

PROOF. It suffices to show the last statement. Let C be the convex hull spanned by $p_1, \ldots, p_k$. Obviously, C is a closed convex set. For any point $p \notin C$, we have the unique point $q \in C$ of minimal distance to p ([2, p. 8, 1.6]). We show that

$$d(p,p_i) > d(q,p_i)$$

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for all $i = 1, \ldots, k$. Let μ be the geodesic from q to p with arc-length parameter, and let $\gamma = \gamma_i$ be the geodesic from q to p_i. We denote by $\alpha = \alpha_i$ the angle at q formed by the initial tangent vectors to μ and γ. The first variation formula

$$\frac{d}{dt}L[V_t]|_{t=0} = \langle Y, X \rangle |_0^\ell - \int_0^\ell \langle Y, \nabla_X X \rangle \, dt,$$

where $\ell = d(q,p)$ and V, X and Y are as in (2.1) (cf. [3, p. 5, (1.3)]), implies $-\cos\alpha \geq 0$, and hence $\alpha \geq \pi/2$ (cf. [2, p. 9, Exercise (i)]). Now we consider a triangle QPP_i in the Euclidean plane such that $d(Q,P) = d(q,p)$, $d(Q,P_i) = d(q,p_i)$ and the angle $\angle PQP_i = \alpha$. Since $\alpha \geq \pi/2$, we have $d(Q,P_i) < d(P,P_i)$. Then by Toponogov comparison theorem ([2, pp. 5–6, 1.4]) we get $d(p,p_i) \geq d(P,P_i)$. Putting all this together, we obtain $d(p,p_i) > d(q,p_i)$. Summing up these inequalities for $i = 1, \ldots, k$, we have $f(p) > f(q)$, which implies the last statement in Proposition 2.4.

If the vertices $p_1, \ldots, p_k$ are in general position, then the null space of the Hessian $H(f)$ of f at $q \in M - \{p_1, \ldots, p_k\}$ is $\{0\}$ because of $H(f) = \sum_{i=1}^k H\big(d(-,p_i)\big)$ and Lemma 2.2. In this case, the restriction $f|_{M-\{p_1,\ldots,p_k\}}$ of f to $M - \{p_1, \ldots, p_k\}$ is strictly convex.

LEMMA 2.5 (THEOREM 1 (I)). *If the vertices $p_1, \ldots, p_k$ are in general position, then the Fermat set consists of exactly one point.*

PROOF. Suppose that the vertices are in general position. If the Fermat set has two distinct points, then the minimal geodesic segment joining these points is contained in the Fermat set. This, however contradicts the strict convexity of $f|_{M-\{p_1,\ldots,p_k\}}$.

LEMMA 2.6.

(1) A point p of M at which f takes a (locally) minimal value, must be the Fermat point.

(2) A critical point of the smooth function $f|_{M-\{p_1,\ldots,p_k\}}$ must be the Fermat point.

PROOF. This can be easily shown from the convexity of f. We omit the details.

At $p \in M - \{p_1, \ldots, p_k\}$, the gradient vector of f coincides with $-\sum_{i=1}^k X(p,p_i)$. By Lemma 2.6 (2), p is the Fermat point if and only if $\sum_{i=1}^k X(p,p_i) = 0$. This proves Theorem 1(II).

LEMMA 2.7. *For a vertex p_h, $f(p_h)$ is a minimal value of f if and only if p_h is singular.*

PROOF OF THE IF PART. Suppose the contrary. Then there is a point p in M with $f(p) < f(p_h)$ and $d(p,p_h) < d(p_h,p_i)$ for all $i \neq h$. Take a geodesic $\gamma\colon [0,\ell] \longrightarrow M$ with $\gamma(0) = p_h$ and $\gamma(\ell) = p$. Define a function $F\colon [0,\ell] \longrightarrow \mathbb{R}$ to be the composite $f \cdot \gamma$. By $d(p,p_h) < d(p_h,p_i)$, F is smooth on $[0,\ell]$. Since $f(p_h) > f(p)$, there exists a real number t_0 with $0 < t_0 < \ell$ and $F'(t_0) < 0$. On the other hand, we have

$$F'(0) = \|\dot\gamma(0)\| - \sum_{i=1, i\neq h}^{k} \langle \dot\gamma(0), X_i \rangle, \tag{2.8}$$

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where $X_i = X(p_h, p_i)$. Since p_h is singular, $\| \sum_{i=1, i\neq h}^{k} X_i \|$ is not greater than 1. By the Cauchy-Schwarz inequality, we get $F'(0) \geq 0$. From the convexity of f, we obtain $F'(t) \geq 0$ for all $t \in (0, \ell)$. This contradicts $F'(t_0) < 0$. Hence $f(p_h)$ is minimal.

PROOF OF THE ONLY IF PART. It suffices to prove that f does not take a minimal value at p_h if $\| \sum_{i=1, i\neq h}^{k} X_i \| > 1, X_i = X(p_h, p_i)$. Let $\gamma\colon \mathbb{R} \longrightarrow M$ be the geodesic with $\gamma(0) = p_h$ and $\dot{\gamma}(0) = \sum_{i=1, i\neq h}^{h} X_i$, and suppose $\|\dot{\gamma}(0)\| > 1$. Define a function $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ to be the composite $f \cdot \gamma$. Then $F'(0)$ is given by (2.8), and $F'(0) = \|\dot{\gamma}(0)\| - \|\dot{\gamma}(0)\|^2 < 0$. Thus $f(p_h) = F(0)$ is not minimal.

Putting Lemmas 2.6 and 2.7 together, we see that a vertex p_h is the Fermat point if and only if it is singular. This completes the proof of Theorem 1 (III).

3. **Proof of Propositions 2 and 4.**

PROOF OF PROPOSITION 2. We put $X_i = X(p_h, p_i)$ for $i \neq h$ and $X = \sum_{i=1, i\neq h}^{k} X_i$. We recall that X_i are unit vectors. Suppose that the inequality (1.1) holds, namely $\|X\| \leq 1$. Then the inequality $(1.\,1)'$ is obtained from the equalities

$$\langle X, X \rangle = \sum_{\ell=1, \ell\neq h}^{k} \langle X_\ell, X_\ell \rangle + 2 \sum_{1\leq i<j\leq k, i\neq h, j\neq h} \langle X_i, X_j \rangle, \text{ and}$$

$$\sum_{\ell=1, \ell\neq h}^{k} \langle X_\ell, X_\ell \rangle = k - 1.$$

The converse is also obtained from these equalities.

PROOF OF PROPOSITION 4. We put $X_i = X(p, p_i)$ for $i = 1, \ldots, k$ and $Y = X_1 + X_2 + \cdots + X_k$. Then we get

$$\langle Y, Y \rangle = \sum_{\ell=1}^{k} \langle X_\ell, X_\ell \rangle + 2 \sum_{1\leq i<j\leq k} \langle X_i, X_j \rangle, \text{and}$$

$$\langle X_h, Y \rangle = \langle X_h, X_h \rangle + \sum_{i=1, i\neq h}^{k} \langle X_h, X_i \rangle.$$

These give the equality

$$\langle Y, Y \rangle - 2\langle X_h, Y \rangle = (k-2) + 2 \sum_{1\leq i<j\leq k, i\neq h, j\neq h} \langle X_i, X_j \rangle. \tag{3.1}$$

Suppose that the equality (1.2) holds, namely $Y = 0$. Then the left-hand side of (3.1) is equal to 0, and we obtain the equality $(1.\,2)'$. The converse is obtained from (3.1) and

$$\sum_{h=1}^{k} (\langle Y, Y \rangle - 2\langle X_h, Y \rangle) = -\langle Y, Y \rangle.$$

4. **Another characterization of the Fermat point in the 3-dimensional Euclidean space.** We begin with

DEFINITION 4.1. Let $A_1, A_2, \ldots, A_{n+1}$ be $n+1$ vertices in the n-dimensional Euclidean space $\mathbb{R}^n$ which are not contained in any $(n-1)$-dimensional subspace, where n is an integer ≥ 2. A point P of $\mathbb{R}^n$ is said to be an *equiangular point* for $A_1, A_2, \ldots, A_{n+1}$ if the $n+1$ (n-dimensional) solid angles at P spanned by n vertices of $A_1, A_2, \ldots, A_{n+1}$ are well-defined and partition equally the total solid angle around P.

In $\mathbb{R}^2$ an equiangular point for three given vertices A_1, A_2 and A_3 exists if and only if there is no singular vertex. In this case, the equiangular point is precisely the Fermat point for them and a vertex, say A_1, is singular if and only if $\angle A_2A_1A_3 \geq 2\pi/3$. The following theorem asserts that similar results hold in $\mathbb{R}^3$.

THEOREM 4.2. *Let A_1, A_2, A_3 and A_4 be vertices in $\mathbb{R}^3$ which are not contained in a plane. Then the following (I) and (II) hold.*

(I) There exists an equiangular point for A_1, A_2, A_3 and A_4 (in the sense of Definition 4.1) if and only if there is no singular vertex. Furthermore in this case a point P is an equiangular point for A_1, A_2, A_3 and A_4 if and only if it is the Fermat point for them. In particular an equiangular point, if it exists, is unique.

(II) A vertex, say A_1, is singular if and only if the solid angle at A_1 of the tetrahedron $A_1A_2A_3A_4$ is greater than or equal to $\pi (= 4\pi/4)$.

In order to prove this theorem, we prepare

PROPOSITION 4.3. *Let v_i $(1 \leq i \leq 3)$ be unit vectors in $\mathbb{R}^3$, and T the spherical triangle on the 2-dimensional sphere S^2 spanned by v_i $(1 \leq i \leq 3)$. Then the area* Area(T) *of T is given by the formula*

$$\text{Area}(T) = 2\cos^{-1}\frac{\|v_1 + v_2 + v_3\|^2 - 1}{2\sqrt{2\prod_{i<j}(1 + \langle v_i, v_j\rangle)}}, \tag{4.4}$$

where i and j range over the integers with $1 \leq i < j \leq 3$.

PROOF. We assume the reader to be familiar with elementary formulae on spherical trigonometry. Denote lengths of sides of a spherical triangle ABC by a, b and c. Then Euler's formula (see [5, pp. 176–177] or [6, p. 76] for example) gives

$$\begin{aligned}\cos\frac{\text{Area}(T)}{2} &= \frac{1 + \cos a + \cos b + \cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}} \\ &= \frac{1 + \cos a + \cos b + \cos c}{\sqrt{2(1 + \cos a)(1 + \cos b)(1 + \cos c)}}.\end{aligned}$$

Since $0 \leq \text{Area}(T)/2 \leq \pi$, we obtain the formula (4.4).

PROOF OF THEOREM 4.2. The assertion (II) is an immediate consequence of Proposition 4.3.

We prove the assertion (I). Suppose that P is an equiangular point for A_1, A_2, A_3 and A_4. Choose arbitrarily three vectors from the four vectors $X_i = X(P, A_i)$, $i = 1, \ldots, 4$, and regard them as v_1, v_2 and v_3 in Proposition 4.3. Then in the formula (4.4), Area$(T) = 4\pi/4$ and hence $\|v_1 + v_2 + v_3\|^2 = 1$ for any choice. Thus, for any h with $1 \leq h \leq 4$,

$$\sum_{1 \leq i,j \leq 4,\, i \neq h, j \neq h} \langle X_i, X_j \rangle = -1. \tag{4.5}$$

By Proposition 4 (where $k = 4$), we get $X_1 + X_2 + X_3 + X_4 = 0$. By Theorem 1(II), P must be the Fermat point for the vertices $A_1, \ldots, A_4$. Since the Fermat point is unique (see Theorem 1(I)), any of the vertices $A_1, \ldots, A_4$ is not the Fermat point. By Theorem 1(III), there is no singular vertex among $A_1, \ldots, A_4$.

Next suppose that the vertices $A_1, \ldots, A_4$ are not singular. Then there exists uniquely the Fermat point P for the vertices $A_1, \ldots, A_4$. The point P is distinct from $A_1, \ldots, A_4$ and $X_1 + X_2 + X_3 + X_4 = 0$. By Proposition 4, we obtain the equality (4.5). Proposition 4.3 implies that all solid angles at P spanned by three vertices of $A_1, \ldots, A_4$ are equal to π, if they are well-defined. On this occasion, P is an equiangular point for $A_1, \ldots, A_4$. For the well-definedness of the solid angles, we have to show that $\angle A_i P A_j \neq \pi$ for all i and j with $1 \leq i < j \leq 4$. For example suppose $\angle A_1 P A_2 = \pi$. Since $X_1 + X_2 + X_3 + X_4 = 0$, we have

$$\langle X_1 + X_2, X_1 + X_2 \rangle = \langle X_3 + X_4, X_3 + X_4 \rangle.$$

This yields $\langle X_1, X_2 \rangle = \langle X_3, X_4 \rangle$, hence $\angle A_3 P A_4 = \pi$. Thus, $A_1, \ldots, A_4$ must be in a plane. This contradicts our assumption in Theorem 4.2. Similarly we can prove $\angle A_i P A_j \neq \pi$ for general i and j. We omit the details.

There may naturally arise a question whether the above theorem can be generalized to higher dimensional Euclidean spaces. But the answer is "no". We conclude this section with such a counterexample in $\mathbb{R}^4$.

EXAMPLE 4.6. Let $A_1 = (1,0,0,0)$, $A_2 = (0,1,0,0)$, $A_3 = (0,0,1,0)$, $A_4 = (-1/2, -1/2, -1/2, -1/2)$ and $A_5 = (-1/2, -1/2, -1/2, 1/2)$ be vertices in $\mathbb{R}^4$. These A_i can be regarded as unit vectors in $\mathbb{R}^4$. Since $A_1 + A_2 + \cdots + A_5 = 0$, by Theorem 1 the origin 0 is the Fermat point for A_i, $1 \leq i \leq 5$. But the origin is not an equiangular point for them as is seen in the following.

The 4-dimensional total solid angle around the origin is $2\pi^2$, and the solid angle ω at the origin spanned by the four vertices A_1, A_2, A_3 and $B_4 = (0,0,0,1)$ is $\pi^2/8$. The complementary solid angle of ω is divided equally into four parts, namely the solid angles at the origin spanned by the vertex A_4 and three of A_1, A_2, A_3 and B_4. Hence the solid angle at the origin spanned by A_1, A_2, A_3 and A_4 is equal to $(2\pi^2 - \pi^2/8)/4 \quad (= 15\pi^2/32)$ which is not equal to $2\pi^2/5$.

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