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ON THE RADIUS OF CURVATURE FOR CONVEX ANALYTIC FUNCTIONS

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1. Introduction.

Definition 1.1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| < 1$. If f is univalent, we say that f belongs to the class S.

Definition 1.2. Let $f \in S$, $0 \leqq \alpha < 1$. Then f belongs to the class of convex functions of order α, denoted by K_α, provided

$$\text{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha \quad \text{for } |z| < 1, \tag{1}$$

and if $\epsilon > 0$ is given, there exists z_0, $|z_0| < 1$, such that

$$\text{Re}\left[1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right] \leqq \alpha + \epsilon.$$

Let $f \in K_\alpha$ and consider the Jordan curve $\gamma_r = f(|z| = r)$, $0 < r < 1$. Let $s(r, \theta)$ measure the arc length along γ_r; and let $\phi(r, \theta)$ measure the angle (in the anti-clockwise sense) that the tangent line to γ_r at $f(re^{i\theta})$ makes with the positive real axis. Then the radius of curvature of γ_r at $f(re^{i\theta})$ is $\rho(r, \theta) \equiv (\partial\phi/\partial s)^{-1}$. It is known [**3**, p. 359] that

$$\rho(r, \theta) = \frac{r|f'(z)|}{\text{Re}[1 + zf''(z)/f'(z)]}, \qquad z = re^{i\theta}. \tag{2}$$

Keogh has shown [**4**] that if f is convex, then

$$\max_\theta \rho(r, \theta) \leqq \frac{r}{1 - r^2}, \tag{3}$$

with equality holding for the function $f(z) = z/(1 - z)$.

Our aim in this paper is to determine all functions in K_0 for which equality holds in (3), and also to give a corresponding result for the class K_α.

2. Main results. In order to determine which functions of K_0 yield equality in (3), I include the proof of Keogh's result, which I state as a lemma.

LEMMA 2.1. *If f is convex, then*

$$\max_\theta \rho(r, \theta) \leqq \frac{r}{1 - r^2}.$$

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Proof. Since f is convex,

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right] > 0 \quad \text{in } |z| < 1.$$

Thus, there exists a monotone increasing function $u(t)$ defined on $[0, 2\pi]$ such that $\int_0^{2\pi} du(t) = 1$,

$$1+\frac{zf''(z)}{f'(z)} = \int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\, du(t). \tag{4}$$

Hence,

$$\log f'(z) = \int_0^{2\pi} \log(1-ze^{-it})^{-2}\, du(t). \tag{5}$$

Therefore, $\log|f'(z)| = \int_0^{2\pi} \log|1-ze^{-it}|^{-2}\, du(t)$. Thus,

$$|f'(z)| = \exp \int_0^{2\pi} \log \frac{1}{\Delta^2}\, du(t),$$

where $\Delta^2 = \Delta^2(r, t-\theta) = 1 - 2r\cos(t-\theta) + r^2$, $z = re^{i\theta}$. From (2) and (4) it follows that

$$\rho(r,\theta) = r\left[\exp \int_0^{2\pi} \log \frac{1}{\Delta^2}\, du(t)\right]\left[\int_0^{2\pi} \frac{1-r^2}{\Delta^2}\, du(t)\right]^{-1}.$$

By the arithmetic geometric mean inequality [**1**, p. 156] we have

$$\exp \int_0^{2\pi} \log \frac{1}{\Delta^2}\, du(t) \leqq \int_0^{2\pi} \frac{1}{\Delta^2}\, du(t). \tag{6}$$

Thus, $\rho(r,\theta) \leqq r/(1-r^2)$.

THEOREM 2.2. *Equality holds in Lemma* 2.1 *only for the following functions* (*and rotations*)†:

(A) $f(z) = \dfrac{z}{1-z}$,

(B) $f(z) = \dfrac{1}{2i\sin t} \log\left[\dfrac{1-ze^{-it}}{1-ze^{it}}\right], \qquad \sin t \neq 0,$

(C) $f(z) = \dfrac{1}{(1-2\lambda)2i\sin t}\left[\left(\dfrac{1-ze^{-it}}{1-ze^{it}}\right)^{1-2\lambda} - 1\right],$

$$\sin t \neq 0, \qquad 0 < \lambda < 1, \lambda \neq \tfrac{1}{2}.$$

Proof. By the proof of the lemma, we have equality if and only if equality holds in (6) for some $\theta = \theta(r)$. Now, this occurs only if $\log(1/\Delta^2)$ is constant, except possibly on a set of du-measure zero [**1**, p. 156]. By the nature of

†If g is a rotation of f, then $g(z) = e^{-i\beta}f(ze^{i\beta})$ for some β, $0 \leqslant \beta < 2\pi$.

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$\log(1/\Delta^2)$, this is the case only if either (a) $u(t)$ has a single jump, or (b) $u(t)$ has exactly two jumps located at points t_1, t_2 such that $\cos(t_1 - \theta(r)) = \cos(t_2 - \theta(r))$. It follows from (a) and (b) that equality in (6) for $\theta = \theta(r)$ is independent of r; i.e. we have that $\theta(r)$ is a constant, say $\theta(r) = \bar{\theta}$. Note that if case (b) occurs, then we also have equality in (6) for $\theta = \bar{\theta} + \pi$, but not for any other values of θ $(\bar{\theta} \leqq \theta < \bar{\theta} + 2\pi)$. Now, if $f(|z| = r)$ has its maximum radius of curvature at $f(re^{i\bar{\theta}})$ (i.e., equality in (6) for $\theta = \bar{\theta}$), then the function $g(z) = e^{-i\bar{\theta}}f(ze^{i\bar{\theta}})$ will have the property that $g(|z| = r)$ has its maximum radius of curvature at $g(r)$. Thus, we need only assume that equality holds in (6) for $\bar{\theta} = 0$. Then for each function which yields a sharp result in this case, we must also include its rotations. Let us now examine cases (a) and (b).

Case (a). $u(t)$ has a single jump, say at t. From (5) we have

$$\log f'(z) = \log(1 - ze^{-it})^{-2}.$$

Hence, $f(z) = z/(1 - ze^{-it})$, and so $f(z)$ is a rotation of $z/(1 - z)$. This yields part (A) in the statement of the theorem.

Case (b). $u(t)$ has two jumps, located at points t_1, t_2, where $\cos t_1 = \cos t_2$. Let λ and $1 - \lambda$ denote the lengths of the jumps at t_1 and t_2, respectively. From (5) we have

$$\log f'(z) = \lambda \log(1 - ze^{-it_1})^{-2} + (1 - \lambda)\log(1 - ze^{-it_2})^{-2}.$$

Since $\cos t_1 = \cos t_2$, $e^{-it_2} = e^{it_1}$. Letting $t_1 = t$, we have

$$f(z) = \begin{cases} \dfrac{1}{2i \sin t} \log\left[\dfrac{1 - ze^{-it}}{1 - ze^{it}}\right] & \text{if } \lambda = \tfrac{1}{2}, \\ \dfrac{1}{(1 - 2\lambda)2i \sin t}\left[\left(\dfrac{1 - ze^{-it}}{1 - ze^{it}}\right)^{1-2\lambda} - 1\right] & \text{if } \lambda \neq \tfrac{1}{2}. \end{cases}$$

This yields parts (B) and (C) in the statement of the theorem.

THEOREM 2.3. *If* $f \in K_\alpha$ $(0 < \alpha < 1)$, *then*

$$\max_\theta \rho(r, \theta) \leqq \frac{r}{(1 - r^2)^{1-\alpha}}$$

with equality holding only for the function (*and rotations*):

$$f(z) = \begin{cases} -\log(1 - z) & \textit{if } \alpha = \tfrac{1}{2}, \\ \dfrac{1 - (1 - z)^{2\alpha-1}}{2\alpha - 1} & \textit{if } \alpha \neq \tfrac{1}{2}. \end{cases}$$

Proof. Let $f \in K_\alpha$. Then $g(z) \equiv \int_0^z [f'(t)]^{1/(1-\alpha)}\, dt \in K_0$. By Lemma 2.1 we have

$$\rho_g(r, \theta) \equiv \frac{|g'(z)|}{\mathrm{Re}[1 + zg''(z)/g'(z)]} \leqq \frac{r}{1 - r^2}, \qquad z = re^{i\theta}. \tag{7}$$

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One easily checks the following relations:

$$|g'(z)| = |f'(z)|^{1/(1-\alpha)},$$

(8) $$\operatorname{Re}\left[1 + \frac{zg''(z)}{g'(z)}\right] = 1 + \frac{1}{1-\alpha}\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}\right].$$

Thus, from (7) it follows that

(9) $$\frac{r|f'(z)|^{1/(1-\alpha)}}{1 + (1-\alpha)^{-1}\operatorname{Re}[zf''(z)/f'(z)]} \leqq \frac{r}{1-r^2}, \qquad z = re^{i\theta}.$$

Note that since $f \in K_\alpha$, $\operatorname{Re}[zf''(z)/f'(z)] > \alpha - 1$. Fix z and let

$$t = \operatorname{Re}[zf''(z)/f'(z)].$$

For $t > \alpha - 1$ we have

(10) $$1 + \frac{1}{1-\alpha}t \leqq (1+t)^{1/(1-\alpha)},$$

with equality holding only when $t = 0$. Applying (10) to (9) we have

$$\rho_f(r,\theta) = \frac{r|f'(z)|}{\operatorname{Re}[1 + zf''(z)/f'(z)]} \leqq \frac{r}{(1-r^2)^{1-\alpha}}, \qquad z = re^{i\theta}.$$

Hence,

(11) $$\max_\theta \rho_f(r,\theta) \leqq \frac{r}{(1-r^2)^{1-\alpha}}.$$

To determine when equality occurs in (11), we need only, by (7), examine those functions $f \in K_\alpha$ such that $g(z) = \int_0^z [f'(t)]^{1/(1-\alpha)}\,dt$ is a function given in (A), (B), or (C) of Theorem 2.2. Suppose first that g has the form (A); i.e., g is a rotation of $z/(1-z)$. Without loss of generality we can assume that $g(z) = z/(1-z)$. It follows that

(12) $$f(z) = \begin{cases} -\log(1-z) & \text{if } \alpha = \frac{1}{2}, \\ \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha-1} & \text{if } \alpha \neq \frac{1}{2}. \end{cases}$$

Since $g(z) = z/(1-z)$ maps the circle $|z| = r$ onto a circle of radius $r/(1-r^2)$, we have equality in (7) for all θ. Hence, $f(z)$ yields equality in (9) for all θ. By (10) it follows that equality holds in (11), provided for each r, $0 < r < 1$, there exists $\theta = \theta(r)$, such that

(13) $$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}\right] = 0 \quad \text{for } z = re^{i\theta(r)}.$$

Using (12), one sees that (13) holds if $\theta(r) = \operatorname{arc}\cos r$.

Secondly, suppose that g has the form (B) or (C) of Theorem 2.2. We intend to show that f, defined implicitly by

$$g(z) = \int_0^z [f'(t)]^{1/(1-\alpha)}\,dt,$$

does not yield equality in (11). Without loss of generality we can assume that $g(z)$ is not a rotated form of (B) or (C). Then, equality in (7) can occur only for real values of z. Hence, for equality to hold in (11), by (10) we must have for each r, $0 < r < 1$,

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)}\right] = 0 \quad \text{for } z = r \text{ or for } z = -r. \tag{14}$$

We must show that (14) is impossible. Now,

$$\frac{zg''(z)}{g'(z)} = \frac{2\lambda ze^{-it}}{1 - ze^{-it}} + \frac{2(1-\lambda)ze^{it}}{1 - ze^{it}}.$$

From (8) it follows that

$$2(1-\alpha)\left[\frac{\lambda ze^{-it}}{1 - ze^{-it}} + \frac{(1-\lambda)ze^{it}}{1 - ze^{it}}\right] = \frac{zf''(z)}{f'(z)}.$$

From this expression we see that (14) cannot occur for each r, $0 < r < 1$. The proof of the theorem is complete.

We now give an arc-length result for the class K_α. Keogh [**4**] has proved this result, with $\alpha = 0$, for functions which are convex.

THEOREM 2.4. *Let $f \in K_\alpha$, $0 \leqq \alpha < 1$, and let L_r be the length of*

$$\gamma_r = \{f(re^{i\theta}): 0 \leqq \theta \leqq 2\pi\}.$$

Then

$$L_r \leqq r\int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{2(1-\alpha)}},$$

with equality only for the function (and rotations):

$$f(z) = \begin{cases} -\log(1-z) & \text{if } \alpha = \frac{1}{2}, \\ \dfrac{1-(1-z)^{2\alpha-1}}{2\alpha - 1} & \text{if } \alpha \neq \frac{1}{2}. \end{cases}$$

Proof. Since $f \in K_\alpha$, we have

$$\operatorname{Re}\left[(1-\alpha) + \frac{zf''(z)}{f'(z)}\right] > 0 \quad \text{in } |z| < 1.$$

Thus, there exists a monotone increasing function $u(t)$, defined on $[0, 2\pi]$, $\int_0^{2\pi} du(t) = 1$,

$$(1-\alpha) + \frac{zf''(z)}{f'(z)} = (1-\alpha)\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, du(t).$$

Thus, $\log f'(z) = \int_0^{2\pi} \log(1 - ze^{-it})^{-2(1-\alpha)}\, du(t)$. Hence, $\log|f'(z)| = \int_0^{2\pi} \log|1 - ze^{-it}|^{-2(1-\alpha)}\, du(t)$. It follows by the arithmetic geometric mean inequality that

$$|f'(z)| \leqq \int_0^{2\pi} |1 - ze^{-it}|^{-2(1-\alpha)}\, du(t). \tag{15}$$

Since $L_r = \int_0^{2\pi} r|f'(z)|\, d\theta$, $z = re^{i\theta}$, we have

$$L_r \leqq r \int_0^{2\pi} d\theta \int_0^{2\pi} |1 - ze^{-it}|^{-2(1-\alpha)}\, du(t) \tag{16}$$
$$= r \int_0^{2\pi} du(t) \int_0^{2\pi} |1 - re^{i\theta}|^{-2(1-\alpha)}\, d\theta$$
$$= r \int_0^{2\pi} |1 - re^{i\theta}|^{-2(1-\alpha)}\, d\theta.$$

By the same method used in the proof of Theorem 2.2, we examine (15) for equality. We then find that equality holds in (16) only for the function $f \in K_\alpha$ which is listed in the statement of the theorem. The proof is complete.

Remark. From a result of Hayman [**2**, p. 280], we have the following growth estimates:

$$L_r = \begin{cases} O((1-r)^{2\alpha-1}) & \text{if } 0 \leqq \alpha < \frac{1}{2}, \\ O\left(\log \dfrac{1}{1-r}\right) & \text{if } \alpha = \frac{1}{2}, \\ O(1) & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

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