On Series for calculating Euler's Constant and the Constant in Stirling's Theorem.

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1. Let γ_n denote the value of

(1) $$1+\tfrac{1}{2}+\tfrac{1}{3}\ldots+1/n-\log n$$

where n is a definite integer; and let γ denote the limit of

(2) $$1+\tfrac{1}{2}+\tfrac{1}{3}\ldots+1/n+1/(n+1)\ldots+1/(n+k)-\log(n+k),$$

when the integer k is indefinitely increased. It is known* that the expansion of $\gamma_n-\gamma$ in ascending powers of $1/n$ is

(3) $$\frac{1}{2n}-\frac{B_1}{2n^2}+\frac{B_3}{4n^4}-\frac{B_5}{6n^6}+\ldots,$$

where B_1, B_3, $B_5\ldots$ are the numbers of Bernoulli. The series (3) is, however, divergent, as B_{2r+1} not only increases indefinitely with r, but bears† an infinite ratio to B_{2r-1} in this case. It is proposed to find by elementary methods the expansion of $\gamma_n-\gamma$ up to the term in n^r and to estimate the error (of order $1/n^{r+1}$) made in omitting further terms of series (3). I shall take the case of $r=9$, but the process is quite general.

2. From (2) we obtain $1-\gamma_{n+k}$

$$\begin{aligned}&=\log(n+k)-\tfrac{1}{2}-\tfrac{1}{3}\ldots-1/n\ldots-1/(n+k)\\&=(\log 2-\log 1-\tfrac{1}{2})+(\log 3-\log 2-\tfrac{1}{3})+\ldots\\&\quad+(\log n-\log\overline{n-1}-1/n)+(\log\overline{n+1}-\log n-1/\overline{n+1})+\\&\quad\ldots+\log(n+k)-\log(n+k-1)-1/(n+k).\end{aligned}$$

The first $n-1$ brackets amount to $1-\gamma_n$; hence

$$\gamma_n-\gamma_{n+k}=-\left(\log\frac{n}{n+1}+\frac{1}{n+1}\right)-\left(\log\frac{n+1}{n+2}+\frac{1}{n+2}\right)-\ldots$$

$$\ldots-\left(\log\frac{n+k-1}{n+k}+\frac{1}{n+k}\right).$$

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* Boole, *Finite Diff.* Ch. V. (Euler-Maclaurin Formula); Todhunter, *Integral Calc.* Ch. XII.

† Chrystal, *Algebra*, Ch. XXX.

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The logarithms on the right side can all be expanded in convergent series, as $1/(n+1)$, $1/(n+2)$, $\ldots 1/(n+k)$ are each less than unity; so that

$$\gamma_n - \gamma_{n+k} = \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} + \cdots$$
$$+ \frac{1}{2(n+2)^2} + \frac{1}{3(n+2)^3} + \frac{1}{4(n+2)^4} + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{1}{2(n+k)^2} + \frac{1}{3(n+k)^3} + \frac{1}{4(n+k)^4} + \cdots$$

This doubly infinite series is convergent either way; the columns, therefore, can be written as rows. Hence, making k infinite,

(4) $$\gamma_n - \gamma = \tfrac{1}{2}t_2 + \tfrac{1}{3}t_3 + \tfrac{1}{4}t_4 \ldots \textit{ad inf.},$$

where $t_r = \Sigma_\infty^1 (n+p)^{-r}$. We proceed to expand t_2, t_3, …in powers of $1/n$.

3. Let $\phi\,(m, d, k)$ denote the reciprocal of

$$m\{m+d\}\{m+2d\}\ldots\{m+(k-1)d\};$$

then $$\phi(m-r,\ 1,\ 2r+1) - \phi(m-r+1,\ 1,\ 2r-1)\,.\,\phi(m,\ 0,\ 2)$$
$$= r^2\,.\,\phi(m-r,\ 1,\ 2r+1)\,.\,\phi(m,\ 0,\ 2).$$

Change m to $n+1$ and transpose; we get

(5) $$\phi(n-r+2,\ 1,\ 2r-1)\,.\,\phi(n+1,\ 0,\ 2)$$
$$= \phi(n-r+1, 1, 2r+1) - r^2\phi(n-r+1,\ 1,\ 2r+1)\,.\,\phi(n+1,\ 0,\ 2).$$

Putting $r = 1,\ 2,\ 3,\ \ldots,$ we have

$$\phi(n+1,\ 0,\ 3) = \phi(n,\ 1,\ 3) - 1^2\,.\,\phi(n,\ 1,\ 3)\,,\,\phi(n+1,\ 0,\ 2),$$
$$\phi(n, 1, 3)\phi(n+1, 0, 2) = \phi(n-1, 1, 5) - 2^2.\phi(n-1, 1, 5).\phi(n+1, 0, 2),$$
$$\phi(n-1, 1, 5)\phi(n+1, 0, 2) = \phi(n-2, 1, 7) - 3^2.\phi(n-2, 1, 7).\phi(n+1, 0, 2),$$

and so on. Hence, by continued substitution,

(6) $$\phi(n+1,\ 0,\ 3) = \phi(n,\ 1,\ 3) - 1^2\phi(n-1,\ 1,\ 5) + (1\,.\,2)^2\phi(n-2,\ 1,\ 7)$$
$$- (1\,.\,2\,.\,3)^2\phi(n-3,\ 1,\ 9) + \ldots.$$

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We can stop at any point; the last term contains $\phi(n+1, 0, 2)$ as factor, and is positive or negative according as its number is odd or even. Thus we obtain closer and closer relations of inequality, of which for the present we take the following:

$$\phi(n+1, 0, 3) \text{ is } > \phi(n, 1, 3) - \phi(n-1, 1, 5) + 4\phi(n-2, 1, 7)$$
$$- 36\phi(n-3, 1, 9), \text{ but } < \text{ this expression } + 576\phi(n-4, 1, 11).$$

Changing $n+1$ to $n+2$, $n+3$, ..., and adding up, we get the sum of $\phi(n+1, 0, 3)$, $\phi(n+2, 0, 3)$, $\phi(n+3, 0, 3)$, ... to infinity, *i.e.*, t_3, to be*

$$> \tfrac{1}{2}\phi(n, 1, 2) - \tfrac{1}{4}\phi(n-1, 1, 4) + \tfrac{2}{3}\phi(n-2, 1, 6) - \tfrac{9}{2}\phi(n-3, 1, 8)$$

and < this expression increased by $\frac{288}{5}\phi(n-4, 1, 10)$.

To get t_5, we multiply both sides of (6) by $\phi(n+1, 0, 2)$ and apply equation (5) to the terms on the right side in succession; we thus obtain an equation† like (6), whence similar relations of inequality can be inferred. Thus $\phi(n+1, 0, 5)$ is found to be $> \phi(n-1, 1, 5) - 5\phi(n-2, 1, 7) + 49\phi(n-3, 1, 9) - 820\phi(n-4, 1, 11)$, but < the first three terms of this expression. Hence, as before, changing $n+1$ to $n+2$, $n+3$, ... *ad. inf.*, and adding up, we get t_5 to be

$$> \tfrac{1}{4}\phi(n-1, 1, 4) - \tfrac{5}{6}\phi(n-2, 1, 6) + \tfrac{49}{8}\phi(n-3, 1, 8) - 82\phi(n-4, 1, 10)$$

but < the first three terms of this expression.

We shall similarly obtain $t_7 > \frac{1}{6}\phi(n-2, 1, 6) - \frac{7}{4}\phi(n-3, 1, 8)$, but < this expression $+\frac{273}{10}\phi(n-4, 1, 10)$; $t_9 > \frac{1}{8}\phi(n-3, 1, 8) - 3\phi(n-4, 1, 10)$, but $< \frac{1}{8}\phi(n-3, 1, 8)$; and $t_{11} < \frac{1}{10}\phi(n-4, 1, 10)$. We will not consider $t_{13}, t_{15}, \ldots$, as these when expanded do not affect the term in $1/n^9$ and the previous terms.

To obtain t_2, put $r = \frac{1}{2}$ in equation (5); thus $\phi(n+\frac{3}{2}, 1, 0) \,.\, \phi(n+1, 0, 2)$, *i.e.*, $\phi(n+1, 0, 2) = \phi(n+\frac{1}{2}, 1, 2) - \frac{1}{4}\phi(n+\frac{1}{2}, 1, 2) \,.\, \phi(n+1, 0, 2)$. So also $\phi(n+\frac{1}{2}, 1, 2) \,.\, \phi(n+1, 0, 2) = \phi(n-\frac{1}{2}, 1, 4) - \frac{9}{4}\phi(n-\frac{1}{2}, 1, 4) \,.\, \phi(n+1, 0, 2)$; the last function $= \phi(n-\frac{3}{2}, 1, 6) - \frac{25}{4}\phi(n-\frac{3}{2}, 1, 6) \,.\, \phi(n+1, 0, 2)$; and so on. Hence

* Chrystal, *Algebra* Ch. XXXI.

† The coefficients on the right side may be thus calculated: write down those of the right side of (6); multiply the first by 2^2 and subtract from the second; multiply the result by 3^2 and subtract from the third; and so on. Thus from 1, −1, 4, −36, 576 we obtain successively 1, −5, 49, −820, 21076.

we have $\phi(n+1,\ 0,\ 2)=\phi(n+\frac{1}{2},\ 1,\ 2)-\frac{1}{4}\phi(n-\frac{1}{2},\ 1,\ 4)+\frac{1}{4}\cdot\frac{9}{4}$ $\phi(n-\frac{3}{2},\ 1,\ 6)-\frac{1}{4}\cdot\frac{9}{4}\cdot\frac{25}{4}\phi(n-\frac{5}{2},\ 1,\ 8)+\ldots$. Reasoning exactly as before, we shall therefore have

$$t_2>\phi(n+\tfrac{1}{2},\ 1,\ 1)-\tfrac{1}{12}\phi(n-\tfrac{1}{2},\ 1,\ 3)+\tfrac{9}{80}\phi(n-\tfrac{3}{2},\ 1,\ 5)-$$
$$\tfrac{225}{448}\phi(n-\tfrac{5}{2},\ 1,\ 7)+\tfrac{1225}{256}\phi(n-\tfrac{7}{2},\ 1,\ 9)-\tfrac{945^2}{11264}\phi(n-\tfrac{9}{2},\ 1,\ 11),$$

but < the first five terms of this expression. Multiply both sides of the equality given above by $\phi(n+1,\ 0,\ 2)$ and apply equation (5); we thus obtain* $t_4>\frac{1}{3}\phi(n-\frac{1}{2},\ 1,\ 3)-\frac{1}{2}\phi(n-\frac{3}{2},\ 1,\ 5)$ $+\frac{259}{112}\phi(n-\frac{5}{2},\ 1,\ 7)-\frac{3229}{144}\phi(n-\frac{7}{2},\ 1,\ 9)$, but less than this value increased by $\frac{96111}{256}\phi(n-\frac{9}{2},\ 1,\ 11)$.

We can similarly obtain in succession $t_6>\frac{1}{5}\phi(n-\frac{3}{2},\ 1,\ 5)-$ $\frac{5}{4}\phi(n-\frac{5}{2},1,7)+\frac{329}{24}\phi(n-\frac{7}{2},1,9)-\frac{7855}{32}\phi(n-\frac{9}{2},1,11)$, but < the first three terms; $t_8>\frac{1}{7}\phi(n-\frac{5}{2},\ 1,\ 7)-\frac{7}{3}\phi(n-\frac{7}{2},\ 1,\ 9)$, but < this expression $+\frac{399}{8}\phi(n-\frac{9}{2},1,11)$; $t_{10}>\frac{1}{9}\phi(n-\frac{7}{2},1,9)-\frac{15}{4}\phi(n-\frac{9}{2},1,11)$, but $<\frac{1}{9}\phi(n-\frac{7}{2},\ 1,\ 9)$. We need not consider $t_{12}, t_{14},\ldots$

4. Adding up the results up to that for t_{11} inclusive, we see that $\gamma_n-\gamma$ is certainly greater than

$$(7)\qquad \tfrac{1}{2}\phi(n+\tfrac{1}{2},\ 1,\ 1)+\tfrac{1}{6}\phi(n,\ 1,\ 2)+\tfrac{1}{24}\phi(n-\tfrac{1}{2},\ 1,\ 3)$$
$$-\tfrac{1}{30}\phi(n-1,\ 1,\ 4)-\tfrac{17}{480}\phi(n-\tfrac{3}{2},\ 1,5)+\tfrac{5}{63}\phi(n-2,\ 1,\ 6)$$
$$+\tfrac{367}{2688}\phi(n-\tfrac{5}{2},\ 1,\ 7)-\tfrac{23}{45}\phi(n-3,\ 1,\ 8)-\tfrac{27859}{23040}\phi(n-\tfrac{7}{2},\ 1,\ 9)$$
$$-\tfrac{251}{15}\phi(n-4,\ 1,\ 10)-\tfrac{5469379}{67584}\phi(n-\tfrac{9}{2},\ 1,\ 11).$$

The functions may now be expanded by the Binomial Theorem; all the series will be absolutely convergent for $n=5$. But as in the expansions of the last three, only terms up to n^{10} or n^{11} have to be retained, the ratio of convergency for the inequality will be much greater than 5; it will, however, be found in any case not to be greater than 10. With this restriction we see that the expression (7) is greater than

$$\frac{1}{2n}-\frac{1}{12n^2}+\frac{1}{120n^4}-\frac{1}{252n^6}+\frac{1}{240n^8}-\frac{1387}{60n^{10}}-\frac{1012575}{5632n^{11}}.$$

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* The co-efficients can again be calculated by a simple rule: write down those of the first equality – $1,\ -\frac{1}{4},\ +\frac{9}{16},\ -\frac{225}{64},\ +\frac{11025}{256},\ldots$; multiply the first by $\frac{9}{4}$ and subtract from the second; multiply the result by $\frac{25}{4}$ and subtract from the third; and so on, Thus we get $1,\ -\frac{5}{4},\ +\frac{259}{16},\ -\frac{3229}{16},\ +\frac{1057221}{256},\ldots$

If $n > 10$, the last two terms are numerically less than $42/n^{10}$; but for $n \gtrapprox 100$, they are less than $25/n^{10}$.

This determines an inferior limit of $\gamma_n - \gamma$: a superior limit can be similarly found. The first ten terms of the right side of (4) are found to be less than $\frac{1}{2}\phi(n+\frac{1}{2}, 1, 1) + \frac{1}{6}\phi(n, 1, 2)$
$+ \frac{1}{24}\phi(n-\frac{1}{2}, 1, 3) - \frac{1}{30}\phi(n-1, 1, 4) - \frac{17}{480}\phi(n-\frac{3}{2}, 1, 5) + \frac{5}{63}\phi(n-2, 1, 6)$
$+ \frac{367}{2688}\phi(n-\frac{5}{2}, 1, 7) - \frac{23}{45}\phi(n-3, 1, 8) - \frac{27859}{23040}\phi(n-\frac{7}{2}, 1, 9)$
$+ \frac{1271}{55}\phi(n-4, 1, 10) + \frac{102495}{1024}\phi(n-\frac{9}{2}, 1, 11)$. Expanding till we get positive terms in n^{10} or n^{11}, we see that this sum is less than

$$(8) \qquad \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8} + \frac{11039}{660n^{10}} + \frac{325111}{1536n^{11}}.$$

We have still to assign a superior limit to the terms omitted. Now $\frac{1}{12}t_{12} + \frac{1}{13}t_{13} + \frac{1}{14}t_{14} \ldots$

$$< \frac{1}{12}(t_{12} + t_{13} + t_{14} \ldots), \; i.e.$$

$$< \frac{1}{12}\left\{\frac{1}{(n+1)^{12}} + \frac{1}{(n+1)^{13}} + \frac{1}{(n+1)^{14}} \cdots \right.$$
$$+ \frac{1}{(n+2)^{12}} + \frac{1}{(n+2)^{13}} + \frac{1}{(n+2)^{14}} \cdots$$
$$+ \frac{1}{(n+3)^{12}} + \frac{1}{(n+3)^{13}} + \frac{1}{(n+3)^{14}} \cdots$$
$$\left. \cdots\cdots\cdots\cdots\cdots\cdots \right\}, \text{ or}$$

$$< \frac{1}{12}\left\{\frac{1}{n(n+1)^{11}} + \frac{1}{(n+1)(n+2)^{11}} + \frac{1}{(n+2)(n+3)^{11}} \cdots\right\};$$

and, therefore, *a fortiori*, the terms omitted are

$$< \frac{1}{12(n+1)^{10}}\left\{\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} \cdots\right\},$$

i.e., $< \frac{1}{12n(n+1)^{10}}$, or, finally, $< \frac{1}{12n^{11}}$. The sum of this quantity and the last two, terms of (8) is found to be less than $38/n^{10}$ for $n \gtrapprox 10$, and less than $19/n^{10}$ for $n \gtrapprox 100$. In the latter case, we conclude that $\gamma_n - \gamma$ lies between

$$\left(\frac{1}{2n} - \frac{1}{12n^2} \cdots + \frac{1}{240n^8}\right) - \frac{25}{n^{10}} \text{ and } \left(\frac{1}{2n} - \frac{1}{12n^2} \cdots + \frac{1}{240n^8}\right) + \frac{19}{n^{10}},$$

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the expression within brackets coinciding with the first five terms of series (3). The difference between the two values is less than $50/n^{10}$, *i.e.*, $\frac{1}{2}(10)^{-18}$, and the value of γ thus obtained will be true to seventeen * places of decimals.

5. The constant in Stirling's Theorem, or rather its logarithm, can be dealt with in the same way. If δ_n denote $\underline{|n} \,.\, e^n \div n^{n+\frac{1}{2}}$, and δ be the limit of $\underline{|n+k} \,.\, e^{n+k} \div (n+k)^{n+k+\frac{1}{2}}$ when the integer k is indefinitely increased, it is known that $\log\delta_n - \log\delta$ can be expanded in the following series:—

$$\text{(9)} \qquad \frac{B_1}{1\,.\,2n} - \frac{B_3}{3\,.\,4n^3} + \frac{B_5}{5\,.\,6n^5} - \ldots.$$

Denoting the logarithms by λ_n, λ, we have

$$\begin{aligned}
\lambda_{n+k} &= (n+k) + \log 2 + \log 3 + \ldots + \log(n+k-1) \\
&\qquad + \log(n+k) - (n+k+\tfrac{1}{2})\log(n+k) \\
&= (n+k) - (n+k-\tfrac{1}{2})\log(n+k) + (n+k-\tfrac{1}{2})\log(n+k-1) \\
&\qquad - (n+k-\tfrac{3}{2})\log(n+k-1) + (n+k-\tfrac{3}{2})\log(n+k-2) \\
&\qquad - (n+k-\tfrac{5}{2})\log(n+k-2) + (n+k-\tfrac{5}{2})\log(n+k-3) \\
&\qquad - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad + \tfrac{7}{2}\log 3 - \tfrac{5}{2}\log 3 + \tfrac{5}{2}\log 2 - \tfrac{3}{2}\log 2 \\
&= 1 - \{(n+k-\tfrac{1}{2})\log(n+k)/(n+k-1) - 1\} \\
&\qquad - \{(n+k-\tfrac{3}{2})\log(n+k-1)/(n+k-2) - 1\} \\
&\qquad - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad - \{\tfrac{5}{2}\log\tfrac{3}{2} - 1\} - \{\tfrac{3}{2}\log\tfrac{2}{1} - 1\}.
\end{aligned}$$

So also
$$\begin{aligned}
\lambda_n &= 1 - \{(n-\tfrac{1}{2})\log n/(n-1) - 1\} \\
&\qquad - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad - \{\tfrac{5}{2}\log\tfrac{3}{2} - 1\} - \{\tfrac{3}{2}\log\tfrac{2}{1} - 1\};
\end{aligned}$$

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* As a matter of fact the series up to n^8 gives in this case a value correct to 18 places; we are, however, able as shown above to prove that the remaining terms of the series can at most affect the 18th place. For the use of the convergent portion only of series (3), see Boole, *Finite Diff.* Ch. VIII., and Bromwich, *Infinite Series* Ch. XI. The latter has proved the approximation to three terms of the series by definite integration in *Mess. of Math.*, Vol. XXXVI., 6.

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so that

$$(10) \qquad \lambda_n - \lambda_{n+k} = (n+\tfrac{1}{2})\log\frac{n+1}{n} - 1 + (n+\tfrac{3}{2})\log\frac{n+2}{n+1} - 1 \ldots$$
$$+ (n+k-\tfrac{1}{2})\log\frac{n+k}{n+k-1} - 1.$$

Now it can be proved that

$$(p+\tfrac{1}{2})\log(p+1)/p = -(p+\tfrac{1}{2})\log\{1-1/(p+1)\}$$
$$= 1 + \left(\frac{1}{3} - \frac{1}{2\cdot 2}\right)\frac{1}{(p+1)^2} + \left(\frac{1}{4} - \frac{1}{2\cdot 3}\right)\frac{1\cdot}{(p+1)^3} + \ldots$$
$$= 1 + \frac{b_2}{(p+1)^2} + \frac{b_3}{(p+1)^3} + \ldots \textit{ ad inf.},$$

where $br = 1/(r+1) - 1/(2r)$. Hence from (10), making k infinitely large, we get

$$\lambda_n - \lambda = \frac{b_2}{(n+1)^2} + \frac{b_3}{(n+1)^3} + \frac{b_4}{(n+1)^4} + \ldots$$
$$+ \frac{b_2}{(n+2)^2} + \frac{b_3}{(n+2)^3} + \frac{b_4}{(n+2)^4} + \ldots$$
$$+ \frac{b_2}{(n+3)^2} + \frac{b_3}{(n+3)^3} + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots, \textit{ad inf.},$$

which may be written thus :—

$$(11) \qquad \lambda_n - \lambda = b_2t_2 + b_3t_3 + b_4t_4 + \ldots \textit{ad inf.}$$

We now expand the t's as in §§ 3 and 4, and give to the b's their arithmetical values. Keeping only terms which affect $1/n^8$ and previous powers, we get for the inferior limit the following expression :—

$$\tfrac{1}{12}\phi(n+\tfrac{1}{2}, 1, 1) + \tfrac{1}{24}\phi(n, 1, 2) + \tfrac{13}{720}\phi(n-\tfrac{1}{2}, 1, 3)$$
$$- \tfrac{1}{240}\phi(n-1, 1, 4) - \tfrac{109}{6720}\phi(n-\tfrac{3}{2}, 1, 5) + \tfrac{1}{112}\phi(n-2, 1, 6)$$
$$+ \tfrac{5171}{80640}\phi(n-\tfrac{5}{2}, 1, 7) - \tfrac{79}{1440}\phi(n-3, 1, 8) - \tfrac{31021}{17280}\phi(n-\tfrac{7}{2}, 1, 9)$$

$-\tfrac{28}{5}\phi(n-4, 1, 10$. On expanding as before, this gives

$$\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7} - \frac{168437}{138240n^9} - \frac{77785}{6144n^{10}}.$$

The last two terms are numerically less than $25/10n^9$ when $n \geqq 10$.

The first eight terms of the series in (11) are similarly found to be less than

$$\tfrac{1}{12}\phi(n+\tfrac{1}{2},\ 1,\ 1)+\tfrac{1}{24}\phi(n,1,2)+\tfrac{13}{720}\phi(n-\tfrac{1}{2},1,3)-\tfrac{1}{240}\phi(n-1,\ 1,\ 4)$$
$$-\tfrac{109}{6720}\phi(n-\tfrac{3}{2},1,\ 5)+\tfrac{1}{112}\phi(n-2,\ 1,\ 6)+\tfrac{5171}{80640}\phi(n-\tfrac{5}{2},\ 1,\ 7)$$
$$-\tfrac{79}{1440}\phi(n-3,\ 1,\ 8)+\tfrac{11195}{9216}\phi(n-\tfrac{7}{2},\ 1,\ 9)+\tfrac{501}{80}\phi(n-4,\ 1,\ 10),$$

which gives

$$\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}-\frac{1}{1680n^7}+\frac{10319}{5760n^9}+\frac{121897}{15360n^{10}}.$$

A superior limit of the terms omitted will be found to be b_{10}/n^9, *i.e.*, $9/220n^9$. This term and the last two terms above are seen to be less than $27/10n^9$ for $n \geqq 10$. Thus, for instance, when n is 10, the value of $\lambda_n - \lambda$ derived from the first four terms of series (9) will differ from the true value by a quantity less than $52/10n^9$, or by about $\frac{1}{2}(10)^{-8}$.

6. I conclude by obtaining algebraically other series for γ and establishing analogous series for λ.

From (2) we have

$$\gamma_{n+k}=(1-\log 2)+(\tfrac{1}{2}-\log 3/2)+(\tfrac{1}{3}-\log 4/3)+\dots$$
$$+\{1/(n+k-1)-\log(n+k)/(n+k-1)\}+1/(n+k).$$

Now $$\frac{1}{p}-\log(1+p)/p=\frac{1}{2\,.\,p^2}-\frac{1}{3\,.\,p^3}+\frac{1}{4\,.\,p^4}\dots\textit{ad inf.};$$

hence $$\gamma_{n+k}=\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\dots$$
$$+\frac{1}{2\,.\,2^2}-\frac{1}{3\,.\,2^2}+\frac{1}{4\,.\,2^4}-\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+\frac{1}{2(n+k-1)^2}-\frac{1}{3(n+k-1)^3}+\frac{1}{4(n+k-1)^4}\dots+\frac{1}{n+k}.$$

Making k infinite we get

$$\gamma=\tfrac{1}{2}(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\textit{ad inf.})-\tfrac{1}{3}(1+\frac{1}{2^3}+\frac{1}{3^3}\dots\dots\dots\textit{ad inf.})$$
$$+\tfrac{1}{4}(1+\frac{1}{2^4}+\frac{1}{3^4}\dots\textit{ad inf.})\dots\dots\dots\dots\textit{ad inf.}$$
$$=\tfrac{1}{2}s_2-\tfrac{1}{3}s_3+\tfrac{1}{4}s_4-\tfrac{1}{5}s_5\dots\dots\dots\dots\dots\text{(A)},$$

where $$s_r=1+\frac{1}{2^r}+\frac{1}{3^r}+\dots\dots\dots\dots\textit{ad inf.}$$

In (4) make $n=1$; then, as $\gamma_1=1$,

$$1-\gamma=\tfrac{1}{2}(s_2-1)+\tfrac{1}{3}(s_3-1)+\tfrac{1}{4}(s_4-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}.$$

It can be shown that $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}\ldots\ldots\ldots\ldots\ldots\ldots\textit{ad inf.}$

i.e., unity $=(s_2-1)+(s_3-1)+(s_4-1)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$;
hence from (B),

$$\gamma=\tfrac{1}{2}(s_2-1)+\tfrac{2}{3}(s_3-1)+\tfrac{3}{4}(s_4-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

Supposing n to be very large and taking terms up to s_{2n} in (A) and (B), we find on addition that unity is the limit of

$$s_2-\frac{1}{2}-\frac{1}{3}+\tfrac{1}{2}s_4-\frac{1}{4}-\frac{1}{5}+\ldots+\frac{1}{n}s_{2n}-\frac{1}{2n}-\frac{1}{2n+1};$$

hence $s_2-1+\frac{1}{2}(s_4-1)+\frac{1}{3}(s_6-1)\ldots+\frac{1}{n}(s_{2n}-1)$ is the limit of $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}+\frac{1}{2n+1}$, when n becomes infinitely large.

This limit can be shown to be log2, so that

$$\log 2=\tfrac{2}{2}(s_2-1)+\tfrac{2}{4}(s_4-1)+\tfrac{2}{6}(s_6-1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b).$$

Hence by help of (B) we obtain

$$2-2\gamma-\log 2=\tfrac{2}{3}(s_3-1)+\tfrac{2}{5}(s_5-1)+\tfrac{2}{7}(s_7-1)\ldots\ldots\ldots\ldots\text{(D)}.$$

Again log2 γ_{n+k}

$$=\log(2n+2k)-(1+\tfrac{1}{2}+\tfrac{1}{3}\ldots+1\div\overline{n+k})$$

$$=\log\frac{2n+2k}{2n+2k-1}-\frac{1}{n+k}+\left(\log\frac{2n+2k-1}{2n+2k-3}-\frac{1}{n+k-1}\right)\ldots$$

$$\ldots+(\log\tfrac{5}{3}-\tfrac{1}{2})+(\log\tfrac{3}{1}-1).$$

But $\log\{1+1/(2p)\}-\log\{1-1/(2p)\}-1/p$

$$=\frac{2}{3(2p)^3}+\frac{2}{5(2p)^5}+\frac{2}{7(2p)^7}+\ldots\ldots\ldots\ldots\ldots\ldots\textit{ad inf.}\ (c);$$

and the limit of the first terms on the right side is really zero when k is infinite. Therefore, we deduce

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$$\log 2 - \gamma = \frac{2}{3 \cdot 2^3} + \frac{2}{5 \cdot 2^5} + \frac{2}{7 \cdot 2^7} + \dots\dots\dots \textit{ad inf.}$$

$$+ \frac{2}{3 \cdot 4^3} + \frac{2}{5 \cdot 4^5} + \frac{2}{7 \cdot 4^7} + \dots\dots\dots \;\; ,, \;\; ,,$$

$$+ \frac{2}{3 \cdot 6^3} + \frac{2}{5 \cdot 6^5} + \frac{2}{7 \cdot 6^7} + \dots\dots\dots \;\; ,, \;\; ,,$$

$$+ \dots\dots\dots\dots\dots\dots \;\; ,, \;\; ,,$$

$$= \frac{s_3}{3 \cdot 2^2} + \frac{s_5}{5 \cdot 2^4} + \frac{s_7}{7 \cdot 2^6} + \dots\dots\dots\dots (\mathrm{E}).$$

Also, from (c), $\log 3 - 1 = 1/3 \cdot 2^2 + 1/5 \cdot 2^4 + 1/7 \cdot 2^6 + \dots\dots\dots\dots$;
so that $\log 2 - \gamma - \log 3 + 1$, or, $1 - \log\frac{3}{2} - \gamma$

$$= \frac{s_3 - 1}{3 \cdot 2^2} + \frac{s_5 - 1}{5 \cdot 2^4} + \frac{s_7 - 1}{7 \cdot 2^6} + \dots\dots\dots\dots (\mathrm{F}).$$

Euler employed the formulæ (B), (E), (F) in calculating * γ and Legendre the formula (D). They can be obtained * from the well-known series for $\log\Gamma(1+x)$,

$$\tfrac{1}{2}\log\frac{x\pi}{\sin x\pi} - (\gamma x + \tfrac{1}{3}s_3 x^3 + \tfrac{1}{5}s_5 x^5 + \dots)$$

and

$$\tfrac{1}{2}\log\frac{x\pi}{\sin x\pi} - \tfrac{1}{2}\log\frac{1+x}{1-x} + c_1 x - c_3 x^3 - c_5 x^5 - \dots .$$

7. In (10) make $n = 1$; then, as $\lambda_1 = 1$,

$$1 - \lambda = b_2(s_2 - 1) + b_3(s_3 - 1) + b_4(s_4 - 1) + \dots\dots\dots\dots (\mathrm{B}_1).$$

In §5 expand $(p + \frac{1}{2})\{\log(p+1)/p\}$ in the form

$$1 + \frac{b_2}{p^2} - \frac{b_3}{p^3} + \frac{b_4}{p^4} - \dots\dots\dots\dots (d),$$

and proceed as before; we thus obtain

$$1 - \lambda = b_2 s_2 - b_3 s_3 + b_4 s_4 - b_5 s_5 + \dots\dots\dots\dots (\mathrm{A}_1).$$

From (B_1) and (a) –

$$\lambda = (1 - b_2)(s_2 - 1) + (1 - b_3)(s_3 - 1) + (1 - b_4)(s_4 - 1) \dots\dots (\mathrm{C}_1).$$

From (d) we get

$$\tfrac{3}{2}\log 2 - 1 = b_2 - b_3 + b_4 - b_5 + \dots\dots\dots\dots \textit{ad inf.};$$

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* See *Ency. Brit.* ed. IX., *s.v.* *Infinitesimal Calculus* (B. Williamson); also *Mess. of Math.*, Vol. I. (G. W. L. Glaisher).

and from (A_1) and (B_1)

$$2b_3s_3 + 2b_5s_5 + \ldots = b_2 + b_3 + b_4 + b_5 \ldots$$

Thus $\quad 2b_3(s_3 - 1) + 2b_5(s_5 - 1) + 2b_7(s_7 - 1)\ldots$

$$= b_2 - b_3 + b_4 - b_5 + \ldots = \tfrac{3}{2}\log 2 - 1.$$

Combining with (B_1), we obtain $2 - 2\lambda - \frac{3}{2}\log 2 + 1$,

i.e., $3 - \frac{3}{2}\log 2 - 2\lambda = 2b_2(s_2 - 1) + 2b_4(s_4 - 1) + 2b_6(s_6 - 1) \ldots\ldots (D_1)$.

Substituting here the numerical values of the b's, we find the right side

$$= 2\{\tfrac{1}{3}(s_2 - 1) + \tfrac{1}{5}(s_4 - 1) + \tfrac{1}{7}(s_6 - 1)\ldots\} - \{\tfrac{1}{2}(s_2 - 1) + \tfrac{1}{4}(s_4 - 1)\ldots\}.$$

Hence and from (b),

$$3 - \log 2 - 2\lambda = \tfrac{2}{3}(s_2 - 1) + \tfrac{2}{5}(s_4 - 1) + \tfrac{2}{7}(s_6 - 1) + \ldots\ldots\ldots\ldots (D_2).$$

From (B_1) $\quad 1 - \lambda$

$$= \frac{s_2 - 1}{3} + \frac{s_3 - 1}{4} + \frac{s_4 - 1}{5} \ldots - \tfrac{1}{2}\left(\frac{s_2 - 1}{2} + \frac{s_3 - 1}{3} + \frac{s_4 - 1}{4} \ldots\right);$$

therefore

$$3 - \gamma - 2\lambda = \tfrac{2}{3}(s_2 - 1) + \tfrac{2}{4}(s_3 - 1) + \tfrac{2}{5}(s_4 - 1) + \ldots\ldots\ldots\ldots\ldots\ldots (D_3).$$

By Wallis's Theorem $(\pi/2)^{\frac{1}{2}}$ is the limit* where m is made infinitely large of

$$\frac{2 \,.\, 4 \ldots 2m \sqrt{(2m+1)}}{3 \,.\, 5 \ldots (2m+1)}, \text{ i.e., of } 2^{2m}(\underline{|m})^2 \sqrt{(2m+1)} \div \underline{|2m+1}.$$

Now in this case e^{λ} is the limit of $\underline{|m} \,.\, e^m \div m^{m+\frac{1}{2}}$. Thus $(\pi/2)^{\frac{1}{2}}$ is the limit of $2^{2m} \,.\, m^{2m+1} \,.\, e^{\lambda+1} \div (2m+1)^{2m+1}$, or of $\left(\dfrac{2m}{2m+1}\right)^{2m} \cdot \dfrac{me^{\lambda+1}}{2m+1}$.

Now the limit of $m/(2m+1)$ is $\frac{1}{2}$; that of $\{2m \div (2m+1)\}^{2m}$ is e^{-1}. Thus $\sqrt{(\pi/2)} = e^{\lambda}/2$, or $\lambda = \frac{1}{2}\log 2 + \frac{1}{2}\log\pi$.

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* Chrystal, *Algebra*, Ch. XXX.

Therefore $\lambda - \log 2 = \log \surd(\pi/2)$

$= \log\{\text{lt. of } 2.4\ldots 2m \surd(2m+1) \div 3.5\ldots(2m+1)\}$

$= \frac{1}{2}\text{lt. of } \{(2\log 2 - \log 1 - \log 3) + (2\log 4 - \log 3 - \log 5)$

$+ (2\log 6 - \log 5 - \log 7) \ldots + 2\log 2m - \log(2m-1) - \log(2m+1)\}$

$$= \tfrac{1}{2}\left\{\frac{1}{2^2} + \frac{1}{2.2^4} + \frac{1}{3.2^6} + \ldots + \frac{1}{4^2} + \frac{1}{2.4^4} + \frac{1}{3.4^6} + \ldots \right.$$

$$\left. + \frac{1}{6^2} + \frac{1}{2.6^4} + \frac{1}{2.6^6} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textit{ad inf.}\right\}.$$

Hence we get $\lambda - \log 2 = \dfrac{s_2}{2.2^2} + \dfrac{s_4}{4.2^4} + \dfrac{s_6}{6.2^6} + \ldots\ldots\ldots\ldots\ldots\ldots (E_1)$.

Also $\log 2 - \frac{1}{2}\log 3$, or $\frac{1}{2}(2\log 2 - \log 1 - \log 3)$

$$= \frac{1}{2.2^2} + \frac{1}{4.2^4} + \frac{1}{6.2^6} + \ldots;$$

so that

$$\lambda - 2\log 2 + \tfrac{1}{2}\log 3 = \frac{s_2 - 1}{2.2^2} + \frac{s_4 - 1}{4.2^4} + \frac{s_6 - 1}{6.2^6} + \ldots\ldots\ldots\ldots\ldots\ldots (F_1).$$

The equalities (D_1), (E_1), (F_1), (B_1) are closely analogous to (D), (E), (F), (B), and may be employed in calculating λ or in fact $\log\pi$. They can be obtained from the well-known * results—

$$\log\Gamma(1-x) = Cx + \tfrac{1}{2}s_2x^2 + \tfrac{1}{3}s_3x^3 + \ldots,$$

$$\log\Gamma(1+x) = \tfrac{1}{2}\log\frac{x\pi}{\sin x\pi} - \tfrac{1}{2}\log\frac{1+x}{1-x} + (1-C)x - \tfrac{1}{3}(s_3 - 1)x^3 - \ldots.$$

* *Ency. Brit.*, *loc. cit.*