A NOTE ON d-GROUPS

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This note concerns subgroups of the general linear group $\mathrm{GL}(n, F)$, where n is finite and F algebraically closed. $g \in \mathrm{GL}(n, F)$ is called a d-element if there exists an $x \in \mathrm{GL}(n, \mathrm{F})$ such that $x^{-1}gx$ is diagonal, and a u-element if $(g - 1)^n = 0$. A subgroup G of $\mathrm{GL}(n, F)$ is called a d-group (or a u-group) if every element of G is a d-element (or a u-element). In view of the Jordan decomposition of the elements of $\mathrm{GL}(n, F)$ into products of d-elements and u-elements it is important to know the structure of d-groups and u-groups. u-groups present very little difficulty and their structure is well known (**1**, 19.4), but d-groups seem to have a more complicated structure.

If G is any subgroup of $\mathrm{GL}(n, F)$, we denote by G_0 its connected component of the identity in the topology induced on G by the Zariski topology. G is called *non-modular* if char $F = 0$ or if char $F = p$ and G contains no subgroup of finite index divisible by p. We say that G is *weakly non-modular* if char $F = 0$ or if char $F = p$ and p does not divide $(G:G_0)$. Note that $(G:G_0)$ is always finite; cf. (**3**, Chapter 4). G is *locally completely reducible* if every finitely generated subgroup of G is completely reducible. Note that this is equivalent to saying that every subgroup of G is completely reducible. In (**2**, Theorem 1), J. D. Dixon proves the following result.

Let G be a soluble non-modular subgroup of $\mathrm{GL}(n, F)$. *Then G is a d-group if and only if G is completely reducible.*

The object of this note is to prove the following generalization of this result.

THEOREM. *Let G be a soluble subgroup of* $\mathrm{GL}(n, F)$. *Then the following are equivalent:*

(a) *G is a d-group,*
(b) *G is locally completely reducible.*
(c) *G is completely reducible and weakly non-modular.*

To prove the theorem we use two lemmas. The first lemma is a rewriting of certain classical results of *I.* Schur (**5**) on periodic linear groups.

LEMMA 1. *Let G be a periodic subgroup of* $\mathrm{GL}(n, F)$. *Then the following are equivalent:*

(a) *G is a d-group,*
(b) *G is locally completely reducible,*
(c) *if* char $F \neq 0$, *then G contains no elements of order* char F.

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The following result is well known.

LEMMA 2. *Let G be an abelian subgroup of* GL(n, F). *Then the following are equivalent:*

(a) *G is a d-group,*
(b) *G is locally completely reducible,*
(c) *G is completely reducible.*

We are led therefore to make two *conjectures:*

(1) *If G is a subgroup of* GL(n, F), *then G is a d-group if and only if G is locally completely reducible.*
(2) *Every locally completely reducible subgroup of* GL(n, F) *has an abelian normal subgroup of finite index.*

Trivially a locally completely reducible subgroup of GL(n, F) is a d-group, and it is easily seen from the proof of the theorem given below that a d-group with an abelian normal subgroup of finite index is locally completely reducible.

Proof of the Theorem. Suppose that (a) holds. G_0 is triangulizable by the Lie-Kolchin theorem and so G_0 is abelian, (**1**, 10.2). Hence being a d-group, G_0 is completely reducible by Lemma 2. Now G/G_0 is isomorphic to a finite linear d-group over a field of the same characteristic as F; see (**4** or **1**, 5.10.1 and 8.4). Then Lemma 1 implies that char $F = 0$ or char $F \nmid (G:G_0)$, and so G is weakly non-modular. G_0 is completely reducible and so, by an extension of Maschke's theorem (**2**, Lemma 1), G is completely reducible. We have therefore shown that (a) implies (c). As every subgroup of a d-group is a d-group, this also shows that (a) implies (b). We know that (b) implies (a), and it remains only to show that (c) implies (a).

Let G be a completely reducible weakly non-modular soluble subgroup of GL(n, F). We wish to prove that G is a d-group. G has an abelian normal subgroup of finite index, A say, by Mal'cev's theorem (**6**, Theorem 15). The closure of A in G, $\bar{A}$ say, is also an abelian normal subgroup of finite index in G by (**1**, 3.5 and 4.5). Because $\bar{A}$ is closed and of finite index in G, $G_0 \leqslant \bar{A}$ and thus G_0 is abelian. By Clifford's theorem, G_0 is completely reducible and so G_0 is a d-group by Lemma 2. It follows easily now from (**2**, Lemma 2) and the weak non-modularity of G that G is a d-group. (The author is indebted to the referee for suggesting the use of this lemma, thus shortening the argument.)

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