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Models with dimension

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It is well known that, if AC is assumed, then any vector space has a basis, any independent subset can be extended to a basis, any two bases of a vector space have the same cardinality and a one-one map between independent subsets can be extended to a monomorphism between the subspaces they generate.

We follow [1] in using Marsh's notions of minimal formula and dimension to generalize the notion of vector space, and in defining algebraic closure, independence and basis. The models which we consider obey the Exchange Lemma and if they also have dimension, then one-one maps between independent subsets are elementary.

We first show that, if M is a model with dimension and the algebraic closure of any finite subset is finite, then $T(M)$ is $\aleph_0$-categorical, and if M is atomic and has dimension, then either M has a finite basis or the algebraic closure of any finite set is finite.

We then investigate whether our models have the properties mentioned in the first paragraph in the situations of not assuming AC and of assuming the axiom of choice for sets of finite sets (ACF). Consistency results are established by constructing permutation models of set theory with atoms (ZFA) and then using the Jech-Sochor transfer theorem. Our results are as follows.

It is consistent with ZF that there is a vector space with Dedekind domain and bases of incomparable cardinalities. It is consistent with ZF that there is a vector space with Dedekind domain, a basis and a "maximal" independent subset which is not a basis. Without AC one-one maps between

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independent subsets extend if M has degree 1 .

It is consistent with ZF and ACF that there is a vector space with Dedekind domain and no basis. Assuming ACF, if M has a Dedekind basis and a well-ordered language then any two bases have the same cardinality. However, in the order Mostowski model of ZFA + ACF , for any model M , all bases are of the same cardinality. In ZF and ACF, if M is atomic, has finite closure and dimension, and $p : A \to B$ is a one-one map between independent subsets, then p extends to an elementary map $q : \bigcup_{a \in A} \mathrm{cl}\, a \to \bigcup_{b \in B} \mathrm{cl}\, b$.

As a corollary to these results we show that if M is atomic, has dimension, degree 1 , and an infinite basis then, assuming ACF various definitions of a model being Dedekind are equivalent. Dekker [2] has a similar result for vector spaces in a recursive setting.

References

[1] J.N. Crossley, Anil Nerode, *Combinatorial functors* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 81. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[2] J.C.E. Dekker, "Countable vector spaces with recursive operations. Part I", *J. Symbolic Logic* 34 (1969), 363-387.