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Embedding Coverings in Bundles

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Abstract. If $V \to X$ is a vector bundle of fiber dimension k and $Y \to X$ is a finite sheeted covering map of degree d, the implications for the Euler class $e(V)$ in $H^k(X)$ of V implied by the existence of an embedding $Y \to V$ lifting the covering map are explored. In particular it is proved that $dd'e(V) = 0$ where d' is a certain divisor of $d - 1$, and often $d' = 1$.

1 Introduction

It is a natural and intuitively attractive problem to decide whether a given "abstract" covering space $Y \to X$ can be made "concrete" by embedding it in a vector bundle $V \to X$, especially in the trivial bundle $X \times \mathbf{R}^k$. In this paper we examine a particular aspect of the general problem, previously studied in some detail by V. L. Hansen [4], P. Duvall and L. S. Husch [2], [3], and P. Zhang [8]. In much of the work cited it was assumed that the vector bundle in question admitted a non-vanishing section. The question remained just how necessary or appropriate that assumption actually was. We offer the following answer to this question.

Theorem 1 *Let $p\colon V \to X$ be the projection of an oriented k-plane vector bundle over a space X, with Euler class $e(V) \in H^k(X;\mathbf{Z})$. Let $f\colon Y \to X$ be a degree d covering map, where Y and X are connected and $d \geq 2$. If there is an embedding $g\colon Y \to V$ such that $pg = f$, then $dd'e(V) = 0$ in $H^k(X;\mathbf{Z})$, for some integer d' dividing $d - 1$.*

Remark 2 In addition, there is a covering $Z \to Y$ of degree d' such that the pullback of V all the way to Z admits a non-vanishing section. The integer d' will be described more precisely in Section 3.

In the case when V is an n-plane bundle over the n-manifold X (the case $k = n$) and $f\colon Y \to X$ is a *regular* covering, this is due, in essence, to P. A. Duvall and L. S. Husch [3]. In their case, the conclusion was that the pullback of the bundle V to Y itself admits a non-vanishing section. Duvall and Husch explicitly ask whether their result is true for irregular coverings. We see that it is, since the Euler class in question is a multiple of a top class living in a torsion free cohomology group. But for other combinations of dimensions more subtle phenomena occur. We will give a simple example to show that the Euler class need not vanish. For general irregular coverings we have been unable to decide whether the Euler class of the bundle V is of order d, or whether the extra factor d' is actually necessary in the theorem, in general.

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2 An Example and a Question

Here we give an example that shows that in general a vector bundle into which a d-fold covering space embeds need not have a non-vanishing section or a vanishing Euler class.

Example 1 *A lifting of a covering to an embedding in a 2-plane bundle with nontrivial Euler class.* Consider the 2-fold covering map $f\colon S^3 \to \mathbf{R}P^3$. Let $W = S^3 \times \mathbf{R}^2$, and V be the quotient of W by the involution $(x, v) \to (-x, -v)$. If $v \in \mathbf{R}^2$ is any nonzero vector, then the quotient map restricted to $S^3 \times \{v\}$ yields an embedding $g\colon S^3 \to V$ lifting f. On the other hand, the Borsuk-Ulam theorem implies that $V \to \mathbf{R}P^3$ is not a trivial bundle. In fact V is the Whitney sum of two copies of the canonical line bundle over $\mathbf{R}P^3$ and its Euler class is the nonzero element in $H^2(\mathbf{R}P^3; \mathbf{Z}) = \mathbf{Z}/2$. In this case the Euler class is necessarily d-torsion, where $d = 2$. We refer the reader to [7] for basic facts about vector bundles and their Euler classes.

Question 1 Is there an example of a lifting of a degree d covering to an embedding in a nontrivial bundle whose Euler class is not d-torsion?

3 Proof of Theorem

We will need to make use of the transfer homomorphism associated with a covering map $f\colon Y \to X$ of degree $d > 0$ that is not necessarily the orbit map of a group action. The transfer is a "backwards" homomorphism $\mathrm{tr}\colon H^*(Y) \to H^*(X)$, with the crucial property that $\mathrm{tr} \circ f^* =$ multiplication by d. See [1] for an elementary exposition of the transfer map.

Proof of Theorem The proof will proceed by induction on the degree d of the covering $Y \to X$. We may assume that d is at least 2. The initial case when $d = 2$ is a consequence of a more general result in the following construction that will precede and be used in the inductive step.

Choose a base point $x_0 \in X$ and $y_0 \in Y$ such that $f(y_0) = x_0$. Set $v_0 = g(y_0)$ in V. Let $\tilde{p}\colon W \to Y$ be the pullback of $p\colon V \to X$. We identify W explicitly as $\{(y, v) \mid f(y) = p(v)\}$, where $\tilde{p}$ is induced by projection on the first coordinate, and $\tilde{f}\colon W \to V$ is induced by projection on the second coordinate. Let $w_0 \in W$ be a base point such that $w_0 = (y_0, v_0)$ where $f(y_0) = p(v_0)$, so that $\tilde{p}(w_0) = y_0$. Note that $\tilde{f}(w_0) = v_0$ and also that $p\tilde{f} = f\tilde{p}$. Moreover, $\tilde{f}$ is the covering map corresponding to $p_*^{-1} f_*\bigl(\pi_1(Y, y_0)\bigr) \subset \pi_1(V, v_0)$.

Set $Z = \tilde{f}^{-1}\bigl(g(Y)\bigr) \subset W$, that is $Z = \{\bigl(y', g(y)\bigr) \mid f(y) = f(y')\}$. Then $\tilde{p}$ maps Z to Y as a d-fold covering, since, identifying $g(Y)$ with Y, we can recognize $Z \to Y$ as the pullback of the covering $f\colon Y \to X$ over itself.

Now covering space theory implies that the component Z_0 of Z containing w_0 is actually homeomorphic to Y via the map $\tilde{p}$.

Suppose there is a second component Z_1 of Z homeomorphic to Y via $\tilde{p}$ (*e.g.*, $d = 2$, the initial case of our underlying inductive argument).

Then Z_0 and Z_1 are two disjoint sections of the bundle $\tilde{p}\colon W \to Y$. It follows that $e(W) = 0$, and more, that W admits a nonvanishing section. Naturality of Euler classes and the existence of the transfer then show that $0 = \mathrm{tr}\bigl(e(W)\bigr) = \mathrm{tr}\bigl(e(f^*V)\bigr) = \mathrm{tr}(f^*\bigl(e(V)\bigr) = d \cdot e(V) = 0$.

(More generally, if the lift g takes its values in the complement of the zero-section of $V \to X$, then one can also show $e(W) = 0$, since then Z_0 is a section of W in the complement of the zero-section of W living over the 0-section of V. In particular $W \to Y$ admits a non-vanishing section in this case.)

Otherwise, there is a second component Z_i of Z mapping to Y via $\tilde{p}$ as a d_i-fold covering for some d_i, where $1 < d_i < d$. Note that the covering Z_i of Y is naturally lifted to an embedding in the bundle W over Y. Moreover, the lifting of the covering $Z_i \to Y$ to an embedding $Z_i \to W$ takes its values in $W - Z_0$. Thus the preceding observation shows that the pullback of W to Z_i has trivial Euler class and in fact admits a nonvanishing section. Naturality of Euler classes and the same transfer argument as above then imply that $dd_i e(V) = 0$. But this holds for the degree of each component Z_i other than Z_0 itself. These d_i thus satisfy $\sum d_i = d-1$. In particular we see that $dd'e(V) = 0$ for $d' = \gcd\{d_1, \dots d_r\}$, since we can write $d' = \sum n_i d_i$. In particular, $dd'e(V) = 0$ for some divisor d' of $d-1$, as required. ■

Remark 3 If the embedding $g\colon Y \to V$ takes its values in the complement of the 0-section, then the pullback bundle $W \to Y$ itself admits a nonvanishing section, so that $e(W) = 0$, hence $de(V) = 0$, in this case.

Remark 4 Using basic covering space theory one can interpret the integer d' as follows. Let $G = \pi_1(X, x_0)$ and $H = f_*\big(\pi_1(Y, y_0)\big) < G$. Then $d = |G/H|$ and $d' = \gcd\{|H/(H \cap gHg^{-1})| : g \notin H\}$. In particular, $d' = 1$ if H is not equal to its own normalizer in G.

4 Applications

Note that the pullback of a *regular* covering over itself is a trivial covering. In this case in the proof of the theorem there are indeed already two trivial sheets in Z over Y. Thus $e(W) = 0$ and, so, $0 = \operatorname{tr}\big(e(W)\big) = \operatorname{tr}\Big(f^*\big(e(V)\big)\Big) = de(V)$. Thus we have the following corollary:

Corollary 5 *Let $p\colon V \to X$ be the projection of an oriented k-plane vector bundle over a space X, and let $f\colon Y \to X$ be a degree d* regular *covering map. If there is an embedding $g\colon Y \to V$ such that $pg = f$, then $d\, e(V) = 0$ in $H^k(X)$.*

Alternatively, one just observes that $d' = 1$ in this case. In particular, the conclusion of the corollary holds if the image $f_*\big(\pi_1(Y)\big)$ has nontrivial normalizer in $\pi_1(X)$.

Now we specialize to the case of an n-plane bundle over an n-manifold. In particular we show that the assumption that V admits a non-vanishing section (which appears in the work of Duvall and Husch, of Zhang, and of Hansen) is completely justified in this case.

Corollary 6 *Let V be the total space of an oriented n-plane vector bundle over a closed oriented n-manifold X, with Euler class $e(V) \in H^n(X)$, and let $p\colon V \to X$ be the bundle projection. Let $f\colon Y \to X$ be a connected d-fold covering space. If there is an embedding $g\colon Y \to V$ such that $pg = f$, then $e(V) = 0$ in $H^n(X)$.*

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Proof In this case $H^n(X) \approx H^n(Y) = \mathbf{Z}$. By the Theorem $dd'e(V) = 0$, so that $e(V) = 0$ since $H^n(X)$ is torsion free. For n-plane bundles over n-dimensional spaces, the Euler class is both the primary and only obstruction to constructing a nowhere vanishing section, and the result follows. ■

In general there would be no reason for a bundle to admit a nonvanishing section just because the primary obstruction to the existence of a section vanishes.

Corollary 7 *If F is a connected Riemann surface, $f\colon G \to F$ is a connected d-fold covering, $d > 1$, and $p\colon V \to F$ is an orientable 2-plane bundle over F and if there exists an embedding $g\colon G \to V$ such that $pg = f$, then V is a trivial bundle.*

Proof By Corollary 6, $e(V) = 0$. But oriented 2-plane bundles are completely classified by the Euler classes. ■

A quite different proof of Corollary 7 follows from the work in [6], where the bundle V is completed to a closed 4-manifold and results of Roklin are applied to the multiples of the homology class carried by the zero section.

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