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UNIQUENESS FOR SINGULAR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS II

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Abstract. We prove uniqueness of positive solutions for the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, λ is a large positive parameter, $f : (0, \infty) \to [0, \infty)$ is nonincreasing for large t and is allowed to be singular at 0.

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1. Introduction. Consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, λ is a large positive parameter and $f : (0, \infty) \to (0, \infty)$.

The existence and uniqueness of a classical positive solution to (1.1) for all $\lambda > 0$ was obtained in [**3**] when f is nonincreasing and $\lim_{t\to 0^+} f(t) = \infty$. We refer to [**4**, **7**, **9**, **11**, **12**] for uniqueness results to (1.1) when λ is large and f is nonsingular. Note that $f(t) \sim t^\beta$ at ∞ for some $\beta \in [0, 1)$ in [**4**, **9**, **11**], $f(t) \sim t^\beta \ln(1 + t)$ for some $\beta \in (0, 1)$ is allowed in [**7**], while $f(t) = (1 + t)^{-\gamma}$ with $\gamma > 0$ small is permitted in [**12**]. We are interested here in studying uniqueness of solutions to (1.1) for λ large when $f(t)$ is nonincreasing for t large and is possibly singular at 0. Our results complement the uniqueness result in [**6**], where $f(t)$ is possibly singular at 0 and nondecreasing for t large, and the result in [**12**] mentioned above. Our approach is based on sharp upper and lower estimates on the solutions of (1.1).

We shall make the following assumptions:

(A1) $f : (0, \infty) \to (0, \infty)$ is continuous.

(A2) There exist constants $A > 0$ and $\alpha \in (0, 1)$ such that

$$c^{-\alpha} f(t) \le f(ct) \le f(t)$$

for all $c > 1$, $t > A$.

(A3) $\liminf_{t\to 0^+} \frac{f(t)}{t} > 0$

(A4) For each constant $B > 0$, there exists a constant $C_B > 0$ such that

$$|f(t) - f(s)| \leq \frac{C_B|t-s|}{\min^{\alpha+1}(s,t)}$$

for $s, t \leq B$.

REMARK 1.1. Condition (A2) is equivalent to the assumption that $f(t)$ is nonincreasing and $t^{\alpha} f(t)$is nondecreasing for $t > A$.

REMARK 1.2. (i) It is easily seen that condition (A4) is satisfied if f is of class C^1 on $(0, \infty)$and

$$\limsup_{t \to 0^+} t^{\alpha+1}|f'(t)| < \infty.$$

(ii) Note that (A4) implies

(A5) $\limsup_{t\to 0^+} t^{\alpha}|f(t)| < \infty$.

To see this, let $B > 0$ and $t \in (0, B]$. Let $n_0 \in N$ be the largest number such that $n_0 t < B$. Then, by (A4),

$$|f(t) - f(B)| \leq \ |f(n_0 t) - f(B)| \ + \sum_{k=1}^{n_0 - 1} |f(kt) - f((k+1)t)|$$

$$\leq \frac{C_B}{t^{\alpha}} \sum_{k=1}^{n_0} \frac{1}{k^{\alpha+1}} \leq \frac{\tilde{C}_B}{t^{\alpha}},$$

for $t \leq B$, where $\tilde{C}_B = C_B \sum_{k=1}^{\infty} \frac{1}{k^{\alpha+1}}$. Hence, (A5) follows.

EXAMPLE 1.1. The following nonlinearities satisfy (A1)–(A4):

(a) $f(t) = \frac{\ln(a+t^p)}{t^{\alpha}}$, *where* $\alpha \in (0, 1)$, $a \geq 1$, $p \geq 0$ if $a > 1$ *and* $0 \leq p \leq \alpha + 1$ *if* $a = 1$.

(b) $f(t) = \frac{t^{\beta}}{(1+t)^{\alpha}}$, *where* $\alpha \in (0, 1)$, $0 \leq \beta < \alpha$.

(c) $f(t) = \frac{t^{\delta}+|\sin(t^{\gamma})|}{t^{\alpha}}$, *where* $\alpha \in (0, 1)$, $0 < \gamma < \delta < \alpha$. *Note that this function is not differentiable* on $(0, \infty)$.

By a solution of (1.1), we mean a function $u \in C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0, 1)$ which satisfies (1.1). By the strong maximum principle [**1**], any solution u of (1.1) is positive with $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν denotes the outer unit normal vector on $\partial\Omega$. Our main result is

THEOREM 1.1. *Let (A1)–(A4) hold. Then, there exists a positive constant λ_0 such that (1.1) has a unique solution for $\lambda > \lambda_0$.*

THEOREM 1.2. *Let (A1), (A3), (A5) hold and suppose that there exists a constant $C > 0$ such that $\lim_{t\to\infty} t^{\alpha} f(t) = C$. Let u_{λ} be a solution of (1.1). Then,*

$$\lim_{\lambda \to \infty} \frac{u_{\lambda}(x)}{(\lambda C)^{1/(1+\alpha)} w(x)} = 1$$

uniformly in Ω, where w denotes the unique solution of

$$-\Delta w = w^{-\alpha} \ \ in \ \ \Omega, \ \ w = 0 \ \ on \ \ \partial\Omega.$$

2. Preliminary results. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 be the normalized positive eigenfunction associated with λ_1 i.e. $||\phi_1||_\infty = 1$.

We shall denote the norms in $L^2(\Omega)$, $C^1(\bar{\Omega})$, and $C^{1,\beta}(\bar{\Omega})$ by $||.||_2$, $|.|_1$, and $|.|_{1,\beta}$ respectively.

We first recall the following regularity result in [**5**, Lemma 3.1]

LEMMA 2.1. *Let $h \in L^1(\Omega)$ and suppose that there exist constants $\gamma \in (0, 1)$ and $C > 0$ such that*

$$|h(x)| \leq \frac{C}{\phi_1^\gamma(x)}$$

for a.e. $x \in \Omega$. Then, the problem

$$\begin{cases} -\Delta u = h \ \ in \ \ \Omega, \\ \quad u = 0 \ on \ \partial\Omega \end{cases}$$

has a unique solution $u \in H_0^1(\Omega)$. Furthermore, there exist constants $\beta \in (0, 1)$ and $M >$ 0depending only on C, γ, Ω such that $u \in C^{1,\beta}(\bar{\Omega})$ and $|u|_{1,\beta} < M$.

COROLLARY 2.1. *Let h and u be given as in Lemma 2.1. Then, there exists a constant $k > 0$ such that $|u| \leq k\phi_1$ in Ω.*

Proof. By Lemma 2.1, there exist constants $\beta \in (0, 1)$ and $M > 0$ such that $|u|_{1,\beta} < M$. Hence, by the Mean Value Theorem, $|u(x)| \leq Md(x)$ for $x \in \Omega$, where $d(x)$ denotes the distance from x to $\partial\Omega$. Since $\phi_1 > 0$ in Ω and $\frac{\partial\phi_1}{\partial\nu} < 0$ on $\partial\Omega$, there exists a constant $k_0 > 0$ such that $\phi_1(x) \geq k_0 d(x)$ for $x \in \Omega$ (see e.g. [9, Proposition 2.1 (i)]. Consequently, $|u| \leq k\phi_1$ in Ω, where $k = M/k_0$. □

LEMMA 2.2. *Let D be an open set in Ω with $\bar{D} \subset \Omega$. Let $\gamma \in (0, 1)$ and z be the solution of*

$$\begin{cases} -\Delta z = \frac{1}{\phi_1^\gamma}\chi_D \ \ in \ \ \Omega, \\ \quad z = 0 \qquad on \ \partial\Omega, \end{cases} \tag{2.1}$$

Then, $|z|_1 \to 0$ as $|D| \to 0$. Here, χ_D denotes the characteristic function on D and $|D|$ the Lebesgue measure of D.

Proof. By Lemma 2.1, there exist $\beta \in (0, 1)$ and $M > 0$ independent of z such that $z \in C^{1,\beta}(\bar{\Omega})$ and $|z|_{1,\beta} < M$.

Multiplying the equation in (2.1) by z and integrating gives

$$||\nabla z||_2^2 = \int_D \frac{z}{\phi_1^\gamma}dx \leq M \int_D \frac{1}{\phi_1^\gamma}dx.$$

Since $1/\phi_1^\gamma \in L^1(\Omega)$ (see [**8**]), it follows that $||\nabla z||_2 \to 0$ as $|D| \to 0$.

By the interpolation result in [2, Corollary 1.3], there exist constants $c > 0$ and $\theta \in (0, 1)$ independent of z such that

$$|z|_1 \leq c|z|_{1,\beta}^{1-\theta}||\nabla z||_2^\theta \leq cM^{1-\theta}||\nabla z||_2^\theta,$$

which implies $|z|_1 \to 0$ as $|D| \to 0$. □

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LEMMA 2.3. *Let (A1), (A3), (A5) hold and let u be a solution of (1.1). Then, for* λ *large enough, there exists a constant* $c_\lambda > 0$ *with* $\lim_{\lambda\to\infty} c_\lambda = \infty$ *such that*

$$u \geq c_\lambda \phi_1 \text{ in } \Omega.$$

Proof. Let u be a solution of (1.1) and $M > 0$. By (A1) and (A3), there exists a constant $K > 0$ such that

$$f(t) \geq Kt$$

for $t \in (0, M]$. By the strong maximum principle, there exists a constant $\delta > 0$ such that $u \geq \delta\phi_1$ in Ω. Let δ_0 be the largest of those δ. Then, $u \geq \delta_0\phi_1$ in Ω. Suppose $\lambda > \lambda_1/K$. We claim that $\delta_0 \geq M$. Suppose to the contrary that $\delta_0 < M$. Let $D = \{x \in \Omega : u(x) < M\}$ and $a = \min(\lambda K, \lambda_1 M/\delta_0\} > \lambda_1$. Then,

$$\begin{cases} -\Delta u \geq \lambda K u \geq \lambda K \delta_0 \phi_1 \geq a\delta_0\phi_1 & \text{in } D, \\ \quad u = M \geq a\delta_0/\lambda_1 & \text{on } \partial D. \end{cases}$$

By the weak comparison principle [**10**, Lemma A2], $u \geq (a\delta_0/\lambda_1)\phi_1$ in Ω, which contradicts the maximality of δ_0. Hence, $u \geq M\phi_1$ in D, and since $u \geq M \geq M\phi_1$ in $\Omega\backslash D$, this completes the proof. □

LEMMA 2.4. *Let (A1), (A3), (A5) hold and suppose that there exist positive constants* A, M_0, M_1 *such that*

$$M_0 c^{-\alpha} f(t) \leq f(ct) \leq M_1 f(t) \tag{2.2}$$

for $c > 1, t > A$. *Then, there exist positive constants* $\bar{\lambda}$, K_0 *and* c_λ *with* $\lim_{\lambda\to\infty} c_\lambda = \infty$ *such that if u is a solution of (1.1) with* $\lambda \geq \bar{\lambda}$ *then*

$$c_\lambda\phi_1 \leq u \leq K_0 c_\lambda \phi_1 \quad \text{in } \Omega.$$

Proof. Let u be a solution of (1.1) and λ be large enough so that Lemma 2.2 holds. Let c_λ be the largest number so that $u \geq c_\lambda\phi_1$ in Ω.

For $c_\lambda\phi_1 > A$, it follows from (2.2) that

$$f(u) \leq M_1 f(c_\lambda\phi_1) \leq \frac{M_2 f(c_\lambda)}{\phi_1^\alpha}, \tag{2.3}$$

where $M_2 = M_1 M_0^{-1}$. For $u > A$ and $c_\lambda\phi_1 \leq A$,

$$f(u) \leq M_1 f(A) \leq \frac{M_1 A^\alpha f(A)}{(c_\lambda\phi_1)^\alpha}, \tag{2.4}$$

while it follows from (A5) that there exists a constant $B > 0$ such that

$$f(u) \leq \frac{B}{u^\alpha} \leq \frac{B}{(c_\lambda\phi_1)^\alpha}. \tag{2.5}$$

for $u \leq A$. Since $\liminf_{\lambda\to\infty} c_\lambda^\alpha f(c_\lambda) > 0$, it follows from (2.2)–(2.4) that there exists a constant $M > 0$ such that

$$-\Delta u = \lambda f(u) \leq \frac{\lambda M f(c_\lambda)}{\phi_1^\alpha} \text{ in } \Omega \tag{2.6}$$

for λ large. Let ϕ be the solution of

$$-\Delta \phi = \frac{1}{\phi_1^\alpha} \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega. \tag{2.7}$$

By Corollary 2.1, there exists a constant $k > 0$ such that $\phi \leq k\phi_1$ in Ω. Then (2.5) and the weak comparison principle imply

$$u \leq d_\lambda \phi_1 \quad \text{in } \Omega, \tag{2.8}$$

where $d_\lambda = \lambda k M f(c_\lambda)$.

Let $D_0 = \{x \in \Omega : \phi_1(x) > 1/2\}$. Then for λ large,

$$u \geq c_\lambda/2 > A \text{ in } D_0,$$

which implies

$$-\Delta u \geq \begin{cases} \lambda M_1^{-1} f(d_\lambda) & \text{in } D_0, \\ 0 & \text{in } \Omega\backslash D_0. \end{cases}$$

Hence,

$$u \geq \lambda M_1^{-1} f(d_\lambda)\phi_0 \ \geq \lambda k_0 f(d_\lambda)\phi_1 \ \text{ in } \Omega, \tag{2.9}$$

where ϕ_0 is the solution of

$$-\Delta\phi_0 = \begin{cases} 1 & \text{in } D_0, \\ 0 & \text{in } \Omega\backslash D_0, \end{cases}$$

and $k_0 > 0$ is such that $M_1^{-1}\phi_0 \geq k_0\phi_1$ in Ω. By (2.2),

$$d_\lambda^\alpha f(d_\lambda) \geq M_0 c_\lambda^\alpha f(c_\lambda),$$

which, together with (2.8) and the maximality of c_λ, implies

$$c_\lambda \geq \lambda k_0 f(d_\lambda) \geq \frac{\lambda k_0 M_0 c_\lambda^\alpha f(c_\lambda)}{(\lambda k M f(c_\lambda))^\alpha}.$$

Consequently,

$$c_\lambda \geq \lambda k_1 f(c_\lambda), \tag{2.10}$$

where $k_1 = (k_0 M_0/(kM)^\alpha)^{1/(1-\alpha)}$. Hence,

$$d_\lambda = \lambda k M f(c_\lambda) \leq K_0 c_\lambda, \tag{2.11}$$

where $K_0 = kM/k_1$. This, together with (2.7), completes the proof of Lemma 2.3. □

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REMARK 2.1. Let (A1), (A3), (A5) hold and suppose that there exists a constant $C > 0$ such that $\lim_{t\to\infty} t^\alpha f(t) = C$. Then (2.2) hold and we deduce from (2.9) and (2.10) that for λ large, there exist positive constants m_1, m_2 such that any solution u of (1.1) satisfies

$$m_1\lambda^{1/(\alpha+1)}\phi_1 \le u \le m_2\lambda^{1/(\alpha+1)}\phi_1 \text{ in } \Omega.$$

3. Proof of the main results.

Proof of Theorem 1.1 Since f is sublinear at ∞ and $\liminf_{t\to\infty} t^\alpha f(t) > 0$, the existence of a solution to (1.1) for λ large follows from [**5**, Theorem 2.1]. Let u_1 and u_2 be solutions of (1.1) with λ large. By Lemma 2.3, $c_0u_2 \le u_1 \le c_0^{-1}u_2$ in Ω, where $c_0 = K_0^{-1}$. Let c be the largest number such that $cu_2 \le u_1 \le c^{-1}u_2$ in Ω and suppose that $c < 1$. Then,

$$|u_1 - u_2| \le (c^{-1} - 1)u_2 \text{ in } \Omega.$$

Let $a > 0$ be such that

$$c^\alpha - c \ge a(1 - c) \text{ for } c \in [c_0, 1]. \tag{3.1}$$

If $u_2 > AK_0$, then $u_1 > A$ and it follows from (3.1), (A2) and Lemma 2.3 that

$$\begin{aligned} f(u_1) - cf(u_2) &\ge f(c^{-1}u_2) - cf(u_2) \ge (c^\alpha - c)f(u_2) \\ &\ge \frac{B_0(c^\alpha - c)}{u_2^\alpha} \ge \frac{B_0a(1-c)}{(K_0c_\lambda\phi_1)^\alpha} = \frac{B_1(1-c)}{(c_\lambda\phi_1)^\alpha}, \end{aligned} \tag{3.2}$$

where $B_0 = (AK_0)^\alpha f(AK_0)$, $B_1 = B_0a/K_0^\alpha$.

On the other hand, if $u_2 \le AK_0$ then $u_1 \le AK_0^2$ and it follows from (A4) with $B = AK_0^2$ that

$$\begin{aligned} |f(u_1) - f(u_2)| &\le \frac{C_B|u_1 - u_2|}{\min^{\alpha+1}(u_1, u_2)} \le \frac{C_B(c^{-1} - 1)u_2}{(cu_2)^{\alpha+1}} \\ &\le \frac{C_B(1-c)}{c^{\alpha+2}(c_\lambda\phi_1)^\alpha} = \frac{B_2(1-c)}{(c_\lambda\phi_1)^\alpha}, \end{aligned}$$

where $B_2 = C_B/c_0^{2+\alpha}$. In particular,

$$f(u_1) - cf(u_2) \ge -\frac{B_2(1-c)}{(c_\lambda\phi_1)^\alpha}. \tag{3.3}$$

Let $D_\lambda = \{x \in \Omega : \phi_1(x) > AK_0/c_\lambda\}$. Then, $u_2 \ge c_\lambda\phi_1 > AK_0$ in D_λ and it follows from (3.2)–(3.3) that

$$\begin{aligned} -\Delta(u_1 - cu_2) &= \lambda(f(u_1) - cf(u_2)) \\ &\ge \frac{\lambda B_1(1-c)}{(c_\lambda\phi_1)^\alpha} - \frac{\lambda B_3(1-c)}{(c_\lambda\phi_1)^\alpha}\chi_{\Omega\setminus D_\lambda} \text{ in } \Omega, \end{aligned} \tag{3.4}$$

where $B_3 = B_1 + B_2$. Let z be the solution of (2.1) with $D = \Omega\setminus D_\lambda$ and $\gamma = \alpha$. Since

$$\Omega\setminus D_\lambda \subset \{x \in \Omega : \phi_1(x) \le AK_0/c_\lambda\}$$

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and $c_\lambda \to \infty$ as $\lambda \to \infty$, it follows that $|\Omega\backslash D_\lambda| \to 0$ as $\lambda \to \infty$. Hence, Lemma 2.1 with $D = \Omega\backslash D_\lambda$ gives $|z|_1 \to 0$ as $\lambda \to \infty$. This, together with (3.4), gives

$$u_1 - cu_2 \geq \frac{\lambda(1-c)}{c_\lambda^\alpha}(B_1\phi - B_3 z) \geq \frac{\lambda B_1(1-c)}{2c_\lambda^\alpha}\phi \text{ in } \Omega$$

if λ is large enough, where ϕ is defined in (2.6). This contradicts the maximality of c and therefore $c = 1$, which completes the proof. □

Proof of Theorem 2.2. Without loss of generality, we assume $C = 1$. Let u_λ be a solution of (1.1) with λ large, and let $v_\lambda = \lambda^{1/(1+\alpha)} w$. Note that v_λ satisfies

$$-\Delta v_\lambda = \lambda v_\lambda^{-\alpha} \text{ in } \Omega,\ \ v_\lambda = 0 \text{ on } \partial\Omega.$$

By Remark 2.1,

$$m_1 \lambda^{1/(\alpha+1)} \phi_1 \leq u_\lambda, v_\lambda \leq m_2 \lambda^{1/(\alpha+1)} \phi_1 \ \text{ in } \Omega,$$

which implies $c_0 v_\lambda \leq u_\lambda \leq c_0^{-1} v_\lambda$ in Ω, where $c_0 = m_1/m_2$. Let c be the largest number such that $cv_\lambda \leq u_\lambda \leq c^{-1} v_\lambda$ in Ω. Let $\varepsilon \in (0, 1)$ and suppose that $c \leq (1-\varepsilon)^{1/(1-\alpha)} \equiv \varepsilon_0$. Since $\lim_{t\to\infty} t^\alpha f(t) = 1$, there exists a constant $A > 0$ such that

$$\frac{1-\varepsilon/2}{t^\alpha} \leq f(t) \leq \frac{(1-\varepsilon/2)^{-1}}{t^\alpha}$$

for $t > A$. Hence, for $u_\lambda > A$,

$$\begin{aligned} f(u_\lambda) - \frac{c}{v_\lambda^\alpha} &\geq \frac{1-\varepsilon/2}{u_\lambda^\alpha} - \frac{c}{v_\lambda^\alpha} \geq \frac{(1-\varepsilon/2)c^\alpha - c}{v_\lambda^\alpha} \\ &\geq \frac{m_3}{v_\lambda^\alpha} \geq \frac{m_4}{\lambda^{\alpha/(1+\alpha)}\phi_1^\alpha}\ , \end{aligned} \tag{3.5}$$

where $m_3 = \min_{c_0\leq c\leq\varepsilon_0}((1-\varepsilon/2)c^\alpha - c) > 0$, $m_4 = m_3 m_2^{-\alpha}$, and

$$\begin{aligned} f(u_\lambda) - \frac{c^{-1}}{v_\lambda^\alpha} &\leq \frac{(1-\varepsilon/2)^{-1}}{u_\lambda^\alpha} - \frac{c^{-1}}{v_\lambda^\alpha} \leq \frac{(1-\varepsilon/2)^{-1}c^{-\alpha} - c^{-1}}{v_\lambda^\alpha} \\ &\leq -\frac{m_5}{v_\lambda^\alpha} \leq -\frac{m_6}{\lambda^{\alpha/(1+\alpha)}\phi_1^\alpha}, \end{aligned} \tag{3.6}$$

where $m_5 = \min_{c_0\leq c\leq\varepsilon_0}(c^{-1} - (1-\varepsilon/2)^{-1}c^{-\alpha}) > 0$, $m_6 = m_5 m_2^{-\alpha}$.

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On the other hand, it follows from (A1) and (A5) that there exists a constant $B > 0$ such that

$$0 < f(t) \leq Bt^{-\alpha} \quad \text{for } t \in (0, A].$$

Hence, for $u_\lambda \leq A$,

$$f(u_\lambda) - \frac{c}{v_\lambda^\alpha} \geq -\frac{c}{v_\lambda^\alpha} \geq -\frac{\varepsilon_0}{v_\lambda^\alpha} \geq -\frac{m_7}{\lambda^{\frac{\alpha}{\alpha+1}}\phi_1^\alpha}, \tag{3.7}$$

where $m_7 = \varepsilon_0 m_1^{-\alpha}$, and

$$f(u_\lambda) - \frac{c^{-1}}{v_\lambda^\alpha} \leq \frac{B}{u_\lambda^\alpha} \leq \frac{m_8}{\lambda^{\frac{\alpha}{\alpha+1}}\phi_1^\alpha}, \tag{3.8}$$

where $m_8 = Bm_1^{-\alpha}$.

Let $D_\lambda = \{x \in \Omega : \phi_1(x) > Am_1^{-1}\lambda^{-1/(\alpha+1)}\}$. Note that $u_\lambda > A$ in D_λ and $|\Omega \backslash D_\lambda| \to 0$ as $\lambda \to \infty$.

From (3.5) and (3.7), it follows that

$$-\Delta(u_\lambda - cv_\lambda) = \lambda\left(f(u_\lambda) - \frac{c}{v_\lambda^\alpha}\right) \geq \lambda^{1/(\alpha+1)}\left(\frac{m_4}{\phi_1^\alpha} - \frac{m_9}{\phi_1^\alpha}\chi_{\Omega\backslash D_\lambda}\right),$$

where $m_9 = m_4 + m_7$. On the other hand, (3.6) and (3.8) give

$$-\Delta(u_\lambda - c^{-1}v_\lambda) = \lambda\left(f(u_\lambda) - \frac{c^{-1}}{v_\lambda^\alpha}\right) \leq -\lambda^{1/(\alpha+1)}\left(\frac{m_6}{\phi_1^\alpha} - \frac{m_{10}}{\phi_1^\alpha}\chi_{\Omega\backslash D_\lambda}\right),$$

where $m_{10} = m_6 + m_8$. Hence, Lemma 2.1 and the weak comparison principle give

$$u_\lambda - cv_\lambda \geq \lambda^{1/(\alpha+1)}(m_4\phi - m_9 z) \geq \lambda^{1/(\alpha+1)}(m_4/2)\phi \quad \text{in } \Omega,$$

and

$$u_\lambda - c^{-1}v_\lambda \leq -\lambda^{1/(\alpha+1)}(m_6\phi - m_{10}z) \leq -\lambda^{1/(\alpha+1)}(m_6/2)\phi \quad \text{in } \Omega$$

for $\lambda \gg 1$, where ϕ is defined in (2.6) and z is defined in Lemma 2.1 with $D = \Omega\backslash D_\lambda$. This contradicts the maximality of c and therefore $c \geq (1-\varepsilon)^{1/(1-\alpha)}$ for $\lambda \gg 1$ i.e.

$$(1-\varepsilon)^{1/(1-\alpha)}v_\lambda \leq u_\lambda \leq (1-\varepsilon)^{1/(\alpha-1)}v_\lambda \quad \text{in } \Omega.$$

This completes the proof of Theorem 1.2. □

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