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EXTENSIONS OF ORDERABLE GROUPS

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Introduction. The purpose of this note is to unify Theorem 4 of G. Baumslag [2] and a result of D. M. Smirnov in [6] in a more general setting. We prove the following result.

THEOREM. *Let A be a normal subgroup of a non-abelian free group F, V a proper fully invariant subgroup of A and $\underline{\underline{V}}$ the variety generated by A/V. If A/V is orderable and F/A has an infrainvariant system with factors in $\underline{\underline{V}}$ and right-orderable then F/V is orderable.*

Let $G=F/V$, $X=A/V$ and $Y=F/A$. Also let RO denote the class of right-orderable groups. We fix this notation throughout the paper. Observe that if X is a non-trivial orderable group, then $V \leq A'$ and $\underline{\underline{V}}$ contains all abelian groups. Thus torsion-free abelian groups are in $RO \cap \underline{\underline{V}}$ and the hypothesis of the theorem holds if Y has an infrainvariant system with torsion-free abelian factors and $V=A'$. This is Smirnov's result in [6]. If Y is an ordered group, the convex subgroups of Y form an infrainvariant system with torsion-free abelian factors. Thus the hypothesis of the theorem holds when X and Y are orderable. This is Theorem 4 in [2].

DEFINITIONS. We say that a group H has an infrainvariant system with factors in a class $\underline{\underline{X}}$ if H has a set of subgroups $S=\{H_\lambda;\ \lambda \in \Lambda\}$ such that (i) Λ is a complete totally ordered set, (ii) $\langle e \rangle$, $G \in S$, (iii) if $\lambda < \mu$ then $H_\lambda \leq H_\mu$ and if μ is an immediate successor of λ in the ordering of Λ then $H_\lambda \quad H_\mu$, $H_\mu/H_\lambda \in X$, and (iv) for any $\lambda \in \Lambda$ and any $h \in H$, $H_\lambda^h \in S$. A group $H \in RO$ is the set H can be ordered in such a way that for all g, h, x in H, $g<h$ implies $gx<hx$. This is equivalent to saying that H is isomorphic to a subgroup of the group of order preserving permutations of an ordered set (see [3]). If H is a group and Z a subset of H then $S_H(Z)$ denotes the semigroup generated by $\{z^h;\ z \in Z, h \in H\}$. If K is a normal subgroup of a group H then we say that K is H-orderable if the set K can be ordered in such a way that for all x, y, z in K, h in H, $x<y$ implies $xz<yz$ and $x^h<y^h$. This is equivalent to saying that given any finite set $x_1, \ldots, x_n$ in $K\backslash\langle e \rangle$, $e \notin S_H(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ for a suitable choice of signs $\varepsilon_i = \pm 1$. If H is H-orderable then we simply say H is orderable and denote the class of such groups by O.

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Proofs. We will use the following result which is a consequence of Theorem 1 in [2].

PROPOSITION 1. (G. Baumslag). *If $W = X \text{ wr } Y$, the standard restricted wreath product of X and Y, and B the base group of W then given any finite set of elements $x_1, \ldots, x_n \in X \backslash \langle e \rangle$, there exists a homomorphism ϕ of G into W such that $\phi x_1, \ldots, \phi x_n \in B \backslash \langle e \rangle$.*

LEMMA 1. *If $Y = F/A \in RO$ and $X = A/V \in O$ then A/V is G orderable. If $V = A'$ then the converse is also true, that is if A/V is F/V-orderable then $F/A \in RO$.*

Proof. Let $W = X \text{ wr } Y$ and B the base group of W. Under the given hypothesis, B is W-orderable, since given any order on X and right-order on Y the corresponding lexicographic order of B is a W-order. Suppose that X is not G-orderable. Then there exist elements $x_1, \ldots, x_n \in X \backslash \langle e \rangle$ such that $e \in S_G(x_1^{\varepsilon_1}, \ldots, x_n^{\varepsilon_n})$ for all choices of signs $\varepsilon_i = \pm 1$. By Proposition 1 there is a homomorphism ψ of G into W such that $\psi x_1, \ldots, \psi x_n \in B \backslash \langle e \rangle$. But then $S_W((\psi x_1)^{\varepsilon_1}, \ldots, (\psi x_n)^{\varepsilon_n})$ contains e for all choices of signs $\varepsilon_1 = \pm 1$, contradicting the fact that B is W-orderable.

Conversely, in the case $V = A'$, let A/A' be F/A'-orderable. Since $C_F(A/A') = A$ ([1], Theorem 1), F/A is a group of order-preserving permutations of the ordered set A/A' and consequently is an RO-group.

Proof of the theorem. By hypothesis there exists an infrainvariant system $\sum = \{F_\lambda, \lambda \in \Lambda\}$ of subgroups connecting A to F with factors in $RO \cap \underline{\underline{V}}$. Let $\sum_1 = \{v(F_\lambda), \lambda \in \Lambda\}$ where $v(F_\lambda)$ is the verbal subgroup of F_λ corresponding to the variety $\underline{\underline{V}}$. Then $\sum_1$ is an infrainvariant system connecting V to F. In fact

(i) $v(F_\lambda^g) = (v(F_\lambda))^g$,
(ii) For any $B \subseteq \Lambda$, if $\bigcup_{\lambda \in B} F_\lambda = F_\gamma$ then $\bigcup_{\lambda \in B} v(F_\lambda) = v(F_\gamma)$ and
(iii) If $\bigcap_{\lambda \in B} F_\lambda = F_\gamma$ then $\bigcap_{\lambda \in B} v(F_\lambda) = v(F_\gamma)$.

(i) and (ii) are obvious and (iii) follows from a result of Dunwoody in [4]. Note that $\sum_1$ does not contain repetitions for $v(F_\lambda) = v(F_\mu)$ implies $F_\lambda = F_\mu$. Let $v(F_\alpha) \prec v(F_{\alpha+1})$ be a jump in $\sum_1$ and let $N_\alpha = N_F(v(F_\alpha)) = N_F(v(F_{\alpha+1}))$. Then $F_\alpha \prec F_{\alpha+1}$ is a jump in $\sum$. Since $v(F_\alpha)$ is fully invariant in F_α, $N_F(F_\alpha) \leq N_\alpha$, conversely if $g \in N_\alpha$ and $F_\beta = F_\alpha^g$ then $v(F_\beta) = v(F_\alpha)$ and $\alpha = \beta$. Thus $N_F(F_\alpha) = N_\alpha$. Since F/A has a system with factors in RO, passing through F_α, N_α/F_α also has such a system and is therefore in RO. By Lemma 1, $F_\alpha/v(F_\alpha)$ is $N_\alpha/v(F_\alpha)$-orderable. Since $F_{\alpha+1}/F_\alpha \in \underline{\underline{V}}$, $F_\alpha \geq v(F_{\alpha+1})$ and therefore $v(F_{\alpha+1})/v(F_\alpha)$ is also $N_\alpha/v(F_\alpha)$-orderable. By a theorem of Kokorin in [5], the system $\sum_1$ assures the orderability of F/V.

In the same way Theorem 1 of Smirnov in [7] can be modified to the following.

If Y has an infrainvariant system with factors in $\underline{\underline{V}}$ then G has an infrainvariant system whose factors are subgroups of free $\underline{\underline{V}}$-groups.

References

1. M. Auslander and R. C. Lyndon, *Commutator subgroups of free groups.* Amer. J. Math. **77,** 929–931 (1955).

2. G. Baumslag, *Wreath products and extensions.* Math. Z. **81,** 286–299 (1963).

3. P. M. Cohn, *Groups of order automorphisms of ordered sets.* Mathematika **4,** 41–50 (1957).

4. M. J. Dunwoody, *On verbal subgroups of free groups.* Arch. Math. (Basel) **16,** 153–157 (1965).

5. A. I. Kokorin, *Γ-fully orderable and relatively convex subgroups of orderable groups.* (In Russian) Sibirsk. Mat. Z. **7,** 713–717 (1966).

6. D. M. Smirnov, *Right-ordered groups.* Algebra i Logika **5:6,** 41–59 (1966).

7. D. M. Smirnov, *On generalized solvable groups and their group rings.* Mat. Sb. **109** (N.S. 67), 366–383 (1965).

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