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A SPECIALISED NET OF QUADRICS HAVING SELF-POLAR POLYHEDRA, WITH DETAILS OF THE FIVE-DIMENSIONAL EXAMPLE

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1. If $x_0, x_1, \ldots, x_n$ are homogeneous coordinates in $[n]$, projective space of n dimensions, the *prime* (to use the standard name for a hyperplane)

$$x_0 + x_1\theta + \ldots + x_n\theta^n = 0$$

osculates, as θ varies, the rational normal curve C whose parametric form is [**2**, p. 347]

$$x_r = (-1)^r \binom{n}{r} \theta^{n-r}.$$

Take a set of $n + 2$ points on C for which $\theta = \eta^j \zeta$ where ζ is any complex number and

$$\eta = \exp\,[2\pi i/(n + 2)]$$

so that the η^j, for $0 \leqq j < n + 2$, are the $(n + 2)$th roots of unity. The $n + 2$ primes osculating C at these points bound an $(n + 2)$-hedron H which varies with ζ, and H is polar for all the quadrics

$$(1.1) \quad \sum \alpha_j P_j^{\,2} \equiv \sum_{j=0}^{n+1} \alpha_j (x_0 + x_1\eta^j\zeta + x_2\eta^{2j}\zeta^2 + \ldots + x_n\eta^{nj}\zeta^n)^2 = 0$$

in the sense that the polar of any vertex, common to n of its $n + 2$ bounding primes, contains the opposite $[n - 2]$ common to the residual pair. Call this vertex and opposite $[n - 2]$ *conjugate* with respect to the quadric.

If the α_j can be chosen so that (1.1) does not depend on ζ the various H will all be polar for the same quadric. In attempting to achieve this it is natural, since $\sum_{j=0}^{n+1} \eta^{mj}$ is zero save when m is zero or a multiple of $n + 2$, to take $\alpha_j = \eta^{kj}$ with k some positive integer. But then, the power of ζ appearing in any term in the expanded form of (1.1) being less by k than the power of η^j, if ζ is to be cancellable the $2n + 1$ consecutive integers

$$k, k + 1, \ldots, k + 2n$$

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must not involve two multiples (zero included) of $n + 2$. Hence k is 1, 2 or 3. Then

$$\sum \eta^j P_j^2 \equiv (n+2)\zeta^{n+1}(x_1x_n + x_2x_{n-1} + \ldots) \equiv (n+2)\zeta^{n+1}\sum_{r=1}^{n} x_r x_{n+1-r},$$

$$\sum \eta^{2j} P_j^2 \equiv (n+2)\zeta^{n}(x_0x_n + x_1x_{n-1} + \ldots) \equiv (n+2)\zeta^{n}\sum_{r=0}^{n} x_r x_{n-r},$$

$$\sum \eta^{3j} P_j^2 \equiv (n+2)\zeta^{n-1}(x_0x_{n-1} + x_1x_{n-2} + \ldots) \equiv (n+2)\zeta^{n-1}\sum_{r=0}^{n-1} x_r x_{n-1-r}.$$

So one obtains a net N of quadrics

(1.2) $\lambda Q_0 + \mu Q_1 + \nu Q_2 = 0$

all of which have all ∞^1 H as polars: it is based on the quadratic forms

$$Q_0 \equiv \sum_{r=1}^{n} x_r x_{n+1-r}, \quad Q_1 \equiv \sum_{r=0}^{n} x_r x_{n-r}, \quad Q_2 \equiv \sum_{r=0}^{n-1} x_r x_{n-1-r}.$$

Note that any x_r^2 occurring in a sum only appears once, while any product appears twice. It is sometimes convenient to speak of the quadric $Q_i = 0$ merely as Q_i. Q_0 is a cone with vertex X_0, Q_2 a cone with vertex X_n; here, as is customary, X_r denotes the vertex of the simplex of reference opposite to $x_r = 0$. X_0 and X_n are not only vertices of cones: they are on the octavic $(n - 3)$-fold B, the base locus of N, and so may be expected to be singular points of the Jacobian curve $\mathcal{J}$ of N.

2. If η^p and η^q are not equal

(2.1) $\eta^p\eta^q Q_0 - (\eta^p + \eta^q)\zeta Q_1 + \zeta^2 Q_2$

lacks the terms in P_p^2 and P_q^2 so that the quadric is a cone; its vertex, common to all $P_j = 0$ save for $j = p, q$ is a vertex of H; all vertices of all H are on $\mathcal{J}$. Each edge of an H, common to $n - 1$ of its $n + 2$ bounding primes, is a trisecant of $\mathcal{J}$ because it contains three vertices of H. Likewise a bounding plane, common to $n - 2$ bounding primes, is met by the other four in the sides of a quadrilateral and meets $\mathcal{J}$ at the six vertices of this; a bounding solid meets $\mathcal{J}$ at the ten vertices of a pentahedron; and so on.

If $\eta^p + \eta^q = 0$ the cones (2.1) are just the pencil in N spanned by Q_0 and Q_2. This cannot happen unless n is even, requiring as it does the presence of -1 among the $(n + 2)$th roots of unity; but in this event (2.1) is

(2.2) $\displaystyle\sum_{r=1}^{n} (\zeta^2 x_{r-1}x_{n-r} - \eta^{2p} x_r x_{n+1-r})$ (n even).

The sum of the pair of suffixes in every product here is odd, so that one suffix of every pair is odd and the other even and (2.2) is

$$2\sum_{s=1}^{n/2} x_{2s-1}(\zeta^2 x_{n-2s} - \eta^{2p} x_{n-2s+2}),$$

a quadratic form not in $n + 1$ but in only n variables. It represents a cone whose vertex

$$(1, 0, u, 0, u^2, \ldots, 0, u^{n/2})$$

is the common zero of these n variables; here $u = \zeta^2/\eta^{2p}$. As the cone varies in the pencil its vertex traces a rational normal curve of order $\frac{1}{2}n$ in the $[\frac{1}{2}n]\ X_0X_2 \ldots X_n$. This curve, when n is even, is part of $\mathscr{J}$.

When (2.1) is divided by $\eta^{(p+q)/2}$ it becomes

$$\eta^{(p+q)/2}Q_0 - 2\zeta \cos\frac{(p-q)\pi}{n+2} Q_1 + \zeta^2\eta^{-(p+q)/2} Q_2;$$

the cones are just those quadrics (1.2) for which

$$\frac{\mu^2}{\nu\lambda} = 4\cos^2\frac{(p-q)\pi}{n+2}.$$

They belong to different families according to the value of this squared cosine. Since p, q are unequal integers among $0, 1, 2, \ldots, n + 1$

$$0 < |p - q| \leqq n + 1.$$

But two values of $|p - q|$ whose sum is $n + 2$ yield the same squared cosine, so that none higher than the integral part of $\frac{1}{2}(n + 2)$ need be used. And since $p - q = \frac{1}{2}n + 1$ has, when n is even, been disposed of, the number of different families here is the integral part of $\frac{1}{2}(n + 1)$. Each provides, with ζ varying, a singly-infinite family of cones whose vertices trace a component of $\mathscr{J}$. And every family includes both Q_0 and Q_2.

For the lower values of n the circumstances are easy to describe.

If $n = 3$, $\mu^2/(4\nu\lambda)$ is either $\cos^2\pi/5$ or $\cos^2 2\pi/5$. The geometry of this figure was described in [**4**, pp. 471-480].

If $n = 4$, $\mu^2/(4\nu\lambda)$ is either $\cos^2\pi/6 = 3/4$ or $\cos^2\pi/3 = 1/4$. The The Jacobian curve has, n being even, an additional third component, here a conic in the plane $X_0X_2X_4$. The geometry of this figure has only been described very recently [**5**].

If $n = 5$, $\mu^2/(4\nu\lambda)$ is one of

$$\cos^2\pi/7,\ \cos^2 2\pi/7,\ \cos^2 3\pi/7$$

and $\mathscr{J}$ is tripartite. The following paragraphs are concerned with the geometry of this figure in [5].

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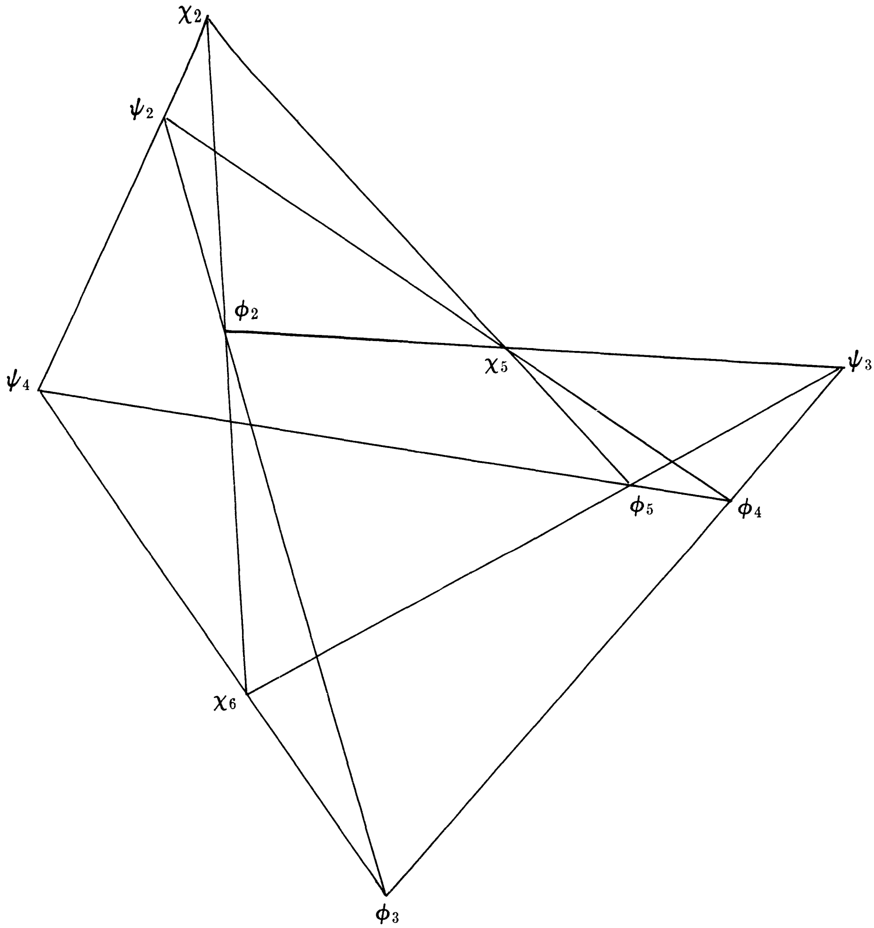

The ten vertices of the pentahedron conjugate to ϕ_0.

3. The geometry in [5]. In [5] the primes

(3.1) $x_0 + x_1\theta + x_2\theta^2 + x_3\theta^3 + x_4\theta^4 + x_5\theta^5 = 0$

osculate, when θ varies, a rational normal quintic C; the vertices X_0, X_5 of the simplex of reference are on C with respective parameters ∞, 0. If (3.1) is looked upon as a restriction on θ when the x_i have been assigned it shows that the point common to five osculating primes of C is

(3.2) $(-e_5, e_4, -e_3, e_2, -e_1, 1)$

where e_i is the elementary symmetric function of degree i in the parameters of the five contacts.

Take $\epsilon = \exp(2\pi i/7)$ and H to be bounded by those seven primes (3.1) for which $\theta = \epsilon^j\zeta$ $(0 \leqq j \leqq 6)$. The different values of ζ afford an infinity of H; each H has 21 vertices whose locus, as ζ varies, is $\mathscr{J}$. The identity

$$(1-\epsilon)(\theta^7-\zeta^7) \equiv (\theta-\zeta)(\theta-\epsilon\zeta)\{(1-\epsilon)\theta^5+(1-\epsilon^2)\theta^4\zeta + (1-\epsilon^3)\theta^3\zeta^2+(1-\epsilon^4)\theta^2\zeta^3+(1-\epsilon^5)\theta\zeta^4+(1-\epsilon^6)\zeta^5\}$$

shows, with (3.2), that when $\zeta = \phi$ the vertex of H opposite to the solid common to the osculating primes of C at $\theta = \phi$, $\epsilon\phi$ is

(3.3) $$[(1-\epsilon^6)\phi^5, (1-\epsilon^5)\phi^4, (1-\epsilon^4)\phi^3, (1-\epsilon^3)\phi^2, (1-\epsilon^2)\phi, 1-\epsilon].$$

So this vertex traces, as ϕ varies, a rational normal quintic Γ_0.

Now only 7 of the 21 vertices of H are here accounted for. Just as a cyclic group of order 7 has three pairs of inverse operations, so the bounding primes of H can succeed each other in three different cycles: ϵ and ϵ^6 correspond to one cycle traversed in opposite senses, ϵ^2 and ϵ^5 to a second, ϵ^4 and ϵ^3 to a third. The phenomenon is mirrored in the Euclidean plane by the linking with a regular convex heptagon of two stellated heptagons, all three heptagons sharing the same vertices. So the vertices of H trace three rational normal quintics:

(3.4) $$\begin{cases} \Gamma_0 : x_{6-j} = (1-\epsilon^j)\phi^{j-1} \\ \Gamma_1 : x_{6-j} = (1-\epsilon^{2j})\psi^{j-1} \qquad j = 6, 5, 4, 3, 2, 1 \\ \Gamma_2 : x_{6-j} = (1-\epsilon^{4j})\chi^{j-1} \end{cases}$$

each Γ_i in the cycle $(\Gamma_0\Gamma_1\Gamma_2)$ being changed into its successor by the operation σ, of period 3, that replaces ϵ by ϵ^2. All three Γ_i contain X_0 and X_5 and share the same osculating spaces there; together they constitute $\mathscr{J}$, the (now tripartite) Jacobian curve.

4. In [5] N is based on

$$Q_0 \equiv 2(x_1x_5+x_2x_4)+x_3^2,\ Q_1 \equiv 2(x_0x_5+x_1x_4+x_2x_3),$$
$$Q_2 \equiv 2(x_0x_4+x_1x_3)+x_2^2.$$

The outcome of substituting the parametric form for Γ_0 in Q_0 is

$$\begin{aligned} &2[(1-\epsilon^5)\phi^4(1-\epsilon)+(1-\epsilon^4)\phi^3(1-\epsilon^2)\phi]+(1-\epsilon^3)^2\phi^4 \\ &= \phi^4[4+4\epsilon^6+1+\epsilon^6-2(\epsilon^5+\epsilon+\epsilon^4+\epsilon^2+\epsilon^3)] \\ &= 7(1+\epsilon^6)\phi^4 = 7\epsilon^3(\epsilon^4+\epsilon^3)\phi^4. \end{aligned}$$

Such procedures show the results of the nine substitutions to be as follows:

	Γ_0	Γ_1	Γ_2
Q_0	$7\epsilon^3(\epsilon^4+\epsilon^3)\phi^4$	$7\epsilon^6(\epsilon+\epsilon^6)\psi^4$	$7\epsilon^5(\epsilon^2+\epsilon^5)\chi^4$
Q_1	$14\phi^5$	$14\psi^5$	$14\chi^5$
Q_2	$7\epsilon^4(\epsilon^3+\epsilon^4)\phi^6$	$7\epsilon(\epsilon^6+\epsilon)\psi^6$	$7\epsilon^2(\epsilon^5+\epsilon^2)\chi^6$

So the ten intersections of (1.2) with any Γ_i lie four at X_0 (parameter ∞), four at X_5 (parameter 0) and two elsewhere, these latter being in general distinct from X_0, X_5 and from each other. The parameters of the latter pair on, say, Γ_0 are zeros of the quadratic

$$\lambda\epsilon^3(\epsilon^4 + \epsilon^3) + 2\mu\phi + \nu\epsilon^4(\epsilon^3 + \epsilon^4)\phi^2$$

and coincide when

$$\mu^2 = \nu\lambda(\epsilon^4 + \epsilon^3)^2 = 4\nu\lambda \cos^2(\pi/7).$$

This agrees with (1.2) then being a cone whose vertex is on Γ_0. Similarly (1.2) is a cone whose vertex is on Γ_1 when

$$\mu^2 = 4\nu\lambda \cos^2(2\pi/7)$$

and a cone whose vertex is on Γ_2 when

$$\mu^2 = 4\nu\lambda \cos^2(4\pi/7).$$

5. The polar primes of (3.3) with respect to Q_0, Q_1, Q_2 are found to be

$$L = M,\ L = \epsilon M,\ L = \epsilon^2 M$$

where

$$L \equiv \sum_{j=0}^{5} \phi^j x_j, \qquad M \equiv \sum_{j=0}^{5} (\epsilon\phi)^j x_j.$$

All three contain the solid $L = M = 0$ which is therefore the space conjugate to (3.3) in the sense of Section 1. This space, as remarked in Section 2, meets $\mathscr{J}$ in the ten vertices of a pentahedron. How are these ten points distributed among the Γ_i?

Take the point with parameter θ on Γ_0. The conjugate solid is $L = M = 0$ with ϕ therein replaced by θ. Substitution from the first member of (3.4) shows that it meets Γ_0 in any points satisfying both the conditions (summations running from $j = 1$ to $j = 6$)

$$\sum\theta^{6-j}(1 - \epsilon^j)\phi^{j-1} = 0 = \sum(\epsilon\theta)^{6-j}(1 - \epsilon^j)\phi^{j-1}$$
$$\sum\theta^{-j}(1 - \epsilon^j)\phi^j = 0 = \sum(\epsilon\theta)^{-j}(1 - \epsilon^j)\phi^j$$
$$\sum(\epsilon\theta^{-1}\phi)^j = \sum(\theta^{-1}\phi)^j = \sum(\epsilon^{-1}\theta^{-1}\phi)^j.$$

These conditions hold, with each sum -1, when

(5.1) $\quad \phi = \epsilon^2\theta,\ \epsilon^3\theta,\ \epsilon^4\theta,\ \epsilon^5\theta$

and the solid conjugate to a point of Γ_0 is quadrisecant to Γ_0.

In order to find where this same solid meets Γ_1 one has to substitute not from the first but from the second member of (3.4).

The resulting conditions are

$$\sum \theta^{6-j}(1 - \epsilon^{2j})\psi^{j-1} = 0 = \sum (\epsilon\theta)^{6-j}(1 - \epsilon^{2j})\psi^{j-1}$$
$$\sum \theta^{-j}(1 - \epsilon^{2j})\psi^{j} = 0 = \sum (\epsilon\theta)^{-j}(1 - \epsilon^{2j})\psi^{j}$$
$$\sum (\theta^{-1}\psi)^{j} = \sum (\epsilon^2\theta^{-1}\psi)^{j}, \sum (\epsilon^{-1}\theta^{-1}\psi)^{j} = \sum (\epsilon\theta^{-1}\psi)^{j}.$$

All four sums here are -1 when

(5.2) $\psi = \epsilon^2\theta, \epsilon^3\theta, \epsilon^4\theta.$

The same solid is found, similarly, to meet Γ_2 where

(5.3) $\chi = \epsilon^2\theta, \epsilon^5\theta, \epsilon^6\theta,$

and all its ten intersections with $\mathscr{J}$ are accounted for.

6. These ten points (5.1, 2, 3) are collinear in threes on the edges of the pentahedron, trisecants of the Jacobian curve. No individual Γ_i, being a rational normal curve, can have any trisecants; but chords of one Γ_i may meet a second, and there may be transversals, trisecant to $\mathscr{J}$ by meeting each Γ_i once. Both possibilities are realised.

The trivial remark that $1 + \epsilon^j - 1 = \epsilon \cdot \epsilon^{j-1}$ leads one to write

$$(1 - \epsilon^{2j})\theta^{j-1} - (1 - \epsilon^j)\theta^{j-1} = \epsilon(1 - \epsilon^j)(\epsilon\theta)^{j-1}$$

which shows that the join of $\psi = \theta$ on Γ_1 to $\phi = \theta$ on Γ_0 meets Γ_0 again at $\phi = \epsilon\theta$. Similarly the join of $\chi = \theta$ on Γ_2 to $\psi = \theta$ on Γ_1 meets Γ_1 again at $\psi = \epsilon^2\theta$, and the join of $\phi = \theta$ on Γ_0 to $\chi = \theta$ on Γ_2 meets Γ_2 again at $\chi = \epsilon^4\theta$.

So six of the ten edges of the above pentahedron are recognized, namely those containing the triads of collinear points

$$\phi = \psi = \epsilon^2\theta, \phi = \epsilon^3\theta; \qquad \psi = \chi = \epsilon^2\theta, \psi = \epsilon^4\theta;$$
$$\phi = \psi = \epsilon^3\theta, \phi = \epsilon^4\theta; \qquad \chi = \phi = \epsilon^2\theta, \chi = \epsilon^6\theta;$$
$$\phi = \psi = \epsilon^4\theta, \phi = \epsilon^5\theta; \qquad \chi = \phi = \epsilon^5\theta, \chi = \epsilon^2\theta.$$

Since $\phi = \psi(=\theta)$ is a (1, 1) correspondence, between the two quintic curves Γ_0 and Γ_1, with united points at X_0 and X_5 the chords of Γ_0 which meet Γ_1 generate a rational scroll R_0 of order [**1**, p. 16] $5 + 5 - 2 = 8$; Γ_0 is a nodal curve on R_0. Similarly for octavic scrolls R_1 and R_2 with nodal curves Γ_1 and Γ_2. These scrolls contribute 24 to the total order [**3**, pp. 205, 209] 40 of the scroll of trisecants of $\mathscr{J}$; the residue, of order 16, is provided by the transversals whose existence is quickly proved. For, when $\epsilon^7 = 1$,

$$1 - \epsilon + \epsilon(1 - \epsilon^2) + \epsilon^3(1 - \epsilon^4) = 0.$$

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Here write ϵ^j for ϵ and multiply by $\epsilon^{a(j-1)}$:

$$(6.1)\quad (1-\epsilon^j)\epsilon^{a(j-1)} + \epsilon(1-\epsilon^{2j})\epsilon^{(a+1)(j-1)} + \epsilon^3(1-\epsilon^{4j})\epsilon^{(a+3)(j-1)} = 0,$$

showing that the points $\phi = \epsilon^a\theta$, $\psi = \epsilon^{(a+1)}\theta$, $\chi = \epsilon^{(a+3)}\theta$ on Γ_0, Γ_1, Γ_2 are collinear. For $a = 2, 3$ one has two edges of the above pentahedron. Moreover there is a second relation

$$1 - \epsilon + \epsilon^{-2}(1-\epsilon^2) + \epsilon^{-6}(1-\epsilon^4) = 0$$

giving rise to

$$(6.2)\quad (1-\epsilon^j)\epsilon^{a(j-1)} + \epsilon^{-2}(1-\epsilon^{2j})\epsilon^{(a-2)(j-1)} + \epsilon^{-6}(1-\epsilon^{4j})\epsilon^{(a-6)(j-1)} = 0$$

showing that
$\phi = \epsilon^a\theta$, $\psi = \epsilon^{a-2}\theta$, $\chi = \epsilon^{a-6}\theta$ on Γ_0, Γ_1, Γ_2 are collinear. For $a = 4, 5$ one notes the remaining two edges of the pentahedron. All three Γ_i are nodal on the scroll of transversals. For instance: through the point $\psi = \epsilon^a\theta$ on Γ_1 pass two transversals, one joining it to $\chi = \epsilon^{a+2}\theta$ on Γ_2 and $\phi = \epsilon^{a-1}\theta$ on Γ_0, the other joining it to $\chi = \epsilon^{a-4}\theta$ on Γ_2 and $\phi = \epsilon^{a+2}\theta$ and Γ_0. These facts are patent on dividing (6.1) by ϵ^{j-1} and multiplying (6.2) by $\epsilon^{2(j-1)}$.

In the figure ψ_k means the point on Γ_1 whose parameter is $\epsilon^k\theta$, and similarly for Γ_2 and Γ_0.

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