



Asymptotic behaviour of large-scale solutions of Hitchin’s equations in higher rank

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ABSTRACT

Let X be a compact Riemann surface. Let (E, θ) be a stable Higgs bundle of degree 0 on X . Let $h_{\det(E)}$ denote a flat metric of the determinant bundle $\det(E)$. For any $t > 0$, there exists a unique harmonic metric h_t of $(E, t\theta)$ such that $\det(h_t) = h_{\det(E)}$. We prove that if the Higgs bundle is induced by a line bundle on the normalization of the spectral curve, then the sequence h_t is convergent to the naturally defined decoupled harmonic metric at the speed of the exponential order. We also obtain a uniform convergence for such a family of Higgs bundles.

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1. Introduction

1.1 Background

Let X be a Riemann surface. Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle of rank r on X . Let h be a Hermitian metric of E . We obtain the Chern connection ∇_h of $(E, \bar{\partial}_E, h)$ and the adjoint θ_h^\dagger of θ with respect to h . Let $R(h)$ denote the curvature of ∇_h . The metric h is called a harmonic metric of $(E, \bar{\partial}_E, \theta)$ if

$$R(h) + [\theta, \theta_h^\dagger] = 0.$$

The metric h is called a decoupled harmonic metric of $(E, \bar{\partial}_E, \theta)$ if

$$R(h) = [\theta, \theta_h^\dagger] = 0.$$

Suppose that X is compact and that $(E, \bar{\partial}_E, \theta)$ is stable of degree 0. Let $\Sigma_{E,\theta}$ denote the spectral curve of (E, θ) . We assume that $(E, \bar{\partial}_E, \theta)$ is generically regular semisimple, i.e. $D(E, \theta) = \{P \in X \mid |T_P^* X \cap \Sigma_{E,\theta}| < r\}$ is a finite subset of X .

Let $h_{\det(E)}$ be a flat metric of $\det(E)$. According to Hitchin [Hit87] and Simpson [Sim88], $(E, \bar{\partial}_E, \theta)$ has a unique harmonic metric h such that $\det(h) = h_{\det(E)}$. Because $(E, \bar{\partial}_E, t\theta)$ is stable of degree 0 for any $t > 0$, there exists a unique harmonic metric h_t of $(E, \bar{\partial}_E, t\theta)$ for any $t > 0$ such that $\det(h_t) = h_{\det(E)}$. We are interested in the behaviour of h_t as $t \rightarrow \infty$. See [GMN10], [KNP15] and [MSW16] for the motivation for this study. It is related to the geometric P=W conjecture [Sza21, Sza22]. See also helpful survey papers [Li19, Swo21].

For any simply connected relatively compact open subset K of $X \setminus D(E, \theta)$, there exists a decomposition of the Higgs bundle

$$(E, \bar{\partial}_E, \theta)|_K = \bigoplus_{i=1}^r (E_{K,i}, \bar{\partial}_{E_{K,i}}, \theta_{K,i}) \tag{1}$$

such that $\text{rank } E_{K,i} = 1$. According to [Moc16], there exist $C(K) > 0, \epsilon(K) > 0$ such that

$$|h_t(u, v)| \leq C(K) \exp(-\epsilon(K)t) |u|_{h_t} |v|_{h_t}$$

for any local sections u and v of $E_{K,i}$ and $E_{K,j}$ ($i \neq j$) in the decomposition (1). This implies that there exist $C'(K) > 0$ and $\epsilon'(K) > 0$ such that

$$|R(h_t)|_K|_{h_t} = |[\theta, \theta_{h_t}^\dagger]|_{h_t} \leq C'(K) \exp(-\epsilon'(K)t).$$

As a result, for any sequence $t(i) \rightarrow \infty$, there exist a subsequence $t'(j) \rightarrow \infty$ and gauge transformations $g_{t'(j)}$ such that the sequence $g_{t'(j)}^* h_{t'(j)}$ is convergent to a decoupled harmonic metric of $(E, \bar{\partial}_E, \theta)|_{X \setminus D(E, \theta)}$ in the C^∞ -sense locally on $X \setminus D(E, \theta)$.

We may ask the following questions under appropriate assumptions.

- Q1** Is there a sequence of gauge transformations g_t such that $g_t^* h_t$ is convergent as $t \rightarrow \infty$ locally on $X \setminus D(E, \theta)$. In other words, is the limit independent of the choice of a subsequence?
- Q2** Let $K \subset X \setminus D(E, \theta)$ be any relatively compact open subset. Then, is the order of the convergence on K dominated by $e^{-\delta(K)t}$ for some $\delta(K) > 0$?

In the rank two case, under the assumption that $\Sigma_{E, \theta}$ is smooth, Mazzeo *et al.* [MSW16] solved the both questions completely. In [Moc16], the question **Q1** was solved without assuming the smoothness of the spectral curve. In the higher rank case, Collier and Li [CL17] solved both questions for cyclic Higgs bundles. Fredrickson [Fred] studied both questions when the spectral curve is smooth, under a mild assumption on the ramification of the spectral curve over X (see Remark 1.2 and [Fred, Proposition 2.2, (2.9)]).

Remark 1.1. Chronologically, the study [Moc16] was inspired by the previous research in [CL17], [KNP15] and [MSW16].

Remark 1.2. Let $Q \in \Sigma_{E, \theta}$ be a critical point of $\pi: \Sigma_{E, \theta} \rightarrow X$. Put $P = \pi(Q)$. Let (X_P, z) be a coordinate neighbourhood around P . By using the holomorphic 1-form dz , we obtain the trivialization $T^*X_P \simeq \mathbb{C} \times X_P$. Let $\Sigma_{E, \theta, Q}$ denote the connected component of $T^*X_P \cap \Sigma_{E, \theta}$ which contains Q . We may assume that $\Sigma_{E, \theta, Q} \cap T_P^*X_P = \{Q\}$ and that $\Sigma_{E, \theta, Q}$ is holomorphically isomorphic to a disc. Let $r(Q)$ denote the degree of $\Sigma_{E, \theta, Q} \rightarrow X_P$. There exist holomorphic functions a_j ($j = 0, \dots, r(Q) - 1$) on X_P such that

$$\Sigma_{E, \theta, Q} = \left\{ (y, z) \in \mathbb{C} \times X_P \mid y^{r(Q)} + \sum_{j=0}^{r(Q)-1} a_j(z) y^j = 0 \right\}.$$

Because $T^*X_P \cap \Sigma_{E, \theta, Q} = \{Q\}$, there exists $\alpha \in \mathbb{C}$ such that

$$y^{r(Q)} + \sum_{j=0}^{r(Q)-1} a_j(0) y^j = (y - \alpha)^{r(Q)}. \quad (2)$$

The smoothness of $\Sigma_{E, \theta, Q}$ is equivalent to the condition that $a_0(z) - (-\alpha)^{r(Q)}$ has a simple 0 at $z = 0$. To study the local property of $\Sigma_{E, \theta, Q}$ around Q and θ around P , we may assume that $\alpha = 0$ by considering $\theta|_{X_P} - \alpha dz \cdot \text{id}_{E|_{X_P}}$. Moreover, we may assume that $a_{r(Q)-1}$ is constantly 0 by considering $\theta|_{X_P} - r(Q)^{-1} a_{r(Q)-1} dz \cdot \text{id}_{E|_{X_P}}$. By changing the coordinate z to $w(z)$ satisfying $w(0) = 0$ and $w(\partial_z w)^{r(Q)} = -a_0(z)$, we may assume that $a_0(z) = -z$. In general, a_j ($1 \leq j \leq r(Q) - 2$) are not constantly 0.

1.2 Main results

1.2.1 The symmetric case As a first main result, let us mention that if $(E, \bar{\partial}_E, \theta)$ has a non-degenerate symmetric pairing C , then both questions **Q1** and **Q2** are extremely easy to answer.

As explained in [LM10b], there exists a unique decoupled harmonic metric h^C of $(E, \theta)|_{X \setminus D(E, \theta)}$ which is compatible with C . By using a variant of Simpson’s main estimate and an elementary linear algebraic argument in §3.1, we can answer both questions Q1 and Q2, and the limit is h^C in this case. The following theorem is a special case of Corollary 3.5.

THEOREM 1.3. *Let K be any relatively compact open subset of $X \setminus D(E, \theta)$. Let $s(h^C, h_t)$ denote the automorphism of $E|_{X \setminus D(E, \theta)}$ determined by $h_t = h^C \cdot s(h^C, h_t)$. For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C(\ell, K)$ and $\epsilon(\ell, K)$ such that the L^2_ℓ -norm of $s(h^C, h_t) - \text{id}$ on K is dominated by $C(\ell, K) \exp(-\epsilon(\ell, K)t)$ as $t \rightarrow \infty$.*

For example, we may apply this theorem to a Higgs bundle contained in the Hitchin section because it has a canonical non-degenerate symmetric pairing.

Indeed, in Theorem 1.3, we do not need to assume that X is compact. See Theorem 3.4 and Corollary 3.5 for the precise statements. These results are also technically useful, which will be applied to the third main result (see §1.2.3 and 1.2.4).

1.2.2 The irreducible case The second main result in this paper is an affirmative answer to question Q1 in the case that the spectral curve is locally and globally irreducible.

We obtain the ideal sheaf $\mathcal{I}(\Sigma_{E, \theta}) \subset \mathcal{O}_{T^*X}$ of $\Sigma_{E, \theta}$. We say that $\Sigma_{E, \theta}$ is locally irreducible if the stalks $\mathcal{I}(\Sigma_{E, \theta})_P$ ($P \in \Sigma_{E, \theta}$) are prime ideals. It is equivalent to the condition that for any $P \in \Sigma_{E, \theta}$ the germ of $\Sigma_{E, \theta}$ at P cannot be expressed as the union of two distinct germs of non-empty complex analytic subsets. (See [GR84, §4.1].) We say that $\Sigma_{E, \theta}$ is globally irreducible if it cannot be expressed as the union of two distinct closed analytic non-empty subsets. The two conditions are independent, in general. Under the assumption that (E, θ) is stable, $\Sigma_{E, \theta}$ is locally irreducible if and only if it is globally irreducible.

THEOREM 1.4 (Corollary 7.7). *Suppose that $\Sigma_{E, \theta}$ is locally irreducible. Then, the sequence h_t is convergent to a decoupled harmonic metric h_∞ in the C^∞ -sense locally on $X \setminus D(E, \theta)$.*

See Theorem 7.5 for the more general statement.

More precisely, we canonically construct a filtered bundle $\mathcal{P}_*(\mathcal{V})$ over $\mathcal{V} = E(*D(E, \theta))$ in an algebraic way from (E, θ) such that (i) $(\mathcal{P}_*(\mathcal{V}), \theta)$ is a decomposable filtered Higgs bundle in the sense of Definition 5.10, (ii) $(\mathcal{P}_*(\mathcal{V}), \theta)$ is stable of degree 0 and (iii) $\det(\mathcal{P}_*\mathcal{V})$ equals the filtered bundle naturally induced by $\det(E)$. There exists a unique decoupled harmonic metric h_∞ of $(E, \theta)|_{X \setminus D(E, \theta)}$ adapted to $\mathcal{P}_*(\mathcal{V})$ such that $\det(h_\infty) = h_{\det(E)}$. We shall prove that the sequence h_t is convergent to h_∞ as $t \rightarrow \infty$ on $X \setminus D(E, \theta)$.

An outline of the proof is as follows. Let $P \in D(E, \theta)$. Let X_P be a small neighbourhood of P in X . By a theorem of Donaldson [Don92], there exists a harmonic metric $h_{P,t}$ of $(E, \bar{\partial}, t\theta)|_{X_P}$ such that $h_{P,t}|_{\partial X_P} = h_\infty|_{\partial X_P}$. According to Proposition 6.6, the sequence $h_{P,t}$ is convergent to $h_\infty|_{X_P \setminus \{P\}}$ in the C^∞ -sense locally on $X_P \setminus \{P\}$ as $t \rightarrow \infty$. As in [MSW16], by patching $h_{P,t}$ and h_∞ , we construct a family of Hermitian metrics \tilde{h}_t ($t > 0$) of E such that (i) $\det(\tilde{h}_t) = h_{\det(E)}$, (ii) $\lim_{t \rightarrow \infty} \tilde{h}_t = h_\infty$ on $X \setminus D(E, \theta)$ and (iii) $\int_X |R(\tilde{h}_t) + [t\theta, (t\theta)^\dagger_{\tilde{h}_t}]| \rightarrow 0$. Let $s(\tilde{h}_t, h_t)$ denote the automorphism of E determined by $h_t = \tilde{h}_t \cdot s(\tilde{h}_t, h_t)$. Then, we shall prove that $\sup_X (s(\tilde{h}_t, h_t) - \text{id}_E) \rightarrow 0$ by essentially the same argument as that in [Moc16].

Because of the assumption of the local irreducibility of $\Sigma_{E, \theta}$, it is easy to find the candidate of ‘the limiting configuration’ h_∞ . In the rank two case, the Higgs bundle $(E, \theta)|_{X_P}$ is easy to understand. There is a homogeneous wild harmonic bundle (E'_P, θ'_P, h'_P) on (\mathbb{P}^1, ∞) such that the restriction of (E'_P, θ'_P) to a neighbourhood of 0 is isomorphic to $(E, \theta)|_{X_P}$, where we consider

an S^1 -action on \mathbb{P}^1 induced by $(a, z) \mapsto a^m z$ for some $m \in \mathbb{Z}_{>0}$. (See [Moc21, §8] for homogeneity of harmonic bundles with respect to an S^1 -action.) The special case is a fiducial solution in [MSW16]. In [Moc16], the restriction of h'_P was useful in the construction of approximate solutions \tilde{h}_t . In the higher rank case, the Higgs bundle $(E, \theta)|_{X_P}$ is more complicated even under the assumption of local irreducibility. It does not seem that the approximation by a homogeneous wild harmonic bundle can work well. Therefore, we develop a way to use the solutions of the boundary-value problem in the construction of approximate solutions.

Remark 1.5. Because we also study the question **Q1** for wild harmonic bundles under a similar assumption on the spectral curve, we also study the Dirichlet problem for wild harmonic bundles (Theorem 2.8).

1.2.3 The order of convergence in the smooth case We study question **Q2** under the following additional condition.

CONDITION 1.6. Let $\rho: \tilde{\Sigma}_{E,\theta} \rightarrow \Sigma_{E,\theta}$ be the normalization. There exists a holomorphic line bundle L with an isomorphism $E \simeq (\pi \circ \rho)_* L$ such that θ is induced by the \mathcal{O}_{T^*X} -action on $\rho_* L$.

For example, this condition is satisfied if $\Sigma_{E,\theta}$ is smooth according to [BNR89, Hit87]. We shall prove the following theorem.

THEOREM 1.7 (Theorem 7.14). Suppose that Condition 1.6 is satisfied. Let $s(h_\infty, h_t)$ be the automorphism of $(E, \theta)|_{X \setminus D(E,\theta)}$ determined by $h_t = h_\infty \cdot s(h_\infty, h_t)$. Let $K \subset X \setminus D(E, \theta)$ be any relatively compact open subset. For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist $C(\ell, K) > 0$ and $\epsilon(\ell, K) > 0$ such that the following holds as $t \rightarrow \infty$:

$$\|(s(h_\infty, h_t) - \text{id})|_K\|_{L^2_\ell} \leq C(\ell, K) \exp(-\epsilon(\ell, K)t).$$

To prove Theorem 1.7, we refine the construction of \tilde{h}_t in §1.2.2. For each $P \in D(E, \theta)$, there exists a non-degenerate symmetric pairing C_P of $(E, \bar{\partial}_E, \theta)|_{X_P}$ such that $C_P|_{X_P \setminus \{P\}}$ is compatible with $h_\infty|_{X_P \setminus \{P\}}$. It is easy to see that the harmonic metric $h_{P,t}$ of $(E, \theta)|_{X_P}$ satisfying $h_{P,t}|_{\partial X_P} = h_\infty|_{\partial X_P}$ is compatible with C_P . Let $s(h_\infty, h_{P,t})$ be the automorphism of $E|_{X_P \setminus \{P\}}$ determined by $h_{P,t} = h_\infty|_{X_P \setminus \{P\}} \cdot s(h_\infty, h_{P,t})$. By the result in the symmetric case mentioned in §1.2.1, on any relatively compact open subset K of $X_P \setminus \{P\}$, $s(h_\infty, h_{P,t}) - \text{id}$ converges to 0 at a speed of the order of $e^{-\delta(K)t}$. Then, the following stronger condition is satisfied:

$$\int_X |R(\tilde{h}_t) + [t\theta, (t\theta)^\dagger]_{\tilde{h}_t}|_{\tilde{h}_t} \leq C e^{-\delta t}.$$

Then, we can obtain the estimate of $\sup |s(\tilde{h}_t, h_t) - \text{id}|$ on any relatively compact open subset in $X \setminus D(E, \theta)$. By a general argument in §4.2, we can obtain the desired estimate of the norms of $s(\tilde{h}_t, h_t) - \text{id}$ and its higher derivatives on X even around $D(E, \theta)$.

1.2.4 A family case The result and the method in §1.2.3 can be generalized to the following family case. Let $p_1: \tilde{\mathcal{X}} \rightarrow \mathcal{S}$ be a smooth proper morphism of complex manifolds such that each fiber is connected and 1-dimensional. We also assume that \mathcal{S} is connected. Let $\pi: \mathcal{S} \times T^*X \rightarrow \mathcal{S} \times X$ and $p_2: \mathcal{S} \times X \rightarrow \mathcal{S}$ denote the projections. Let $\Phi_0: \tilde{\mathcal{X}} \rightarrow \mathcal{S} \times T^*X$ be a morphism of complex manifolds such that $p_2 \circ \pi \circ \Phi_0 = p_1$. We set $\Phi_1 := \pi \circ \Phi_0: \tilde{\mathcal{X}} \rightarrow \mathcal{S} \times X$. We assume the following.

- Φ_1 is proper and finite.
- There exists a closed complex analytic hypersurface $\mathcal{D} \subset \mathcal{S} \times X$ such that (i) \mathcal{D} is finite over \mathcal{S} , (ii) the induced map $\tilde{\mathcal{X}} \setminus \Phi_1^{-1}(\mathcal{D}) \rightarrow (\mathcal{S} \times X) \setminus \mathcal{D}$ is a covering map, and (iii) Φ_0 induces an injection $\tilde{\mathcal{X}} \setminus \Phi_1^{-1}(\mathcal{D}) \rightarrow \mathcal{S} \times T^*X$.

We set $r := |\Phi_1^{-1}(P)|$ for any $P \in (\mathcal{S} \times X) \setminus \mathcal{D}$. Let $g(X)$ and \tilde{g} denote the genus of X and $p_1^{-1}(x)$ ($x \in \mathcal{S}$), respectively. We set $X_x = \{x\} \times X$ and $\mathcal{D}_x = \mathcal{D} \cap X_x$. There exists a natural isomorphism $X_x \simeq X$. We note that $\mathcal{D} \rightarrow \mathcal{S}$ is not assumed to be a covering map, and hence $|\mathcal{D}_x|$ is not necessarily constant on \mathcal{S} .

Let \mathcal{L} be a holomorphic line bundle on $\tilde{\mathcal{X}}$ such that $\deg(\mathcal{L}|_{p_1^{-1}(x)}) = \tilde{g} - rg(X) + r - 1$. We obtain a locally free $\mathcal{O}_{\mathcal{S} \times X}$ -module $\mathcal{E} = \Phi_{1*}(\mathcal{L})$. It is equipped with the morphism $\theta: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{S} \times X/\mathcal{S}}^1$ induced by the $\mathcal{O}_{\mathcal{S} \times T^*X}$ -action on $\Phi_{0*}\mathcal{L}$. For each $x \in \mathcal{S}$, we obtain the Higgs bundle $(\mathcal{E}_x, \theta_x) = (\mathcal{E}, \theta)|_{X_x}$, which is stable of degree 0.

There exists a Hermitian metric $h_{\det \mathcal{E}}$ of $\det(\mathcal{E})$ such that $h_{\det \mathcal{E}|_{X_x}}$ are flat for any $x \in \mathcal{S}$. There exist harmonic metrics $h_{t,x}$ of $(\mathcal{E}_x, t\theta_x)$ ($x \in \mathcal{S}$) such that $\det(h_{t,x}) = h_{\det \mathcal{E}|_{X_x}}$. There also exist decoupled harmonic metrics $h_{\infty,x}$ ($x \in \mathcal{S}$) of $(\mathcal{E}_x, \theta_x)|_{X_x \setminus \mathcal{D}_x}$ such that $\det(h_{\infty,x}) = h_{\det(\mathcal{E})|_{X_x \setminus \mathcal{D}_x}}$.

THEOREM 1.8 (Theorem 7.22). *Let $x_0 \in \mathcal{S}$. Let K be a relatively compact open subset of $X_{x_0} \setminus \mathcal{D}_{x_0}$. Let \mathcal{S}_0 be a neighbourhood of x_0 in \mathcal{S} such that $\mathcal{S}_0 \times K$ is relatively compact in $(\mathcal{S} \times X) \setminus \mathcal{D}$. For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist $C(\ell), \epsilon(\ell) > 0$ such that the L_ℓ^2 -norm of $s(h_{\infty,x}, h_{t,x}) - \text{id}$ ($x \in \mathcal{S}_0, t \geq 1$) on K are dominated by $C(\ell) \exp(-\epsilon(\ell)t)$.*

Remark 1.9. Note that for another Hermitian metric $h'_{\det \mathcal{E}}$ of $\det(\mathcal{E})$ such that $h'_{\det \mathcal{E}|_{X_x}}$ are flat for any $x \in \mathcal{S}$, we obtain an $\mathbb{R}_{>0}$ -valued C^∞ -function β on \mathcal{S} determined by $h'_{\det(\mathcal{E})} = \beta h_{\det(\mathcal{E})}$, and $\beta^{1/r} h_{t,x}$ (respectively $\beta^{1/r} h_{\infty,x}$) are harmonic metrics (respectively decoupled harmonic metrics) of $(\mathcal{E}_x, t\theta_x)$ (respectively $(\mathcal{E}_x, \theta_x)|_{X_x \setminus \mathcal{D}_x}$) such that $\det(\beta^{1/r} h_{t,x}) = h'_{\det \mathcal{E}|_{X_x}}$ (respectively $\det(\beta^{1/r} h_{\infty,x}) = h'_{\det(\mathcal{E})|_{X_x \setminus \mathcal{D}_x}}$). Hence, the claim of Theorem 1.8 is independent of the choice of $h_{\det \mathcal{E}}$.

Remark 1.10. We may apply Theorem 1.8 to obtain a locally uniform estimate for large-scale solutions of the Hitchin equation for a family of stable Higgs bundles of degree 0 whose spectral curves are smooth.

2. Preliminaries

2.1 Some definitions

2.1.1 Decoupled harmonic bundles Let Y be a Riemann surface. Let (V, θ) be a Higgs bundle on Y .

DEFINITION 2.1. A Hermitian metric h of V is called a decoupled harmonic metric of (V, θ) if the following conditions are satisfied.

- (A1)** h is a harmonic metric of the Higgs bundle $(V, \bar{\partial}_V, \theta)$.
- (A2)** h is flat, i.e. the Chern connection ∇_h of $(V, \bar{\partial}_V, h)$ is flat.

Such a (V, θ, h) is called a decoupled harmonic bundle.

Note that the conditions **(A1)** and **(A2)** imply that θ and θ_h^\dagger are commuting.

2.1.2 Symmetric Higgs bundles Let C be a non-degenerate symmetric product of V . It is called a non-degenerate symmetric product of the Higgs bundle (V, θ) if θ is self-adjoint with respect to C . Such a tuple (V, θ, C) is called a symmetric Higgs bundle. Let V^\vee denote the dual bundle of V . Let $\Psi_C: V \rightarrow V^\vee$ be the isomorphism induced by C . Let θ^\vee be the induced Higgs field of V^\vee . The condition is equivalent to Ψ_C inducing an isomorphism of the Higgs bundles $(V, \theta) \simeq (V^\vee, \theta^\vee)$.

A Hermitian metric h of V is said to be compatible with C if Ψ_C is isometric with respect to h and its dual Hermitian metric h^\vee of V^\vee .

2.1.3 Generically regular semisimple Higgs bundles Let $\Sigma_{V,\theta} \subset T^*Y$ denote the spectral curve of (V, θ) . We say that (V, θ) is regular semisimple if the projection $\Sigma_{V,\theta} \rightarrow Y$ is a covering map. We say that (V, θ) is generically regular semisimple if there exists a discrete subset $D \subset Y$ such that $(V, \theta)|_{Y \setminus D}$ is regular semisimple.

Let $\pi: \Sigma_{V,\theta} \rightarrow Y$ denote the projection. If (V, θ) is regular semisimple, there exists a line bundle L_V on $\Sigma_{V,\theta}$ with an isomorphism $\pi_* L_V \simeq V$ such that θ is induced by \mathcal{O}_{T^*Y} -action on L_V .

2.2 Regular semisimple case

2.2.1 Decoupled harmonic metrics Suppose that (V, θ) is regular semisimple. We consider the following condition for a Hermitian metric h of V .

(A3) For any $P \in Y$, the eigen decomposition of θ at P is orthogonal with respect to h .

Note that **(A3)** holds if and only if θ and θ_h^\dagger are commuting. The following lemma is easy to see.

LEMMA 2.2. *If two of the conditions **(A1)**, **(A2)**, and **(A3)** are satisfied for a Hermitian metric h of V , then h is a decoupled harmonic metric of (V, θ) .*

A flat metric h_{L_V} of L_V induces a Hermitian metric $\pi_*(h_{L_V})$ of V . It is easy to check that $\pi_*(h_{L_V})$ is a decoupled harmonic metric of (V, θ) .

PROPOSITION 2.3. *This procedure induces an equivalence between flat metrics of L_V and decoupled harmonic metrics of (V, θ) .*

Remark 2.4. Let (V, θ, h) be a decoupled harmonic bundle. Let $\Sigma_{V,\theta} = \coprod_{i \in \Lambda} \Sigma_{V,\theta,i}$ be the decomposition into connected components. There exists the corresponding decomposition of the Higgs bundle $(V, \theta) = \bigoplus_{i \in \Lambda} (V_i, \theta_i)$ such that $\Sigma_{V_i,\theta_i} = \Sigma_{V,\theta,i}$. Because h is a decoupled harmonic metric, the decomposition is orthogonal with respect to h . Hence, we obtain the decomposition of a decoupled harmonic bundle $(V, \theta, h) = \bigoplus (V_i, \theta_i, h_i)$.

2.2.2 Symmetric products The multiplication of $\mathcal{O}_{\Sigma_{V,\theta}}$ induces a multiplication

$$\pi_* \mathcal{O}_{\Sigma_{V,\theta}} \otimes \pi_* \mathcal{O}_{\Sigma_{V,\theta}} \longrightarrow \pi_* \mathcal{O}_{\Sigma_{V,\theta}}.$$

Any local section f of $\pi_* \mathcal{O}_{\Sigma_{V,\theta}}$ induces an endomorphism F_f of the locally free \mathcal{O}_Y -module $\pi_* \mathcal{O}_{\Sigma_{V,\theta}}$. We obtain the local section $\text{tr}(f) := \text{tr}(F_f)$ of \mathcal{O}_Y .

Let C_{L_V} be a non-degenerate symmetric pairing of L_V . We obtain the non-degenerate pairing C of $V = \pi_* L_V$:

$$V \otimes_{\mathcal{O}_Y} V \xrightarrow{\pi_* C_{L_V}} \pi_* \mathcal{O}_{\Sigma_{V,\theta}} \xrightarrow{\text{tr}} \mathcal{O}_Y. \quad (3)$$

PROPOSITION 2.5. *This procedure induces an equivalence between non-degenerate symmetric pairings of L_V and non-degenerate symmetric pairings of (V, θ) .*

We recall the following proposition.

PROPOSITION 2.6 [LM10b, Proposition 2.30]. *For any non-degenerate symmetric pairing C of (V, θ) , there exists a unique decoupled harmonic metric h^C of (V, θ) which is compatible with C .*

Indeed, let C_{L_V} be the non-degenerate symmetric pairing of L_V corresponding to C . Let h_{L_V} be the unique Hermitian metric of L_V satisfying $h_{L_V}(s, s) = |C_{L_V}(s, s)|$. We obtain the Hermitian metric h^C corresponding to h_{L_V} . Then, h^C is the decoupled harmonic metric compatible with C .

As for the converse, the following holds.

LEMMA 2.7. *Let h be a decoupled harmonic metric of (V, θ) . There exists a non-degenerate symmetric pairing of (V, θ) compatible with h if and only if the following condition is satisfied.*

- *Let h_{L_V} be the corresponding Hermitian metric of L_V , whose Chern connection is flat. Let $\Sigma_{V,\theta,i}$ be any connected component of $\Sigma_{V,\theta}$. Let $\rho_i : \pi_1(\Sigma_{V,\theta,i}) \rightarrow S^1$ be the homomorphism obtained as the monodromy of $(L_V, h_{L_V})|_{\Sigma_{V,\theta,i}}$. Then, the image of ρ_i is contained in $\{\pm 1\}$.*

Proof. There exists a non-degenerate symmetric pairing of (V, θ) compatible with h if and only if there exists a non-degenerate symmetric pairing C_{L_V} of L_V compatible with h_{L_V} . If such a C_{L_V} exists, then each ρ_i comes from an \mathbb{R} -representation. (See [LM10b, §2].) Hence, the image is contained in $\{\pm 1\}$. Conversely, if the image of each ρ_i is contained in $\{\pm 1\}$, then it is easy to construct such a pairing C_{L_V} . □

2.3 Dirichlet problem for wild harmonic bundles on curves

Let Y be a Riemann surface equipped with a Kähler metric g_Y . Let $X \subset Y$ be a connected relatively compact connected open subset whose boundary ∂X is smooth and non-empty. Let $D \subset X$ be a finite subset.

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a good filtered Higgs bundle on (Y, D) of rank r . (See [Moc21, §2.4] for the notion of good filtered Higgs bundles.) We obtain $(\det(\mathcal{P}_*\mathcal{V}), \text{tr}(\theta))$. We set $(V, \theta) = (\mathcal{V}, \theta)|_{Y \setminus D}$. Let $h_{\partial X}$ be a Hermitian metric of $V|_{\partial X}$.

THEOREM 2.8. *There exists a unique harmonic metric h of $(V, \bar{\partial}_V, \theta)|_{X \setminus D}$ such that (i) $h|_{\partial X} = h_{\partial X}$ and (ii) $\mathcal{P}_*^h(V) = \mathcal{P}_*\mathcal{V}$. (See [Moc21, §2.5] for the filtered sheaf $\mathcal{P}_*^h(V)$.)*

Proof. Let us study the case $r = 1$. There exists a Hermitian metric h_0 of V such that (i) $h_0|_{\partial X} = h_{\partial X}$, (ii) h_0 is flat around any point of D , and (iii) $\mathcal{P}_*^{h_0}(V) = \mathcal{P}_*V$. There exists a C^∞ -function $\alpha : X \rightarrow \mathbb{R}$ such that $\alpha|_{\partial X} = 0$ and that $\bar{\partial}\partial\alpha = R(h_0)$. Then, $h = e^{-\alpha}h_0$ is a flat metric of V satisfying the desired conditions. Let h' be another flat metric satisfying the same condition. We obtain the C^∞ -function s on X determined by $h' = e^s h$. Because $\Delta_{g_Y} s = 0$ and $s|_{\partial X} = 0$, we obtain that $s = 0$ on X , and hence $h' = h$.

Let us study the case $r \geq 2$. At each point $P \in D$, let (X_P, z_P) be a holomorphic coordinate neighbourhood around P such that (i) X_P is relatively compact in $X \setminus (D \setminus \{P\})$, (ii) $\bar{X}_P \cap \bar{X}_{P'} = \emptyset$ for any $P, P' \in D$, and (iii) the coordinate z_P induces $(X_P, P) \simeq (\{|z| < 1\}, 0)$. Let $h_{\det(V)}$ be a flat metric of $\det(V)$ adapted to $\det(\mathcal{P}_*V)$ such that $h_{\det(V)}|_{\partial X} = \det(h_{\partial X})$. Let h_0 be a Hermitian metric of V such that (i) $h_0|_{\partial X} = h_{\partial X}$, (ii) $\det(h_0) = h_{\det(V)}$, (iii) $\mathcal{P}_*^{h_0}(V) = \mathcal{P}_*\mathcal{V}$, and

(iv) around $P \in D$, we have $|R(h_0) + [\theta, \theta_{h_0}^\dagger]|_{h, g_Y} = O(|z_P|^{-2+\epsilon})$ for some $\epsilon > 0$. (For example, see [Moc21] for the construction.) We set $F(h_0) = R(h_0) + [\theta, \theta_{h_0}^\dagger]$. There exists $p > 1$ such that $F(h_0)$ is L^p on X . There exists an L^p_2 -function α on X such that (i) $\Delta_{g_Y}(\alpha) = |F(h_0)|_X|_{h_0, g_Y}$ and (ii) $\alpha|_{\partial X} = 0$. There exists $C_0 > 0$ such that $|\alpha| < C_0$ on X .

For $0 < \delta < 1$, we set $X_P(\delta) = \{|z_P| < \delta\}$ and $Z(\delta) := X \setminus \bigcup_{P \in D} X_P(\delta)$. We have $\partial Z(\delta) = \partial X \cup \bigcup_{P \in D} \partial X_P(\delta)$. By the Dirichlet problem for harmonic metrics [Don92, LM10a], there exists a harmonic metric $h_{Z(\delta)}$ of $(V, \bar{\partial}_V, \theta)|_{Z(\delta)}$ such that (i) $\det(h_{Z(\delta)}) = h_{\det(V)|Z(\delta)}$ and (ii) $h_{Z(\delta)}|_{\partial Z(\delta)} = h_0|_{\partial Z(\delta)}$. Let $s_{Z(\delta)}$ be the automorphism of $V|_{Z(\delta)}$ determined by $h_{Z(\delta)} = h_0|_{Z(\delta)} \cdot s_{Z(\delta)}$. According to [Sim88, Lemma 3.1], the following holds on $Z(\delta)$:

$$\Delta_{g_Y} \log \text{Tr}(s_{Z(\delta)}) \leq |F(h_0)|_{Z(\delta)}|_{h_0, g_Y}.$$

Because $\Delta_{g_Y}(\log \text{Tr}(s_{Z(\delta)}) - \alpha) \leq 0$, we obtain

$$\log \text{Tr}(s_{Z(\delta)}) \leq 2C_0 + \log r.$$

Because $\det(s_{Z(\delta)}) = 1$, there exists $C_1 > 0$, which depends only on C_0 and r , such that

$$|s_{Z(\delta)}|_{h_0} + |s_{Z(\delta)}^{-1}|_{h_0} \leq C_1.$$

Then, there exists a sequence $\delta(i) \rightarrow 0$ ($i = 1, 2, \dots$) such that the following hold (see [LM10a, Proposition 2.6]).

- The sequence $h_{Z(\delta(i))}$ is convergent in the C^∞ -sense on any relatively compact open subset of $X \setminus D$. Let h_∞ denote the limit, which is a harmonic metric.
- h_∞ is mutually bounded with h_0 . As a result, $\mathcal{P}_*^{h_\infty}(V) = \mathcal{P}_* \mathcal{V}|_X$.
- $\det(h_\infty) = h_{\det(V)}$.

Let $Z := Z(1/2)$. There exists a harmonic metric $h_{1,Z}$ of $(V, \bar{\partial}_V, \theta)|_Z$ such that (i) $\det(h_{1,Z}) = h_{\det(V)|Z}$, (ii) $h_{1,Z}|_{\partial X_P(1/2)} = h_\infty|_{\partial X_P(1/2)}$ for any $P \in D$, and (iii) $h_{1,Z}|_{\partial X} = h_0|_{\partial X}$. Let i_0 such that $\delta(i_0) < 1/2$. Let $s_{1,\delta(i)}$ be the automorphism of $V|_Z$ determined by $h_{Z(\delta(i))}|_Z = h_{1,Z} \cdot s_{1,\delta(i)}$. We obtain $\Delta_{g_Y} \log \text{Tr}(s_{1,\delta(i)}) \leq 0$ on Z . Hence, we obtain

$$\log(\text{Tr}(s_{1,\delta(i)})/r) \leq \max_{P \in D} \max_{Q \in \partial X_P(1/2)} \{\log(\text{Tr}(s_{1,\delta(i)}|_Q)/r)\}.$$

Because $\log \text{Tr}(s_{1,\delta(i)})/r \rightarrow 0$ on $\bigcup_{P \in D} \partial X_P(1/2)$, we obtain that $s_{1,\delta(i)} \rightarrow \text{id}_V$ on Z . Hence, we obtain $h_\infty|_Z = h_{1,Z}$, which implies that h_∞ satisfies the condition $h_\infty|_{\partial X} = h_0|_{\partial X}$.

Let h' be another harmonic metric satisfying the conditions (i) and (ii). Note that $\det(h') = h_{\det(V)}$. Let s be the automorphism of V determined by $h' = h \cdot s$. By [Sim88, Lemma 3.1], we have the following equality on $X \setminus D$:

$$\Delta_{g_Y} \text{Tr}(s) = -|\bar{\partial}_V(s) \cdot s^{-1/2}|_{h, g_Y}^2 - |[\theta, s]s^{-1/2}|_{h, g_Y}^2.$$

This implies that $\text{Tr}(s)$ is subharmonic on $X \setminus D$. Because $\text{Tr}(s)$ is bounded, we obtain that $\text{Tr}(s)$ is a subharmonic function on X (see [Sim90, Lemma 2.2]). We obtain $\max_X \text{Tr}(s) \leq \max_{\partial X} \text{Tr}(s) = r$. Because $\det(s) = 1$, we have $\text{Tr}(s) \geq r$. Hence, we obtain $\text{Tr}(s) = r$ on X , which implies $s = \text{id}_V$. □

COROLLARY 2.9. *Suppose that $(\mathcal{P}_* \mathcal{V}, \theta)$ is equipped with a perfect symmetric pairing C . If $h_{\partial X}$ is compatible with $C|_{\partial X}$, then h is also compatible with C .*

Proof. Let $h_{\partial X}^\vee$ be the Hermitian metric of $V|_{\partial X}^\vee$ induced by $h_{\partial X}$. Let h^\vee be the Hermitian metric of V^\vee induced by h . Then, h^\vee is the unique harmonic metric of (V^\vee, θ^\vee) satisfying $h_{\partial X}^\vee = h_{\partial X}^\vee$.

Let $\Psi_C : (V, \theta) \simeq (V^\vee, \theta^\vee)$ denote the isomorphism induced by C . Because $h_{\partial X}$ is compatible with C , $h_{\partial X} = \Psi_C^* h_{\partial X}^\vee$ holds on ∂X . By the uniqueness, we obtain $h = \Psi_C^*(h^\vee)$, i.e. h is compatible with C . □

3. Large-scale solutions in the symmetric case

3.1 Preliminary from linear algebra

3.1.1 Hermitian metrics compatible with a non-degenerate symmetric pairing Let V be an r -dimensional \mathbb{C} -vector space. The dual space is denoted by V^\vee . An \mathbb{R} -structure of V is an r -dimensional \mathbb{R} -subspace $V_\mathbb{R}$ such that the natural morphism $\mathbb{C} \otimes_\mathbb{R} V_\mathbb{R} \rightarrow V$ is an isomorphism. A positive definite symmetric bilinear form $C_\mathbb{R}$ of $V_\mathbb{R}$ induces a Hermitian metric h and a non-degenerate symmetric bilinear form C of V by $h(\alpha \otimes u, \beta \otimes v) = \alpha \bar{\beta} C_\mathbb{R}(u, v)$ and $C(\alpha \otimes u, \beta \otimes v) = \alpha \beta C_\mathbb{R}(u, v)$ for any $\alpha, \beta \in \mathbb{C}$ and $u, v \in V_\mathbb{R}$. An orthogonal decomposition $V_\mathbb{R} = \bigoplus V_{\mathbb{R},i}$ with respect to $C_\mathbb{R}$ induces a decomposition $V = \bigoplus V_{\mathbb{R},i} \otimes \mathbb{C}$ which is clearly orthogonal with respect to both h and C .

Let C be a non-degenerate symmetric pairing of V . It induces a \mathbb{C} -linear morphism $\Psi_C : V \rightarrow V^\vee$. A Hermitian metric h of V is called compatible with C if Ψ_C is isometry between (V, h) and (V^\vee, h^\vee) , where h^\vee denotes the Hermitian metric of V^\vee induced by h . If h is compatible with C , there uniquely exists an \mathbb{R} -structure $V_\mathbb{R}$ of V equipped with a positive definite symmetric bilinear form $C_\mathbb{R}$ such that (i) $V_\mathbb{R} \otimes \mathbb{C} = V$ and (ii) h and C are induced by $C_\mathbb{R}$.

3.1.2 An estimate Let C be a non-degenerate symmetric form of V . Let $V = \bigoplus_{i=1}^r V_i$ be an orthogonal decomposition with respect to C such that $\dim V_i = 1$. The following lemma is obvious.

LEMMA 3.1. *There exists a unique Hermitian metric h_0 of V such that (i) h_0 is compatible with C and (ii) the decomposition $V = \bigoplus V_i$ is orthogonal with respect to h_0 .*

For any Hermitian metric h of V compatible with C , let $s(h_0, h)$ be the automorphism of V determined by the condition $h(u, v) = h_0(s(h_0, h)u, v)$ for any $u, v \in V$. Note that $\det(s(h_0, h)) = 1$. Let $\mathcal{H}(C; \epsilon)$ be the set of Hermitian metrics h of V compatible with C such that the following holds for any $u \in V_i, v \in V_j$ ($i \neq j$):

$$|h(u, v)| \leq \epsilon |u|_h \cdot |v|_h. \tag{4}$$

LEMMA 3.2. *There exists $C > 0$, depending only on r , such that the following holds for any $0 \leq \epsilon \leq (2r)^{-1}$ and any $h \in \mathcal{H}(C; \epsilon)$:*

$$|s(h_0, h) - \text{id}_V|_{h_0} + |s(h_0, h)^{-1} - \text{id}_V|_{h_0} \leq C\epsilon.$$

Proof. Let e_i be a base of V_i such that $C(e_i, e_i) = 1$. Note that the tuple (e_1, \dots, e_r) is an orthonormal base with respect to h_0 . Let H be the matrix determined by $H_{i,j} = h(e_i, e_j)$. Then, the linear map $s(h_0, h)$ is represented by the matrix ${}^t H$ with respect to the base (e_1, \dots, e_r) . Because h is compatible with C , ${}^t H \cdot H$ is the identity matrix. We obtain

$$H_{i,i}^2 - 1 = - \sum_{\substack{1 \leq j \leq r \\ j \neq i}} H_{i,j} H_{j,i}. \tag{5}$$

By condition (4), we have $|H_{i,j}| \leq \epsilon H_{i,i}^{1/2} H_{j,j}^{1/2}$ for $i \neq j$. We obtain

$$H_{i,i}^2 - 1 \leq \epsilon \sum_{\substack{1 \leq j \leq r \\ j \neq i}} H_{i,i} H_{j,j}.$$

We set $A = \sum_{j=1}^r H_{j,j}$. We obtain

$$H_{i,i}^2 - 1 - \epsilon H_{i,i} A \leq 0. \quad (6)$$

LEMMA 3.3. The following holds: $H_{i,i} \leq \epsilon A + 1$.

Proof. Let $a > 0$. Let us consider the \mathbb{R} -valued function $f(s) = s^2 - as - 1$ ($s \in \mathbb{R}$). We set $s_{\pm} = 2^{-1}(a \pm \sqrt{a^2 + 4})$, and then we have $f(s_{\pm}) = 0$ and $s_- < s_+$. We obtain $f(s) > 0$ for any $s > s_+$. Hence, if $f(s) \leq 0$, we obtain

$$s \leq s_+ \leq 2^{-1}(a + a + 2) = a + 1. \quad (7)$$

By setting $a = \epsilon A$, we obtain the claim of the lemma from (6) and (7). \square

We obtain $A \leq \epsilon r A + r$, and hence $A \leq (1 - \epsilon r)^{-1} r \leq 2r$. By (4) and (5), we obtain

$$|H_{i,i}^2 - 1| \leq \sum_{\substack{1 \leq j \leq r \\ j \neq i}} |H_{i,j}| \cdot |H_{j,i}| \leq \epsilon \sum_{1 \leq j \leq r} H_{i,i} \cdot H_{j,j} \leq \epsilon A^2 \leq 4r^2 \epsilon.$$

Because $H_{i,i}$ are positive numbers, we obtain $|H_{i,i} - 1| \leq 4r^2 \epsilon$. We also obtain $|H_{i,j}| \leq \epsilon(1 + 4r^2 \epsilon)$. \square

3.2 Harmonic metrics compatible with a non-degenerate symmetric pairing

Let Y be any Riemann surface. Let $(V, \bar{\partial}_V, \theta)$ be a Higgs bundle on Y of rank r , which is regular semisimple. Let C be a non-degenerate symmetric pairing of (V, θ) .

For any $t > 0$, let $\text{Harm}(V, \bar{\partial}_V, t\theta, C)$ denote the set of harmonic metrics of $(V, \bar{\partial}_V, t\theta)$ compatible with C . Let g_Y be a Kähler metric of Y . For any non-negative integer ℓ and $p > 1$, and for any relatively compact open subset K of Y , we define the L_{ℓ}^p -norm $\|f\|_{L_{\ell}^p, K}$ of a section f of $\text{End}(V)$ on K by using g_Y , h^C and the Chern connection of h^C . (See Proposition 2.6 for h^C .)

THEOREM 3.4. *Let K be any relatively compact open subset of Y . There exists $t(K) > 0$ such that:*

- for any $(\ell, p) \in \mathbb{Z}_{>0} \times \mathbb{R}_{>1}$, there exist $A(\ell, p, K) > 0$ and $\epsilon(\ell, p, K) > 0$ such that, for any $h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$),

$$\|s(h^C, h) - \text{id}_E\|_{L_{\ell}^p, K} + \|s(h^C, h)^{-1} - \text{id}_E\|_{L_{\ell}^p, K} \leq A(\ell, p, K) \exp(-\epsilon(\ell, p, K)t). \quad (8)$$

Proof. To simplify the description, we set $s(h) := s(h^C, h)$ in this proof. By [Moc16, Corollary 2.6] and Lemma 3.2, there exist $A(K) > 0$, $\epsilon(K) > 0$, and $t(K) > 0$ such that the following holds for any $h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$):

$$\sup_K |s(h) - \text{id}_V|_{h^C} + \sup_K |s(h)^{-1} - \text{id}_V|_{h^C} \leq A(K) \exp(-\epsilon(K)t). \quad (9)$$

Let $R(h)$ denote the curvature of the Chern connection of $(V, \bar{\partial}_V, h)$. By [Moc16, Theorem 2.9], there exist $A^{(1)}(K) > 0$ and $\epsilon^{(1)}(K) > 0$ such that the following holds for any

$h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$):

$$\sup_K |R(h)|_{h^C, g_Y} \leq A^{(1)}(K) \exp(-\epsilon^{(1)}(K)t). \tag{10}$$

Note that $R(h) = \bar{\partial}_V(s(h)^{-1}\partial_{h^C}s(h))$.

Because $s(h)$ is self-adjoint with respect to h^C and satisfies $\det s(h) = 1$, we have $\text{Tr}(s(h) - \text{id}) \geq 0$, and $\text{Tr}(s(h) - \text{id}_V) = 0$ holds if and only if $s(h) = \text{id}_V$. The following holds on Y (see [Sim88, Lemma 3.1]):

$$\Delta_{g_Y} \text{Tr}(s(h) - \text{id}_V) = \Delta_{g_Y}(\text{Tr}(s(h)) - r) = -|s(h)^{-1/2}\partial_{h^C}s(h)|_{g_Y, h^C}^2 - |[\theta, s(h)]s(h)^{-1/2}|_{g_Y, h^C}^2.$$

Let K_1 be a relatively compact open neighbourhood of \bar{K} in Y . Let $\chi : Y \rightarrow \mathbb{R}_{\geq 0}$ be a function such that $\chi = 1$ on \bar{K} and $\chi = 0$ on $Y \setminus K_1$. We obtain the following:

$$\int_K |s(h)^{-1/2}\partial_{h^C}s(h)|_{g_Y, h^C}^2 \leq \int_Y \text{Tr}(s(h) - \text{id}_E) \cdot |\Delta_{g_Y} \chi|.$$

There exist constants $A^{(2)}(K) > 0$ and $\epsilon^{(2)}(K) > 0$ such that the following holds for any $h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$):

$$\int_K |s(h)^{-1}\partial_{h^C}s(h)|_{g_Y, h^C}^2 \leq A^{(2)}(K) \exp(-\epsilon^{(2)}(K)t). \tag{11}$$

By (10) and (11), there exist $A^{(3)}(p, K) > 0$ and $\epsilon^{(3)}(p, K) > 0$ such that the following holds for any $h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$):

$$\|s(h)^{-1}\partial_{h^C}s(h)\|_{L^p_1, K} \leq A^{(3)}(K) \exp(-\epsilon^{(3)}(p, K)t). \tag{12}$$

By (9) and (12), there exist $A^{(4)}(p, K) > 0$ and $\epsilon^{(4)}(p, K) > 0$ such that the following holds for any $h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$):

$$\|s(h) - \text{id}\|_{L^p_1, K} \leq A^{(4)}(K) \exp(-\epsilon^{(4)}(p, K)t). \tag{13}$$

By (12) and (13), there exist $A^{(5)}(p, K) > 0$ and $\epsilon^{(5)}(p, K) > 0$ such that the following holds for any $h \in \text{Harm}(V, \bar{\partial}_V, t\theta, C)$ ($t \geq t(K)$):

$$\|s(h) - \text{id}\|_{L^p_2, K} \leq A^{(5)}(K) \exp(-\epsilon^{(5)}(p, K)t). \tag{14}$$

Then, by using a standard bootstrapping argument, we obtain the claim of the theorem. \square

COROLLARY 3.5. *Let $t(i) > 0$ be any sequence such that $\lim_{i \rightarrow \infty} t(i) = \infty$. For each $t(i)$, we take any $h_{t(i)} \in \text{Harm}(V, \bar{\partial}_V, t(i)\theta, C)$. Then, the sequence $h_{t(i)}$ is convergent to h^C in the C^∞ -sense on any relatively compact open subsets of Y . The order of the convergence is estimated as in (8).*

4. Some estimates for harmonic bundles on a disc

This section is a preliminary for Theorem 7.17.

4.1 Universal boundedness of higher derivatives of Higgs fields

For any $R > 0$, we set $B(R) = \{z \in \mathbb{C} \mid |z| < R\}$. Let $R_0 > 0$. Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle on $B(R_0)$ of rank r . Let f be the endomorphism of E determined by $\theta = f dz$. Let C_0 be a constant such that

$$|\text{tr}(f^j)| < C_0 \quad (j = 1, \dots, r).$$

Let h be a harmonic metric of $(E, \bar{\partial}_E, \theta)$. Let ∇_h denote the Chern connection of h . Let $R(h)$ denote the curvature of ∇_h . We obtain the endomorphism \mathfrak{R} determined by $R(h) = \mathfrak{R} dz d\bar{z}$. Let f_h^\dagger denote the adjoint of f with respect to h . Because $R(h) + [\theta, \theta_h^\dagger] = 0$, we have $\mathfrak{R} + [f, f_h^\dagger] = 0$.

Let $g_0 = dz d\bar{z}$ denote the standard Euclidean metric. We consider the L_ℓ^p -norm of sections of $\text{End}(E)$ with respect to g_0 and h , and the derivatives with respect to ∇_h .

PROPOSITION 4.1. *Let $0 < R_1 < R_0$. For any $\ell \in \mathbb{Z}_{\geq 0}$ and $p \geq 1$, there exist $C(\ell, p)$, depending only on r, R_0, R_1 and C_0 , such that*

$$\|f|_{B(R_1)}\|_{L_\ell^p} + \|f_h^\dagger|_{B(R_1)}\|_{L_\ell^p} + \|\mathfrak{R}|_{B(R_1)}\|_{L_\ell^p} \leq C(\ell, p).$$

Proof. Let $R_2 = (R_0 + R_1)/2$. By Simpson’s main estimate [Sim90, Sim92], there exists C_1 , depending only on r, R_0, R_1 and C_0 , such that $|f|_h = |f_h^\dagger|_h \leq C_1$ on $B(R_2)$. We also obtain $|R(h)|_{h, g_0} = |\mathfrak{R}|_h \leq 2C_1^2$ on $B(R_2)$.

We recall a result due to Uhlenbeck.

THEOREM 4.2 [Uhl82, Theorem 1.3]. *Let V be a vector bundle on $B(1)$ equipped with a Hermitian metric h_V and a unitary connection ∇_V . Let $R(\nabla_V)$ denote the curvature of ∇_V . For $p \geq 1$, let $\|R(\nabla_V)\|_{L^p, h_V}$ denote the L^p -norm with respect to g_0 and h_V . Then, there exist positive constants c and κ depending only on r and p such that the following holds.*

- If $\|R(\nabla_V)\|_{L^p, h_V} \leq \kappa$, then there exists an orthonormal frame \mathbf{v} of V such that the connection form A of ∇_V with respect to \mathbf{v} satisfies (i) $d^*A = 0$ and (ii) $\|A\|_{L^p_1} \leq c\|R(\nabla_V)\|_{L^p}$.

We choose $T > 0$ such that $100T^{-1}C_1^2 < \kappa$ and $T(R_0 - R_2) > 100$. Let $\varphi_T : \mathbb{C}_w \rightarrow \mathbb{C}_z$ be defined by $\varphi_T(w) = T^{-1}z$. We consider $(\tilde{E}, \tilde{\partial}_{\tilde{E}}, \tilde{\theta}, \tilde{h}) = \varphi_T^*(E, \bar{\partial}_E, \theta, h)$ on $B(TR_0)$. Let $w_0 \in B(TR_2 - 1)$. Let $p > 2$. Let $\mathbf{v}^{(w_0)}$ be an orthonormal frame of $\tilde{E}|_{D(w_0, 1)}$ as in Theorem 4.2 for the metric \tilde{h} and the connection $\nabla_{\tilde{h}}$. Let $\mathcal{A}^{(w_0)}$ and $\mathcal{R}^{(w_0)}$ denote the connection form and the curvature form of $\nabla_{\tilde{h}}$ with respect to $\mathbf{v}^{(w_0)}$. We have

$$d^*\mathcal{A}^{(w_0)} = 0, \quad d\mathcal{A}^{(w_0)} + \mathcal{A}^{(w_0)} \wedge \mathcal{A}^{(w_0)} = \mathcal{R}^{(w_0)}, \tag{15}$$

$$\|\mathcal{A}^{(w_0)}\|_{L^p_1(D(w_0, 1))} \leq c\|\mathcal{R}\|_{L^p(D(w_0, 1))}. \tag{16}$$

Let $\Theta^{(w_0)}$ denote the matrix-valued $(1, 0)$ -form determined by $\tilde{\theta}\mathbf{v}^{(w_0)} = \mathbf{v}^{(w_0)}\Theta^{(w_0)}$. We have the decomposition $\mathcal{A}^{(w_0)} = \mathcal{A}_w^{(w_0)} dw + \mathcal{A}_{\bar{w}}^{(w_0)} d\bar{w}$. We have $\mathcal{A}_w^{(w_0)} = -\overline{t\mathcal{A}_{\bar{w}}^{(w_0)}}$. Because $\tilde{\partial}\tilde{\theta} = 0$, the following holds:

$$\partial_{\bar{w}}\Theta^{(w_0)} + [\mathcal{A}_{\bar{w}}^{(w_0)}, \Theta^{(w_0)}] = 0. \tag{17}$$

We also have

$$\mathcal{R}^{(w_0)} + [\Theta^{(w_0)}, \overline{t\Theta^{(w_0)}}] = 0. \tag{18}$$

Then, by a standard bootstrapping argument, we can prove that for any ℓ there exists $C_2(\ell)$, depending only on ℓ and r such that

$$\|\Theta^{(w_0)}\|_{L_\ell^p(D(w_0, 1/2))} + \|\mathcal{A}^{(w_0)}\|_{L_{\ell+1}^p(D(w_0, 1/2))} \leq C_2(\ell).$$

We obtain a desired estimate for $\|f|_{B(R_1)}\|_{L_\ell^p}$, which implies a desired estimate for $\|f_h^\dagger|_{B(R_1)}\|_{L_\ell^p}$. Because $\mathfrak{R} + [f, f_h^\dagger] = 0$, we also obtain a desired estimate for $\|\mathfrak{R}|_{B(R_1)}\|_{L_\ell^p}$.

4.2 Difference of two families of large-scale solutions on a disc

Let $R_0 > 0$. Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle on $B(R_0)$ of rank r . Let f be the endomorphism of E determined by $\theta = f dz$. Let C_0 be a constant such that

$$|\text{tr}(f^j)| < C_0 \quad (j = 1, \dots, \text{rank}(E)).$$

Let $h_{\det(E)}$ be a flat metric of $\det(E)$. Let $h_{0,t}$ ($t > 0$) be harmonic metrics of $(E, \bar{\partial}_E, t\theta)$ such that $\det(h_{0,t}) = h_{\det(E)}$. Let $\nabla^{0,t}$ denote the Chern connection of $(E, \bar{\partial}_E, h_{0,t})$. For any section u of $\text{End}(E)$ and for any element $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_\ell) \in \{z, \bar{z}\}^\ell$, we set

$$\nabla_{\kappa}^{0,t} u = \nabla_{\kappa_1}^{0,t} \circ \nabla_{\kappa_2}^{0,t} \circ \dots \circ \nabla_{\kappa_\ell}^{0,t}(u).$$

THEOREM 4.3. *Let $0 < R_1 < R_0$. Let $C_1, \epsilon_1 > 0$. For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C(\ell), \epsilon(\ell) > 0$, depending only on r, C_0, C_1, ϵ_1 and ℓ such that the following holds.*

- *Let $t(i) > 0$ be an increasing sequence such that $t(i) \rightarrow \infty$ as $i \rightarrow \infty$. We also assume that $t(1)(R_0 - R_1) > 100$. Let $h_{t(i)}$ be harmonic metrics of $(E, \bar{\partial}_E, t(i)\theta)$ such that $\det(h_{t(i)}) = h_{\det(E)}$. Assume the following on $B(R_0) \setminus B(R_1)$:*

$$|s(h_{0,t(i)}, h_{t(i)}) - \text{id}|_{h_{0,t(i)}} \leq C_1 \exp(-\epsilon_1 t(i)). \tag{19}$$

Then, the following holds on $B(R_1)$ for any $\kappa \in \{z, \bar{z}\}^\ell$:

$$\left| \nabla_{\kappa}^{0,t(i)}(s(h_{0,t(i)}, h_{t(i)}) - \text{id}) \right|_{h_{0,t(i)}} \leq C(\ell) \exp(-\epsilon(\ell)t(i)).$$

4.2.1 The case $\ell = 0$ To simplify the notation we set $s_i = s(h_{0,t(i)}, h_{t(i)})$. By (19), there exist $C'(0), \epsilon'(0) > 0$, depending only on r, C_1 and ϵ_1 , such that the following holds on $B(R_0) \setminus B(R_1)$:

$$\text{Tr}(s_i - \text{id}_E) \leq C'(0) \exp(-\epsilon'(0)t(i)). \tag{20}$$

By [Sim88, Lemma 3.1], we have

$$-\partial_z \partial_{\bar{z}} \text{Tr}(s_i - \text{id}_E) = -|\bar{\partial}(s_i) s_i^{-1/2}|_{h_{0,t(i)}}^2 - |[t(i)\theta, s_i] s_i^{-1/2}|_{h_{0,t(i)}}^2. \tag{21}$$

In particular, $\text{Tr}(s_i - \text{id}_E)$ is a subharmonic function on $B(R_0)$. By the maximum principle of subharmonic functions, (20) holds on $B(R_0)$. Because $\det(s_i) = 1$, we obtain the claim in the case $\ell = 0$.

4.2.2 Estimates for L^2 -norms We set $R_2 = (R_0 + R_1)/2$ and $R_3 = (R_0 + R_2)/2$. Let $\chi : \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$ be a C^∞ -function such that $\chi(z) = 1$ ($|z| \leq R_2$) and $\chi(z) = 0$ ($|z| \geq R_3$). Let $g_z = dz d\bar{z}$ be the standard Euclidean metric. By using [Sim88, Lemma 3.1], we obtain

$$\int_{B(R_2)} \left(|\bar{\partial}(s_i) s_i^{-1/2}|_{h_{0,t(i),g_z}}^2 + |[t(i)\theta, s_i] s_i^{-1/2}|_{h_{0,t(i),g_z}}^2 \right) \text{dvol}_{g_z} \leq \int_{B(R_3) \setminus B(R_2)} |\partial_z \partial_{\bar{z}}(\chi)| \cdot (\text{Tr}(s_i - \text{id}_E)) \text{dvol}_{g_z}. \tag{22}$$

Hence, there exist $C_5 > 0, \epsilon_5 > 0$ such that

$$\int_{B(R_2)} \left(|\bar{\partial}(s_i) s_i^{-1}|_{h_{0,t(i),g_z}}^2 + |s_i^{-1}[t(i)\theta, s_i]|_{h_{0,t(i),g_z}}^2 \right) \text{dvol}_{g_z} \leq C_5 \exp(-\epsilon_5 t(i)). \tag{23}$$

4.2.3 *Rescaling* To study the derivatives, for any $t > t(1)$, we define the map $\rho_t : \mathbb{C}_w \rightarrow \mathbb{C}_z$ by $\rho_t(w) = t^{-1}w$. We have $\rho_t^{-1}(B(R)) = B(tR)$. We use the standard Euclidean metric $g_w = dw \, d\bar{w}$ on \mathbb{C}_w .

We set $\tilde{E}_t = \rho_t^*(E)$ on $B(tR_0)$. It is equipped with the Higgs field $\tilde{\theta}_t = \rho_t^*(t\theta)$. We have $\tilde{\theta}_t = \rho_t^*(f) \, dw$. We have the harmonic metrics $\tilde{h}_{0,t} = \varphi_t^*(h_{0,t})$ of the Higgs bundles $(\tilde{E}_t, \bar{\partial}_{\tilde{E}_t}, \tilde{\theta}_t)$. Let $\tilde{\nabla}^{0,t}$ denote the Chern connection of $(\tilde{E}_t, \bar{\partial}_{\tilde{E}_t}, \tilde{h}_{0,t})$.

By Simpson’s main estimate, there exists $C_{10} > 0$, depending only on r and C_0 , such that the following holds on $B(tR_0 - 1)$:

$$|\tilde{\theta}_t|_{\tilde{h}_{0,t}, g_w} \leq C_{10}. \tag{24}$$

Let $R(\tilde{h}_{0,t})$ denote the curvature of the Chern connection of $(\tilde{E}_t, \bar{\partial}_{\tilde{E}_t}, \tilde{h}_{0,t})$. We have the following equality:

$$R(\tilde{h}_{0,t}) + [\tilde{\theta}_t, (\tilde{\theta}_t)^\dagger_{\tilde{h}_{0,t}}] = 0. \tag{25}$$

By (24) and (25), we have the following on $B(tR_0 - 1)$:

$$|R(\tilde{h}_{0,t})|_{\tilde{h}_{0,t}, g_w} \leq 2C_{10}^2. \tag{26}$$

We also have the universal estimates for higher derivatives of $\tilde{\theta}$ and $R(\tilde{h}_{0,t})$ as in Proposition 4.1.

4.2.4 *Estimates for higher derivatives* We also have the harmonic metrics $\tilde{h}_{t(i)} = \varphi_{t(i)}^*(h_{t(i)})$ of $(\tilde{E}_{t(i)}, \bar{\partial}_{\tilde{E}_{t(i)}}, \tilde{\theta}_{t(i)})$. Let $\tilde{s}_i = \varphi_{t(i)}^*(s_i)$. We have $\tilde{h}_{t(i)} = \tilde{h}_{0,t(i)} \tilde{s}_i$. By (23), we have

$$\int_{B(t(i)R_2)} \left(|\bar{\partial}(\tilde{s}_i) \tilde{s}_i^{-1}|_{\tilde{h}_{0,t(i)}, g_w}^2 + |\tilde{s}_i^{-1}[\tilde{\theta}_{t(i)}, \tilde{s}_i]|_{\tilde{h}_{0,t(i)}, g_w}^2 \right) \, \text{dvol}_{g_w} \leq C_5 \exp(-\epsilon_5 t(i)). \tag{27}$$

This implies

$$\int_{B(t(i)R_2)} |\tilde{s}_i^{-1} \partial_{\tilde{h}_{0,t(i)}}(\tilde{s}_i)|_{\tilde{h}_{0,t(i)}, g_w}^2 \, \text{dvol}_{g_w} \leq C_5 \exp(-\epsilon_5 t(i)). \tag{28}$$

Let $R(\tilde{h}_{t(i)})$ denote the curvature of the Chern connection of $(\tilde{E}_{t(i)}, \bar{\partial}_{\tilde{E}_{t(i)}}, \tilde{h}_{t(i)})$. We have

$$R(\tilde{h}_{t(i)}) + [\tilde{\theta}_{t(i)}, (\tilde{\theta}_{t(i)})^\dagger_{\tilde{h}_{t(i)}}] = 0.$$

Note that

$$(\tilde{\theta}_{t(i)})^\dagger_{\tilde{h}_{t(i)}} = \tilde{s}_i^{-1} (\tilde{\theta}_{t(i)})^\dagger_{\tilde{h}_{0,t(i)}} \tilde{s}_i.$$

We obtain

$$\begin{aligned} \bar{\partial}(\tilde{s}_i^{-1} \partial_{\tilde{h}_{0,t(i)}} \tilde{s}_i) &= R(\tilde{h}_{t(i)}) - R(\tilde{h}_{0,t(i)}) = - \left[\tilde{\theta}_{t(i)}, \tilde{s}_i^{-1} (\tilde{\theta}_{t(i)})^\dagger_{\tilde{h}_{0,t(i)}} \tilde{s}_i - (\tilde{\theta}_{t(i)})^\dagger_{\tilde{h}_{0,t(i)}} \right] \\ &= - \left[\tilde{\theta}_{t(i)}, \tilde{s}_i^{-1} [(\tilde{\theta}_{t(i)})^\dagger_{\tilde{h}_{0,t(i)}}], \tilde{s}_i - \text{id} \right]. \end{aligned} \tag{29}$$

Hence, there exist $C_{11} > 0$ and $\epsilon_{11} > 0$ such that the following holds on $B(t(i)R_0 - 1)$:

$$|\bar{\partial}(\tilde{s}_i^{-1} \partial_{\tilde{h}_{0,t(i)}} \tilde{s}_i)|_{\tilde{h}_{0,t(i)}, g_w} \leq C_{11} \exp(-\epsilon_{11} t(i)). \tag{30}$$

For any $w_0 \in \mathbb{C}_w$, we set $D(w_0, T) = \{|w - w_0| < T\}$. By (26), (28) and (30), for any $p \geq 2$, there exist $C_{12}(p) > 0, \epsilon_{12}(p) > 0$ such that the following holds for any $w_0 \in B(t(i)R_2 - 1)$:

$$|\tilde{s}_i^{-1} \partial_{\tilde{h}_{0,t(i)}} \tilde{s}_i|_{L^p_1(D(w_0, 2/3), \tilde{h}_{0,t(i)}, g_w)} \leq C_{12}(p) \exp(-\epsilon_{12}(p)t(i)). \tag{31}$$

By (31) and the estimate in the case $\ell = 0$, for any $p > 1$, there exist $C_{13}(p) > 0, \epsilon_{13}(p) > 0$ such that the following holds for any $w_0 \in B(t(i)R_2 - 1)$:

$$|\tilde{s}_i - \text{id}|_{L^p_2(D(w_0, 2/3), \tilde{h}_{0,t(i)}, g_w)} \leq C_{13}(p) \exp(-\epsilon_{13}(p)t(i)). \tag{32}$$

By a standard bootstrapping argument, for any $p > 1$ and $\ell \in \mathbb{Z}_{\geq 2}$, there exist $C_{14}(\ell, p) > 0, \epsilon_{14}(\ell, p) > 0$ such that the following holds for any $w_0 \in B(t(i)R_2 - 1)$:

$$|\tilde{s}_i - \text{id}|_{L^p_\ell(D(w_0, 1/2), \tilde{h}_{0,t(i)}, g_w)} \leq C_{14}(\ell, p) \exp(-\epsilon_{14}(\ell, p)t(i)). \tag{33}$$

Then, we obtain the claim of Theorem 4.3.

5. Decomposable filtered extensions

5.1 Meromorphic extensions and filtered extensions

5.1.1 *Vector bundles* Let $U \subset \mathbb{C}$ be a simply connected open neighbourhood of 0. We set $U^* = U \setminus \{0\}$. Let $\iota : U^* \rightarrow U$ denote the inclusion. Let V be a locally free \mathcal{O}_{U^*} -module of rank r . We obtain a locally free $\iota_* \mathcal{O}_{U^*}$ -module $\iota_*(V)$. A meromorphic (respectively smooth) extension of V is defined to be a locally free $\mathcal{O}_U(*0)$ -submodule (respectively \mathcal{O}_U -submodule) $\mathcal{V} \subset \iota_*(V)$ such that $\mathcal{V}|_{U^*} = V$. A filtered extension of V is defined to be a meromorphic extension \mathcal{V} equipped with a filtered bundle $\mathcal{P}_*(\mathcal{V})$ over \mathcal{V} .

Example 5.1. The $\mathcal{O}_U(*0)$ -submodule $\mathcal{O}_U(*0) \exp(z^{-1}) \subset \iota_*(\mathcal{O}_{U^*})$ is a meromorphic extension of \mathcal{O}_{U^*} , which is different from $\mathcal{O}_U(*0) \subset \iota_*(\mathcal{O}_{U^*})$.

For a positive integer ℓ , let $\varphi_\ell : \mathbb{C} \rightarrow \mathbb{C}$ be defined by $\varphi_\ell(\zeta) = \zeta^\ell$. We set $U^{(\ell)} = \varphi_\ell^{-1}(U)$ and $U^{(\ell)*} = U^{(\ell)} \setminus \{0\}$. The induced morphisms $U^{(\ell)} \rightarrow U$ and $U^{(\ell)*} \rightarrow U^*$ are also denoted by φ_ℓ . Let $\text{Gal}(\ell)$ denote the Galois group of the ramified covering φ_ℓ . Namely, we put $\text{Gal}(\ell) = \{a \in \mathbb{C}^* \mid a^\ell = 1\}$, and we consider the action of $\text{Gal}(\ell)$ on $U^{(\ell)}$ by the multiplication on the coordinate ζ . Let $\iota^{(\ell)} : U^{(\ell)*} \rightarrow U^{(\ell)}$ denote the inclusion. We set $V^{(\ell)} := \varphi_\ell^*(V)$, which is naturally $\text{Gal}(\ell)$ -equivariant. The $(\iota^{(\ell)})_* \mathcal{O}_{U^{(\ell)*}}$ -module $(\iota^{(\ell)})_*(V^{(\ell)})$ is also $\text{Gal}(\ell)$ -equivariant. A $\text{Gal}(\ell)$ -equivariant meromorphic extension of $V^{(\ell)}$ is defined to be a locally free $\mathcal{O}_{U^{(\ell)}}(*0)$ -submodule $\mathcal{V}^{(\ell)} \subset (\iota^{(\ell)})_*(V^{(\ell)})$ which is preserved by the $\text{Gal}(\ell)$ -action. A $\text{Gal}(\ell)$ -equivariant filtered extension of $V^{(\ell)}$ is defined to be a filtered bundle $\mathcal{P}_*(\mathcal{V}^{(\ell)})$ over a $\text{Gal}(\ell)$ -equivariant meromorphic extension $\mathcal{V}^{(\ell)}$ of $V^{(\ell)}$ such that each $\mathcal{P}_a \mathcal{V}^{(\ell)}$ is preserved by the $\text{Gal}(\ell)$ -action.

A meromorphic extension \mathcal{V} of V induces a $\text{Gal}(\ell)$ -equivariant meromorphic extension $\varphi_\ell^*(\mathcal{V})$ of $V^{(\ell)}$. Conversely, for any $\text{Gal}(\ell)$ -equivariant meromorphic extension $\mathcal{V}^{(\ell)}$ of $V^{(\ell)}$, we obtain the $\mathcal{O}_U(*0)$ -module $\varphi_{\ell*}(\mathcal{V}^{(\ell)})$ equipped with the $\text{Gal}(\ell)$ -action. The $\text{Gal}(\ell)$ -invariant part $\varphi_{\ell*}(\mathcal{V}^{(\ell)})^{\text{Gal}(\ell)}$ is called the descent of $\mathcal{V}^{(\ell)}$ which is a meromorphic extension of V .

LEMMA 5.2. *For a meromorphic extension \mathcal{V} of V , the descent of $\varphi_\ell^*(\mathcal{V})$ equals \mathcal{V} . For a $\text{Gal}(\ell)$ -equivariant meromorphic extension $\mathcal{V}^{(\ell)}$ of $V^{(\ell)}$, $\varphi_\ell^*(\varphi_{\ell*}(\mathcal{V}^{(\ell)})^{\text{Gal}(\ell)})$ equals $\mathcal{V}^{(\ell)}$. These procedures induce an equivalence between meromorphic extensions of V and $\text{Gal}(\ell)$ -equivariant meromorphic extensions of $V^{(\ell)}$.*

For a filtered extension $\mathcal{P}_*\mathcal{V}$ of V , we obtain a $\text{Gal}(\ell)$ -equivariant filtered extension $\mathcal{P}_*(\varphi_\ell^*(\mathcal{V}))$ over $\mathcal{V}^{(\ell)}$ as follows:

$$\mathcal{P}_a(\varphi_\ell^*\mathcal{V}) = \sum_{\substack{b \in \mathbb{R}, k \in \mathbb{Z} \\ \ell b + k \leq a}} \zeta^{-k} \varphi_\ell^*(\mathcal{P}_b\mathcal{V}) \subset \varphi_\ell^*(\mathcal{V}).$$

The filtered bundle $\mathcal{P}_*(\varphi_\ell^*(\mathcal{V}))$ is denoted by $\varphi_\ell^*(\mathcal{P}_*\mathcal{V})$.

For a $\text{Gal}(\ell)$ -equivariant filtered extension $\mathcal{P}_*(\mathcal{V}^{(\ell)})$ of $V^{(\ell)}$, we obtain a filtered extension $\mathcal{P}_*(\varphi_{\ell*}(\mathcal{V}^{(\ell)})^{\text{Gal}(\ell)})$ as follows:

$$\mathcal{P}_a(\varphi_{\ell*}(\mathcal{V}^{(\ell)})^{\text{Gal}(\ell)}) = \varphi_{\ell*}(\mathcal{P}_{\ell a}\mathcal{V}^{(\ell)})^{\text{Gal}(\ell)}.$$

It is called the descent of $\mathcal{P}_*(\mathcal{V}^{(\ell)})$.

LEMMA 5.3. *These procedures induce an equivalence between filtered extensions of V and $\text{Gal}(\ell)$ -equivariant filtered extensions of $V^{(\ell)}$.*

5.1.2 *Non-degenerate symmetric pairing* For any $b \in \mathbb{R}$, let $\mathcal{P}_*^{(b)}(\mathcal{O}_U(*0))$ denote the filtered bundle over $\mathcal{O}_U(*0)$ defined by

$$\mathcal{P}_a^{(b)}(\mathcal{O}_U(*0)) = z^{-[a-b]} \mathcal{O}_U.$$

Here, we set $[c] := \max\{n \in \mathbb{Z} \mid n \leq c\}$ for any $c \in \mathbb{R}$.

Let $C: V \otimes V \rightarrow \mathcal{O}_{U^*}$ be a holomorphic non-degenerate symmetric pairing. We say that a meromorphic extension \mathcal{V} is compatible with C if C extends to a pairing $\mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{O}_U(*0)$. We say that a filtered extension $\mathcal{P}_*\mathcal{V}$ is compatible with C if C induces $\mathcal{P}_*\mathcal{V} \otimes \mathcal{P}_*\mathcal{V} \rightarrow \mathcal{P}_*^{(0)}(\mathcal{O}_U(*0))$. We say that C is perfect with respect to $\mathcal{P}_*\mathcal{V}$ if C induces an isomorphism $\mathcal{P}_*(\mathcal{V}) \simeq \mathcal{P}_*(\mathcal{V}^\vee)$.

We have the induced symmetric pairing $\det(C)$ of $\det(V)$. If \mathcal{V} (respectively $\mathcal{P}_*\mathcal{V}$) is compatible with C , then $\det(\mathcal{V})$ (respectively $\det(\mathcal{P}_*\mathcal{V})$) is compatible with $\det(C)$.

LEMMA 5.4 [LM10b]. *Suppose that $\mathcal{P}_*\mathcal{V}$ is compatible with C . Then, C is perfect with respect to $\mathcal{P}_*(\mathcal{V})$ if and only if $\det(C)$ is perfect with respect to $\det(\mathcal{P}_*\mathcal{V})$.*

LEMMA 5.5. *There exists a unique meromorphic extension \mathcal{L} of $\det(V)$ which is compatible with $\det(C)$. There exists a unique filtered bundle $\mathcal{P}_*^C\mathcal{L}$ over \mathcal{L} such that $\det(C)$ is perfect with respect to $\mathcal{P}_*^C\mathcal{L}$.*

Proof. We may assume that U is a disc. Let v_0 be a frame of $\det(V)$ on U^* . We obtain a holomorphic function $(\det C)(v_0, v_0)$ on U^* . There exist an integer k and a holomorphic function g_1 such that $(\det C)(v_0, v_0) = z^{-k} \exp(g_1)$. We obtain a frame $v_1 = \exp(-g_1/2)v_0$ of $\det(V)$ on U^* . We set $\mathcal{L} = \mathcal{O}_U(*0)v_1 \subset \iota_*(\det V)$. Then, \mathcal{L} is compatible with $\det(C)$.

We have $\det(C)(v_1, v_1) = z^{-k}$. We define

$$\mathcal{P}_a^C(\mathcal{L}) = z^{-[a-k/2]} \mathcal{O}_U \cdot v_1.$$

Then, $\mathcal{P}_*^C\mathcal{L}$ satisfies the desired condition. The uniqueness is clear. □

We set $C^{(\ell)} := \varphi_\ell^*C$ which is a non-degenerate symmetric pairing of $V^{(\ell)}$.

LEMMA 5.6. *\mathcal{V} (respectively $\mathcal{P}_*\mathcal{V}$) is compatible with C if and only if $\varphi_\ell^*(\mathcal{V})$ (respectively $\varphi_\ell^*(\mathcal{P}_*\mathcal{V})$) is compatible with $C^{(\ell)}$. When $\mathcal{P}_*\mathcal{V}$ and C are compatible, C is perfect with respect to $\mathcal{P}_*\mathcal{V}$ if and only if $C^{(\ell)}$ is perfect with respect to $\varphi_\ell^*(\mathcal{P}_*\mathcal{V})$.*

5.1.3 *Higgs bundles* Let θ be a Higgs field of V , i.e. $\theta: V \rightarrow V \otimes \Omega_U^1$. We obtain $\iota_*(\theta): \iota_*(V) \rightarrow \iota_*(V) \otimes \Omega_U^1$. A meromorphic (respectively smooth) extension of (V, θ) is defined to be a meromorphic (respectively smooth) extension \mathcal{V} of V such that $\iota_*(\theta)(\mathcal{V}) \subset \mathcal{V} \otimes \Omega_U^1$. The induced Higgs field of \mathcal{V} is denoted by θ . A filtered extension of (V, θ) is defined to be a filtered extension $\mathcal{P}_*(\mathcal{V})$ over a meromorphic extension \mathcal{V} of (V, θ) . A filtered extension $(\mathcal{P}_*\mathcal{V}, \theta)$ is said to be regular (respectively good, unramifiedly good) if $(\mathcal{P}_*\mathcal{V}, \theta)$ is a regular (respectively good, unramifiedly good) filtered Higgs bundle. (See [Moc21, §2.4] for the notion of good filtered Higgs bundles and unramifiedly good filtered Higgs bundles.)

LEMMA 5.7. *Let f be the endomorphism of V defined by $\theta = f dz/z$. Let $a_j(z)$ be the holomorphic functions on U^* obtained as the coefficients of the characteristic polynomial $\det(\text{id}_V - f) = \sum_{j=0}^r a_j(z)t^j$.*

- A meromorphic extension of (V, θ) exists if and only if the Higgs bundle (V, θ) is wild, i.e. $a_j(z)$ are meromorphic at $z = 0$. In that case, there exists a good filtered extension.
- A regular filtered extension exists if and only if (V, θ) is tame, i.e. $a_j(z)$ are holomorphic at $z = 0$.

We obtain the Higgs field $\theta^{(\ell)}$ of $V^{(\ell)}$. The following lemmas are clear.

LEMMA 5.8. *The pull back and the descent induce an equivalence between meromorphic extensions of (V, θ) and $\text{Gal}(\ell)$ -equivariant meromorphic extensions of $(V^{(\ell)}, \theta^{(\ell)})$.*

LEMMA 5.9. *The pull back and the descent induce an equivalence between regular (respectively good) filtered extensions of (V, θ) and $\text{Gal}(\ell)$ -equivariant regular (respectively good) filtered meromorphic extensions of $(V^{(\ell)}, \theta^{(\ell)})$.*

5.2 Decomposable filtered extensions of regular semisimple Higgs bundles

5.2.1 *Decomposable filtered extensions* We continue to use the notation in §5.1.1. Let (V, θ) be a regular semisimple Higgs bundle on U^* . Assume that θ is wild. There exist $\ell \in \mathbb{Z}_{>0}$ and the decomposition

$$\varphi_\ell^*(V, \theta) = \bigoplus_{i=1}^r (V_i, \theta_i), \tag{34}$$

where $\text{rank } V_i = 1$, and $\theta_i - \theta_j$ ($i \neq j$) are nowhere vanishing on $U^{(\ell)*}$. Let \mathcal{V} be a meromorphic extension of (V, θ) . The decomposition (34) extends to

$$\varphi_\ell^*(\mathcal{V}, \theta) = \bigoplus_{i=1}^r (\mathcal{V}_i, \theta_i), \tag{35}$$

where each \mathcal{V}_i is a meromorphic extension of V_i .

DEFINITION 5.10. A filtered bundle $\mathcal{P}_*\mathcal{V}$ over \mathcal{V} is called a decomposable filtered extension of (V, θ) if the filtered bundle $\varphi_\ell^*(\mathcal{P}_*\mathcal{V})$ is compatible with the decomposition (35), i.e. the following holds for any $a \in \mathbb{R}$:

$$\mathcal{P}_a(\varphi_\ell^*\mathcal{V}) = \bigoplus_{i=1}^r \mathcal{P}_a(\varphi_\ell^*\mathcal{V}) \cap \mathcal{V}_i.$$

Such a $(\mathcal{P}_*\mathcal{V}, \theta)$ is called a decomposable filtered Higgs bundle.

The following lemma is obvious by definition.

LEMMA 5.11. Suppose that $(\mathcal{P}_*\mathcal{V}, \theta)$ is decomposable:

- $(\mathcal{P}_*\mathcal{V}, \theta)$ is a good filtered Higgs bundle;
- any decomposition $(\mathcal{V}, \theta)|_{U^*} = (V_1, \theta_1) \oplus (V_2, \theta_2)$ extends to a decomposition $(\mathcal{P}_*\mathcal{V}, \theta) = (\mathcal{P}_*\mathcal{V}_1, \theta_1) \oplus (\mathcal{P}_*\mathcal{V}_2, \theta_2)$ such that $\mathcal{V}_i|_{U^*} = V_i$.

5.2.2 *Filtered line bundles and decomposable filtered Higgs bundles* There exists the decomposition

$$(V, \theta) = \bigoplus_{k \in S} (V^{[k]}, \theta^{[k]}), \tag{36}$$

such that $\Sigma_{V^{[k]}, \theta^{[k]}}$ are connected. We set $r_k = \text{rank } V^{[k]}$. For each k , there exists the decomposition of the Higgs bundle

$$\varphi_{r_k}^*(V^{[k]}, \theta^{[k]}) = \bigoplus_{i=1}^{r_k} (V_i^{[k]}, \theta_i^{[k]}), \tag{37}$$

where $\text{rank } V_i^{[k]} = 1$, and $\theta_i^{[k]}$ are 1-forms such that $\theta_i^{[k]} - \theta_j^{[k]}$ ($i \neq j$) are nowhere vanishing on $U^{(r_k)*}$. A decomposable filtered extension $\mathcal{P}_*\mathcal{V}$ of (V, θ) induces filtered extensions $\mathcal{P}_*(\mathcal{V}_i^{[k]})$ of $(V_i^{[k]}, \theta_i^{[k]})$. Note that $\mathcal{P}_*(\mathcal{V}_i^{[k]}) = \sigma^*\mathcal{P}_*(\mathcal{V}_1^{[k]})$ for $\sigma \in \text{Gal}(r_k)$ such that $\sigma^*\theta_1^{[k]} = \theta_i^{[k]}$. Conversely, a filtered extension $\mathcal{P}_*\mathcal{V}_1^{[k]}$ of $V_1^{[k]}$ induces a $\text{Gal}(r_k)$ -equivariant filtered extension $\bigoplus_{\sigma \in \text{Gal}(r_k)} \sigma^*\mathcal{P}_*\mathcal{V}_1^{[k]}$ of $\varphi_{r_k}^*(V^{[k]}) = \bigoplus_{i=1}^{r_k} V_i^{[k]}$, and hence a decomposable filtered extension $\mathcal{P}_*\mathcal{V}^{[k]}$ of $(V^{[k]}, \theta^{[k]})$. Thus, we obtain a decomposable filtered extension $\bigoplus_{k \in S} \mathcal{P}_*\mathcal{V}^{[k]}$ of (V, θ) . Note that $\mathcal{P}_*\mathcal{V}^{[k]}$ is also obtained as $(\varphi_{r_k})_*(\mathcal{P}_*\mathcal{V}_1^{[k]})$ by the natural identification $(\varphi_{r_k})_*(V_1^{[k]}) = V^{[k]}$. The following proposition is easy to see.

PROPOSITION 5.12. This procedure induces an equivalence between decomposable filtered extensions of (V, θ) and a tuple of filtered extensions of $V_1^{[k]}$ ($k \in S$).

5.2.3 *Decomposable filtered extension determined by determinant bundles* Let \mathcal{V} be a meromorphic extension of (V, θ) . The decomposition (36) extends to a decomposition

$$(\mathcal{V}, \theta) = \bigoplus_{k \in S} (\mathcal{V}^{[k]}, \theta^{[k]}). \tag{38}$$

The decomposition (35) extends to a decomposition

$$\varphi_{r_k}^*(\mathcal{V}^{[k]}, \theta^{[k]}) = \bigoplus_{i=1}^{r_k} (\mathcal{V}_i^{[k]}, \theta_i^{[k]}). \tag{39}$$

PROPOSITION 5.13. For a tuple of filtered bundles $\mathcal{P}_*\det(\mathcal{V}^{[k]})$ over $\det(\mathcal{V}^{[k]})$, there uniquely exists a decomposable filtered bundle $\mathcal{P}_*\mathcal{V} = \bigoplus_{k \in S} \mathcal{P}_*\mathcal{V}^{[k]}$ over \mathcal{V} such that $\det(\mathcal{P}_*\mathcal{V}^{[k]}) = \mathcal{P}_*\det(\mathcal{V}^{[k]})$ for any $k \in S$. Moreover, the following hold for any $k \in S$:

- $\dim \text{Gr}_a^{\mathcal{P}_*}(\mathcal{V}^{[k]}) \leq 1$ for any $a \in \mathbb{R}$;
- let d_k be a real number such that $\text{Gr}_{d_k}^{\mathcal{P}_*}(\det(\mathcal{V}^{[k]})) \neq 0$, so then $\text{Gr}_a^{\mathcal{P}_*}(\mathcal{V}^{[k]}) \neq 0$ if and only if $r_k a - d_k \in \mathbb{Z}$ (r_k is odd) or $r_k a - d_k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ (r_k is even).
- $\text{Gr}_a^{\mathcal{P}_*}(\mathcal{V}_i^{[k]}) \neq 0$ if and only if $a - d_k \in \mathbb{Z}$ (r_k is odd) or $a - d_k \in \frac{1}{2}\mathbb{Z}$ (r_k is even).

Proof. It is enough to consider the case where $\Sigma_{V,\theta}$ is connected, i.e. $|S|=1$. We omit the superscript $[k]$ and the subscript k . We set $(V^{(r)}, \theta^{(r)}) = \varphi_r^*(V, \theta)$ and $\mathcal{V}^{(r)} = \varphi_r^*(\mathcal{V})$. There exists the following decomposition of the Higgs bundle on $U^{(r)*}$:

$$(V^{(r)}, \theta^{(r)}) = \bigoplus_{i=1}^r (V_{\beta(i)}, \beta(i) d\zeta). \tag{40}$$

Here, $\beta(i)$ are meromorphic functions on $(U^{(r)}, 0)$ such that $\beta(i) - \beta(j)$ ($i \neq j$) are nowhere vanishing on $U^{(r)*}$. The decomposition (40) extends to a decomposition on $U^{(r)}$:

$$(\mathcal{V}^{(r)}, \theta^{(r)}) = \bigoplus_{i=1}^r (\mathcal{V}_{\beta(i)}, \beta(i) d\zeta). \tag{41}$$

We have $\sigma^* \mathcal{V}_{\beta(i)} = \mathcal{V}_{\sigma^*(\beta(i))}$ for any $\sigma \in \text{Gal}(r)$.

Let $v_{\beta(1)}$ be a frame of $\mathcal{V}_{\beta(1)}$. We obtain frames $v_{\sigma^*(\beta(1))} = \sigma^* v_{\beta(1)}$ of $\mathcal{V}_{\sigma^*\beta(1)}$, and the tuple $v_{\beta(1)}, \dots, v_{\beta(r)}$ is a frame of $\mathcal{V}^{(r)}$. We set

$$b := \min \left\{ c \in \mathbb{R} \mid v_{\beta(1)} \wedge \dots \wedge v_{\beta(r)} \in \mathcal{P}_c(\varphi_r^* \det \mathcal{V}) \right\}.$$

We define the filtered bundles $\mathcal{P}_*^*(\mathcal{V}_{\beta(i)})$ as follows:

$$\mathcal{P}_a^*(\mathcal{V}_{\beta(i)}) = \zeta^{-[a-b/r]} \mathcal{O}_{U^{(r)}} v_{\beta(i)}.$$

They are independent of the choice of $v_{\beta(1)}$. We set $\mathcal{P}_*^*(\mathcal{V}^{(r)}) = \bigoplus \mathcal{P}_*^*(\mathcal{V}_{\beta(i)})$, which is $\text{Gal}(r)$ -equivariant. As the descent, we obtain a filtered bundle $\mathcal{P}_*^*(\mathcal{V})$ over \mathcal{V} , which satisfies the desired condition. The uniqueness is clear. By the construction, $(\mathcal{P}_*^*(\mathcal{V}), \theta)$ is clearly a good filtered Higgs bundle.

Let τ be a frame of $\mathcal{P}_d(\det \mathcal{V})$. There exist an integer m and a nowhere vanishing holomorphic function g on $U^{(r)}$ such that

$$v_{\beta(1)} \wedge \dots \wedge v_{\beta(r)} = \zeta^m g(\zeta) \varphi_r^* \tau.$$

Because a generator σ_0 of $\text{Gal}(r)$ acts on the set $\{\beta(i)\}$ in a cyclic way, we have $\sigma_0^*(v_{\beta(1)} \wedge \dots \wedge v_{\beta(r)}) = (-1)^{(r-1)} v_{\beta(1)} \wedge \dots \wedge v_{\beta(r)}$. Hence, we obtain that $\sigma_0^*(\zeta^m) = (-1)^{r-1} \zeta^m$ and $\sigma_0^* g = g$. This implies that $m/r \in \mathbb{Z}$ if r is odd or that $m/r \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ if r is even. By our choice of b , we have $b = -m + rd$. It is easy to see that $\text{Gr}_c^{\mathcal{P}_*^*}(\mathcal{V}_{\beta(i)}) \neq 0$ if and only if $c - b/r \in \mathbb{Z}$. For each $p \in \mathbb{Z}$, we have the $\text{Gal}(r)$ -invariant sections $\sum_{\sigma \in \text{Gal}(r)} \sigma^*(\zeta^p v_{\beta(1)})$ of $\mathcal{V}^{(r)}$ which induces a section of $\mathcal{P}_{b/r^2 - p/r}^*(\mathcal{V})$. Moreover, it induces a frame of $\text{Gr}_{b/r^2 - p/r}^{\mathcal{P}_*^*}(\mathcal{V})$. Hence, it is easy to see that $\text{Gr}_a^{\mathcal{P}_*^*}(\mathcal{V}) \neq 0$ if and only if $ra - b/r \in \mathbb{Z}$, and that $\dim \text{Gr}_a^{\mathcal{P}_*^*}(\mathcal{V}) \leq 1$. Then, we obtain the last two claims. \square

5.3 Non-degenerate pairings and decomposable filtered extensions

5.3.1 *Non-degenerate symmetric pairings of regular semisimple Higgs bundles* We continue to use the notation in §5.2. Let C be a non-degenerate symmetric pairing of (V, θ) . For any $z_0 \in U^*$, the eigen decomposition of θ at z_0 is orthogonal with respect to C . The decomposition (34) is orthogonal with respect to $\varphi_\ell^* C$.

The decomposition (36) is orthogonal with respect to C . Let $C^{[k]}$ denote the restriction of C to $V^{[k]}$. The decomposition (37) is orthogonal with respect to $\varphi_{r_k}^* C^{[k]}$. Let $C_i^{[k]}$ denote the induced symmetric pairing of $V_i^{[k]}$. We have $C_i^{[k]} = \sigma^* C_1^{[k]}$ for $\sigma \in \text{Gal}(r_k)$ such that $\sigma^* \theta_1^{[k]} =$

$\theta_i^{[k]}$. Conversely, for any non-degenerate symmetric pairings $C_1^{[k]}$ ($k \in S$), we obtain a $\text{Gal}(r_k)$ -equivariant non-degenerate symmetric pairing $\bigoplus_{\sigma \in \text{Gal}(r_k)} \sigma^* C_1^{[k]}$ of $\varphi_{r_k}^* V^{[k]}$. It induces a non-degenerate symmetric pairing $C^{[k]}$ of $(V^{[k]}, \theta^{[k]})$, and a non-degenerate pairing $\bigoplus C^{[k]}$ of (V, θ) . The following lemma is a special case of Proposition 2.5.

LEMMA 5.14. *These procedures induce an equivalence between a non-degenerate symmetric pairing C of (V, θ) and a tuple $(C_1^{[k]})_{k \in S}$ of non-degenerate symmetric pairings of $V_1^{[k]}$.*

5.3.2 *Canonical decomposable filtered extensions in the symmetric case* We recall the following [LM10b, §4.1].

PROPOSITION 5.15. *For a non-degenerate symmetric pairing C of (V, θ) , there uniquely exists a meromorphic extension \mathcal{V}^C of (V, θ) compatible with C . Moreover, there uniquely exists a filtered bundle $\mathcal{P}_*^C(\mathcal{V}^C)$ over \mathcal{V}^C satisfying the following conditions:*

- C is perfect with respect to $\mathcal{P}_*^C(\mathcal{V}^C)$;
- $\mathcal{P}_*^C(\mathcal{V}^C)$ is a decomposable filtered extension of (V, θ) .

We have the non-degenerate symmetric pairing $C_1^{[k]}$ ($k \in S$) of $V_1^{[k]}$ corresponding to C as in Lemma 5.14. There exist unique filtered extensions $\mathcal{P}_*^C((\mathcal{V}_1^{[k]})^C)$ of $V_1^{[k]}$ compatible with $C_1^{[k]}$ as in Lemma 5.5. The decomposable filtered extension $\mathcal{P}_*^C(\mathcal{V}^C)$ of (V, θ) corresponds to the tuple $\mathcal{P}_*^C((\mathcal{V}_1^{[k]})^C)$ ($k \in S$) (Proposition 5.12).

5.3.3 *Comparison of two canonical extensions* Let C be a non-degenerate symmetric pairing of (V, θ) . We have the unique filtered extension $\mathcal{P}_*^C \mathcal{V}^C$ of (V, θ) compatible with C . We have the decomposition

$$(\mathcal{V}^C, \theta) = \bigoplus_{k \in S} ((\mathcal{V}^C)^{[k]}, \theta^{[k]}).$$

Let $\det(C^{[k]})$ denote the induced symmetric pairings of $(\det(V^{[k]}), \text{tr}(\theta^{[k]}))$. Note that $\det((\mathcal{V}^C)^{[k]})$ is a meromorphic extension of $(\det(V^{[k]}), \text{tr}(\theta^{[k]}))$ compatible with $\det(C^{[k]})$. We have the unique filtered extension $\mathcal{P}_*^C \det((\mathcal{V}^C)^{[k]})$ of $(\det(V^{[k]}), \text{tr}(\theta^{[k]}))$ compatible with $\det(C^{[k]})$. We obtain the decomposable filtered Higgs bundle $(\mathcal{P}_*^C(\mathcal{V}^C), \theta)$ determined by the tuple $\mathcal{P}_*^C \det((\mathcal{V}^C)^{[k]})$ as in Proposition 5.13.

PROPOSITION 5.16. The following holds: $\mathcal{P}_*^C(\mathcal{V}^C) = \mathcal{P}_*(\mathcal{V}^C)$.

Proof. The filtered Higgs bundle $(\mathcal{P}_*^C(\mathcal{V}^C), \theta)$ is decomposable. We have $\det(\mathcal{P}_*^C(\mathcal{V}^C)^{[k]}) = \mathcal{P}_*^C \det((\mathcal{V}^C)^{[k]}) = \det \mathcal{P}_*(\mathcal{V}^C)^{[k]}$. Hence, we obtain $\mathcal{P}_*^C(\mathcal{V}^C) = \mathcal{P}_*(\mathcal{V}^C)$ by the uniqueness. \square

COROLLARY 5.17. *Let $\mathcal{P}_*(\mathcal{V}^C)$ be a filtered extension of (V, θ) satisfying the following conditions:*

- C is perfect with respect to $\mathcal{P}_*(\mathcal{V}^C)$;
- $\mathcal{P}_* \mathcal{V}^C = \bigoplus_{k \in S} \mathcal{P}_*((\mathcal{V}^C)^{[k]})$.

Let $\mathcal{P}_(\mathcal{V})$ be the decomposable filtered extension of (V, θ) determined by the filtered bundles $\det(\mathcal{P}_*((\mathcal{V}^C)^{[k]}))$ ($k \in S$). Then, $\mathcal{P}_*^C(\mathcal{V}^C) = \mathcal{P}_*(\mathcal{V}^C)$.*

Proof. It follows from $\det(\mathcal{P}_*((\mathcal{V}^C)^{[k]})) = \mathcal{P}_*^C \det((\mathcal{V}^C)^{[k]})$. □

Let C and C' be non-degenerate symmetric pairings of (V, θ) . Let $C^{[k]}$ and $C'^{[k]}$ ($k \in S$) be the induced non-degenerate symmetric pairings of $(V^{[k]}, \theta^{[k]})$. We have the corresponding symmetric pairings $C_1^{[k]}$ and $C_1'^{[k]}$ of $V_1^{[k]}$.

COROLLARY 5.18. *Suppose that $\det(C^{[k]}) = \det(C'^{[k]})$ for any $k \in S$. Then, $\mathcal{V}^C = \mathcal{V}^{C'}$ holds if and only if $\mathcal{P}_*^C \mathcal{V}^C = \mathcal{P}_*^{C'} \mathcal{V}^{C'}$ holds. It is equivalent to the condition that there exist holomorphic functions $\gamma_1^{[k]}$ ($k \in S$) on $U^{(r_k)}$ satisfying $C_1'^{[k]} = \exp(\gamma_1^{[k]})C_1^{[k]}$ and $\sum_{\sigma \in \text{Gal}(r_k)} \sigma^* \gamma_1^{[k]} = 0$.*

Proof. The ‘if’ part of the claim is clear. The ‘only if’ part of the claim follows from Corollary 5.17. □

5.4 Prolongation of decoupled harmonic bundles

Let (V, θ) be a Higgs bundle on U^* , which is regular semisimple and wild. Let h be a decoupled harmonic metric of (V, θ) . We obtain the good filtered Higgs bundle $(\mathcal{P}_*^h V, \theta)$ on $(U, 0)$.

LEMMA 5.19. *The filtered Higgs bundle $(\mathcal{P}_*^h V, \theta)$ is decomposable.*

Proof. Because the decomposition (34) is orthogonal with respect to $\varphi_\ell^{-1}(h)$, the claim is clear. □

Remark 5.20. If h is a decoupled harmonic metric of (V, θ) , then we obtain that $\mathcal{P}_*^h V$ is a filtered bundle without assuming θ is wild.

We have the decomposition $\mathcal{P}_*^h(V) = \bigoplus_{k \in S} \mathcal{P}_*^h(V^{[k]})$. We obtain the filtered extensions $\det(\mathcal{P}_*^h V^{[k]}) = \mathcal{P}_*^{\det(h)} \det(V^{[k]})$ of $\det(V^{[k]})$. We have the filtered bundle $\mathcal{P}_*^*(\mathcal{V})$ over $\mathcal{V} = \mathcal{P}_*^h V$ determined by $\det(\mathcal{P}_*^h V^{[k]})$ as in Proposition 5.13.

LEMMA 5.21. *We have $\mathcal{P}_*^h(V) = \mathcal{P}_*^*(\mathcal{V})$.*

Proof. This follows from the uniqueness of the decomposable filtered extension $\mathcal{P}_*^*(\mathcal{V})$ of (V, θ) satisfying the condition in Proposition 5.13. □

The decomposition (36) is orthogonal with respect to h . Let $h^{[k]}$ denote the induced decoupled harmonic metric of $(V^{[k]}, \theta^{[k]})$ ($k \in S$). The decomposition (37) is orthogonal with respect to $\varphi_{r_k}^*(h^{[k]})$. Let $h_1^{[k]}$ denote the induced flat metric of $V_1^{[k]}$.

Let h' be another decoupled harmonic metric of (V, θ) . Similarly, we obtain the induced decomposable harmonic metric $h'^{[k]}$ of $(V^{[k]}, \theta^{[k]})$ and the induced flat metric $h_1'^{[k]}$ of $V_1^{[k]}$.

COROLLARY 5.22. *Suppose that $\det(h^{[k]}) = \det(h'^{[k]})$ for any $k \in S$. Then, $\mathcal{P}_*^h V = \mathcal{P}_*^{h'} V$ hold if and only if $\mathcal{P}_*^h(V) = \mathcal{P}_*^{h'}(V)$ holds. This is equivalent to the condition that there uniquely exist holomorphic functions $\gamma_1^{[k]}$ ($k \in S$) on $U^{(r_k)}$ such that (i) $h_1'^{[k]} = \exp(2\text{Re}(\gamma_1^{[k]}))h_1^{[k]}$ and (ii) $\sum_{\sigma \in \text{Gal}(r_k)} \sigma^* \gamma_1^{[k]} = 0$.*

Proof. The ‘if’ part of the claim is clear. The ‘only if’ part of the claim follows from Lemma 5.21. The second claim is clear. □

5.5 Decoupled harmonic metrics and symmetric products

5.5.1 *Comparison of extensions* Let (V, θ) be a Higgs bundle on U^* which is regular semisimple and wild. Let C be a non-degenerate symmetric pairing of (V, θ) . There exists a unique decoupled harmonic metric h^C of (V, θ) compatible with C .

LEMMA 5.23. We have $\mathcal{P}_*^{h^C}(V) = \mathcal{P}_*^C(\mathcal{V}^C)$.

Proof. By the pull back via φ_ℓ , it is enough to consider the case $\text{rank } V = 1$, which is easy to check. \square

5.5.2 *Symmetric products compatible with a decoupled harmonic metric* The following lemma is a special case of Lemma 2.7.

LEMMA 5.24. Suppose $\text{rank } V = 1$. Let h be a flat metric of V . There exists a holomorphic non-degenerate symmetric product C of V which is compatible with h if and only if the monodromy of the Chern connection of h is 1 or -1 . It is equivalent to the condition

$$\{d \in \mathbb{R} \mid \text{Gr}_d^{\mathcal{P}_*^h}(V) \neq 0\} \subset \frac{1}{2}\mathbb{Z}.$$

If C' is another non-degenerate symmetric pairing of V which is compatible with h , there exists a non-zero constant α such that (i) $C' = \alpha C$ and (ii) $|\alpha| = 1$.

PROPOSITION 5.25. Let h be a decoupled harmonic metric of (V, θ) . Suppose that there exist non-degenerate symmetric products $C_{\det(V^{[k]})}$ ($k \in S$) of $\det(V^{[k]})$ which are compatible with $\det(h^{[k]})$.

- There exists a non-degenerate symmetric pairing C of (V, θ) such that (i) C is compatible with h and (ii) $\det(C^{[k]}) = C_{\det(V^{[k]})}$.
- If C' is another non-degenerate symmetric pairing of (V, θ) satisfying the above conditions (i) and (ii), then there exist r_k -roots μ_k of 1 such that $C'^{[k]} = \mu_k C^{[k]}$.

Proof. Let $h_1^{[k]}$ ($k \in S$) be the induced flat metrics of $V_1^{[k]}$. By Lemma 2.7, Proposition 5.13 and Lemma 5.24, there exist non-degenerate symmetric products $C_1^{[k]}$ of $V_1^{[k]}$ compatible with $h_1^{[k]}$ for any $k \in S$. They induce non-degenerate symmetric products $C^{[k]}$ of $(V^{[k]}, \theta^{[k]})$. Because $\det(C^{[k]})$ is compatible with $\det h^{[k]}$, there exist constants α_k such that $\det(C^{[k]}) = \alpha_k \cdot C_{\det(V^{[k]})}$ and $|\alpha_k| = 1$. By replacing $C^{[k]}$ with $\alpha_k^{1/r_k} C^{[k]}$, we obtain the first claim. The second claim is also clear. \square

5.5.3 *Existence* Let \mathcal{V} be a meromorphic extension of (V, θ) .

LEMMA 5.26. Let $C_{\det(V^{[k]})}$ be non-degenerate symmetric pairings of $\det(V^{[k]})$ such that $\det(\mathcal{V}^{[k]})$ is compatible with $C_{\det(V^{[k]})}$. Then, there exists a non-degenerate symmetric pairing C of (V, θ) such that (i) $\det(C^{[k]}) = C_{\det(\mathcal{V}^{[k]})}$ and (ii) $\mathcal{V}^C = \mathcal{V}$.

Proof. It is enough to consider the case $|S| = 1$. We omit the superscript $[k]$ and the subscript k . We use the notation in the proof of Proposition 5.13. Let $C'_{1, \beta(1)}$ be a non-degenerate symmetric pairing of $\mathcal{V}_{\beta(1)}$. We obtain a $\text{Gal}(r)$ -invariant non-degenerate symmetric pairing $\bigoplus_{\sigma \in \text{Gal}(r)} \sigma^* C'_{1, \beta(1)}$ of $\mathcal{V}^{(r)}$. It induces a non-degenerate symmetric pairing C' of \mathcal{V} .

From $C''_{1,\beta(1)} = \zeta C'_{1,\beta(1)}$, we obtain another non-degenerate symmetric pairing C'' , for which we have $\det(C'') = z \det(C')$.

Let α be the holomorphic function on U^* determined by $\det(C') = \alpha \cdot C_{\det(V)}$. By the above consideration, we may assume that α induces a nowhere vanishing holomorphic function on U . By choosing an r th root $\alpha^{1/r}$ of α , and by setting $C = \alpha^{-1/r} C'$, we obtain a desired non-degenerate pairing C . □

We can prove the following lemma similarly.

LEMMA 5.27. *Let $h_{\det(V^{[k]}}$ be flat metrics of $\det(V^{[k]})$ such that $\det(\mathcal{V}^{[k]}) = \mathcal{P}^{h_{\det(V^{[k]})}}(\det(V^{[k]}))$. There exists a decoupled harmonic metric h of (V, θ) such that (i) $\det(h^{[k]}) = h_{\det(V^{[k]})}$ and (ii) $\mathcal{P}^h(V) = \mathcal{V}$.*

5.6 Global case

5.6.1 Meromorphic extensions and filtered extensions Let Y be a Riemann surface with a discrete subset D . Let $\iota_{Y \setminus D}: Y \setminus D \rightarrow Y$ denote the inclusion. For a holomorphic vector bundle V on $Y \setminus D$, a meromorphic extension of V to (Y, D) is defined to be a locally free $\mathcal{O}_Y(*D)$ -submodule \mathcal{V} of $(\iota_{Y \setminus D})_* V$ such that $\mathcal{V}|_{Y \setminus D} = V$. A filtered extension of V to (Y, D) is a filtered bundle $\mathcal{P}_* \mathcal{V}$ over a meromorphic extension \mathcal{V} of V . We use similar terminology for non-degenerate symmetric pairings and Higgs bundles in this situation.

5.6.2 Decomposable filtered extensions Let (V, θ) be a regular semisimple Higgs bundle on $Y \setminus D$ which is wild along D . Let $\mathcal{P}_* \mathcal{V}$ be a filtered extension of (V, θ) to (Y, D) .

DEFINITION 5.28. $\mathcal{P}_* \mathcal{V}$ is called a decomposable filtered extension of (V, θ) if the restriction to a neighbourhood of any $P \in D$ is decomposable.

The following lemma is clear.

LEMMA 5.29. *A decomposable filtered Higgs bundle $(\mathcal{P}_* \mathcal{V}, \theta)$ is a good filtered Higgs bundle. Any decomposition $(\mathcal{V}, \theta)|_{Y \setminus D} = (V_1, \theta_1) \oplus (V_2, \theta_2)$ extends to a decomposition $(\mathcal{P}_* \mathcal{V}, \theta) = (\mathcal{P}_* \mathcal{V}_1, \theta_1) \oplus (\mathcal{P}_* \mathcal{V}_2, \theta_2)$.*

We have the line bundle L_V on $\Sigma_{V,\theta}$ corresponding to (V, θ) . Let $\mathbb{P}(T^*Y)$ be the projective completion of T^*Y . Let Z be the closure of $\Sigma_{V,\theta} \subset T^*(Y \setminus D)$ in $\mathbb{P}(T^*Y)$. Let $\tilde{\Sigma}_{V,\theta} \rightarrow Z$ denote the normalization. We may naturally regard $\tilde{\Sigma}_{V,\theta}$ as a partial compactification of $\Sigma_{V,\theta}$. We set $\tilde{D} = \tilde{\Sigma}_{V,\theta} \setminus \Sigma_{V,\theta}$. The morphism $\pi: \Sigma_{V,\theta} \rightarrow Y \setminus D$ uniquely extends to a morphism $\tilde{\pi}: (\tilde{\Sigma}_{V,\theta}, \tilde{D}) \rightarrow (Y, D)$. From a meromorphic extension \mathcal{L}_V of L_V to $(\tilde{\Sigma}_{V,\theta}, \tilde{D})$, we obtain a meromorphic extension $\tilde{\pi}_*(\mathcal{L}_V)$ of (V, θ) to (Y, D) . From a filtered extension $\mathcal{P}_* \mathcal{L}_V$ of L_V to $(\tilde{\Sigma}_{V,\theta}, \tilde{D})$, we obtain a decomposable filtered extension $\tilde{\pi}_*(\mathcal{P}_* \mathcal{L}_V)$ of (V, θ) to (Y, D) . The following proposition is a reformulation of Proposition 5.12.

PROPOSITION 5.30. *The above procedure induces an equivalence between filtered extensions (respectively meromorphic extensions) of L_V to $(\tilde{\Sigma}_{V,\theta}, \tilde{D})$ and decomposable filtered extensions (respectively meromorphic extensions) of (V, θ) to (Y, D) .*

5.6.3 Symmetric products Let C be a non-degenerate symmetric pairing of (V, θ) . We restate Proposition 5.15 in the global setting.

PROPOSITION 5.31. *For a non-degenerate symmetric pairing C of (V, θ) , there uniquely exists a meromorphic extension \mathcal{V}^C of (V, θ) to (Y, D) compatible with C . Moreover, there uniquely exists a filtered bundle $\mathcal{P}_*^C(\mathcal{V}^C)$ over \mathcal{V}^C satisfying the following conditions;*

- C is perfect with respect to $\mathcal{P}_*^C(\mathcal{V}^C)$;
- $\mathcal{P}_*^C(\mathcal{V}^C)$ is a decomposable filtered extension of (V, θ) .

The decomposable filtered extension $\mathcal{P}_*^C(\mathcal{V}^C)$ is described as follows. Let C_0 be the non-degenerate symmetric pairing of L_V corresponding to C . There exists the unique filtered extension $\mathcal{P}_*^{C_0}(\mathcal{L}_V^{C_0})$ of L_V to $(\tilde{\Sigma}_{V,\theta}, \tilde{D})$. Then, $\mathcal{P}_*^C(\mathcal{V}^C) = \tilde{\pi}_*(\mathcal{P}_*^{C_0}(\mathcal{L}_V^{C_0}))$.

5.6.4 *Decoupled harmonic bundles* Let h be a decoupled harmonic metric of (V, θ) . We obtain the good filtered Higgs bundle $(\mathcal{P}_*^h V, \theta)$ on (Y, D) . We obtain the following lemma from Lemma 5.19.

LEMMA 5.32. *The filtered Higgs bundle $(\mathcal{P}_*^h V, \theta)$ is decomposable.*

We obtain the following lemma from Lemma 5.23.

LEMMA 5.33. *For a non-degenerate symmetric pairing C of (V, θ) , we have $\mathcal{P}_*^{h^C}(V) = \mathcal{P}_*^C(\mathcal{V}^C)$.*

5.7 Kobayashi–Hitchin correspondence for decoupled harmonic bundles

Let X be a compact Riemann surface. Let $D \subset X$ be a finite subset. Let (V, θ) be a regular semisimple Higgs bundle on $X \setminus D$, which is wild along D . For any decoupled harmonic metric h of (V, θ) we obtain a good filtered Higgs bundle $(\mathcal{P}_* \mathcal{V}, \theta)$ on (X, D) which is polystable of degree 0. According to Lemma 5.32, it is decomposable.

Conversely, let $(\mathcal{P}_* \mathcal{V}, \theta)$ be a polystable decomposable filtered Higgs bundle of degree 0 on (X, D) such that $(V, \theta) = (\mathcal{V}, \theta)|_{X \setminus D}$ is regular semisimple. There exists a harmonic metric h of (V, θ) adapted to $\mathcal{P}_* \mathcal{V}$ by [BB04, Moc21, Sim90].

PROPOSITION 5.34. *The harmonic metric h is a decoupled harmonic metric.*

Proof. It is enough to consider the case where $(\mathcal{P}_* \mathcal{V}, \theta)$ is stable. By Lemma 5.29, $\Sigma_{V,\theta}$ is connected. Let $\mathbb{P}(T^*X)$ denote the projective completion of T^*X . Let Z denote the closure of $\Sigma_{V,\theta}$ in $\mathbb{P}(T^*X)$. Let $\tilde{\Sigma}_{V,\theta} \rightarrow Z$ denote the normalization. Let $\rho: \tilde{\Sigma}_{V,\theta} \rightarrow X$ denote the induced morphism. We set $\tilde{D} = \rho^{-1}(D)$. Let L_V be the line bundle on $\Sigma_{V,\theta}$ corresponding to (V, θ) . Because $\mathcal{P}_* \mathcal{V}$ is a decomposable filtered extension of (V, θ) , there exists the corresponding filtered extension $\mathcal{P}_* \mathcal{L}_V$ of L_V on $(\tilde{\Sigma}_{V,\theta}, \tilde{D})$. We have $\rho_*(\mathcal{P}_* \mathcal{L}_V) = \mathcal{P}_* \mathcal{V}$. By Proposition 5.35 below, we have $\deg(\mathcal{P}_* \mathcal{L}_V) = \deg(\mathcal{P}_* \mathcal{V}) = 0$. There exists a flat metric h_{L_V} of L_V adapted to $\mathcal{P}_* \mathcal{L}_V$. We obtain a decoupled harmonic metric h_1 of (V, θ) corresponding to h_{L_V} , which is adapted to $\mathcal{P}_* \mathcal{V}$. By the stability, there exists a positive constant $h = ah_1$, and hence h is also a decoupled harmonic metric. \square

5.7.1 *Degree* Let $\rho: X_1 \rightarrow X_2$ be a non-constant morphism of compact Riemann surfaces. Let $D_2 \subset X_2$ be a finite subset. We set $D_1 = \rho^{-1}(D_2)$. Let $\mathcal{P}_* \mathcal{V}$ be a filtered bundle on (X_1, D_1) . We obtain a filtered bundle $\rho_*(\mathcal{P}_* \mathcal{V})$ on (X_2, D_2) . Let $m(P)$ denote the ramification index of ρ at $P \in X_1$.

PROPOSITION 5.35. *The following holds:*

$$\deg(\rho_*(\mathcal{P}_*\mathcal{V})) = \deg(\mathcal{P}_*\mathcal{V}) - \frac{\text{rank } \mathcal{V}}{2} \sum_{P \in X_1 \setminus D_1} (m(P) - 1).$$

Proof. We have $\mathcal{P}_0(\rho_*\mathcal{V}) = \rho_*(\mathcal{P}_0\mathcal{V})$. By the Grothendieck–Riemann–Roch theorem and the Riemann–Hurwitz formula, we have

$$\deg(\rho_*\mathcal{P}_0\mathcal{V}) = \deg(\mathcal{P}_0\mathcal{V}) - \frac{\text{rank } \mathcal{V}}{2} \sum_{P \in X_1} (m(P) - 1).$$

By the construction of $\rho_*(\mathcal{P}_*\mathcal{V})$, we obtain

$$\begin{aligned} \deg(\rho_*(\mathcal{P}_*\mathcal{V})) &= \deg(\rho_*(\mathcal{P}_0\mathcal{V})) - \sum_{a \in D_1} \sum_{-1 < a \leq 0} \sum_{j=0}^{m(P)-1} \binom{a-j}{m(P)} \dim \text{Gr}_a^{\mathcal{P}}(\mathcal{V}_P) \\ &= \deg(\mathcal{P}_0\mathcal{V}) - \frac{\text{rank } \mathcal{V}}{2} \sum_{P \in X_1} (m(P) - 1) - \sum_{P \in D_1} \sum_{-1 < a \leq 0} \left(a - \frac{1}{2}(m(P) - 1) \right) \dim \text{Gr}_a^{\mathcal{P}}(\mathcal{V}_P) \\ &= \deg(\mathcal{P}_*\mathcal{V}) - \frac{\text{rank } \mathcal{V}}{2} \sum_{P \in X_1 \setminus D_1} (m(P) - 1). \end{aligned} \tag{42}$$

Thus, we are done. □

Remark 5.36. If there is no ramification point in $X_1 \setminus D_1$, we have $\deg(\mathcal{P}_*\mathcal{V}) = \deg(\rho_*\mathcal{P}_*\mathcal{V})$. We can also prove it as follows. Let h_0 be a Hermitian metric of $\mathcal{V}|_{X_1 \setminus D_1}$ such that (i) h_0 is flat around any point of D_1 and (ii) h_0 is adapted to $\mathcal{P}_*\mathcal{V}$. Let $R(h_0)$ be the curvature of the Chern connection of h . Then, we have

$$\deg(\mathcal{P}_*\mathcal{V}) = \frac{\sqrt{-1}}{2\pi} \int_{X_1 \setminus D_1} \text{tr} R(h_0).$$

We have the induced metric $\rho_*(h_0)$ of $\rho_*(\mathcal{V})|_{X_2 \setminus D_2}$. It is flat around any point of D_2 , and it is adapted to $\rho_*(\mathcal{P}_*\mathcal{V})$. Hence, we have

$$\deg(\rho_*(\mathcal{P}_*\mathcal{V})) = \frac{\sqrt{-1}}{2\pi} \int_{X_2 \setminus D_2} \text{tr} R(\rho_*h_0).$$

Then, we obtain $\deg(\mathcal{P}_*\mathcal{V}) = \deg(\rho_*\mathcal{P}_*\mathcal{V})$.

5.8 Dirichlet problem for wild decoupled harmonic bundles

Let Y, X, D and $(\mathcal{P}_*\mathcal{V}, \theta)$ be as in §2.3.

PROPOSITION 5.37. *Assume that (V, θ) is regular semisimple and that $\mathcal{P}_*(\mathcal{V})$ is a decomposable filtered extension. Then, the harmonic metric h in Theorem 2.8 is decoupled.*

Proof. It is enough to consider the case where $\Sigma_{V,\theta}$ is connected. Let $\tilde{\Sigma}_{V,\theta}$ be the partial compactification of $\Sigma_{V,\theta}$ as in §5.6.2. Let \tilde{X} and \tilde{D} denote the inverse images of X and D , respectively, by the natural morphism $\tilde{\Sigma}_{V,\theta} \rightarrow Y$. There exists a line bundle L_V on $\Sigma_{V,\theta}$ corresponding to (V, θ) . Let $\mathcal{P}_*\mathcal{L}_V$ be the filtered line bundle on $(\tilde{\Sigma}_{V,\theta}, \tilde{D})$ corresponding to $(\mathcal{P}_*\mathcal{V}, \theta)$. There exists a Hermitian metric h_0 of L_V such that (i) h_0 is flat around any point of \tilde{D} ,

(ii) h_0 is adapted to $\mathcal{P}_*\mathcal{L}_V$ and (iii) $h_{0|\partial\tilde{X}}$ induces $h_{\partial X}$. Let $R(h_0)$ denote the curvature of the Chern connection of (L_V, h_0) . It vanishes around \tilde{D} . There exists an \mathbb{R} -valued C^∞ -function α on \tilde{X} such that (i) $\bar{\partial}\partial\alpha = R(h_0)|_{\tilde{X}}$ and (ii) $\alpha|_{\partial\tilde{X}} = 0$. Then, $h_1 = e^{-\alpha}h_0$ is a flat metric of $L_V|_{\tilde{X}}$ adapted to $\mathcal{P}_*\mathcal{L}_V$ such that $h_{1|\partial\tilde{X}} = h_{0|\partial\tilde{X}}$. Let h_2 be the decoupled harmonic metric of $(V, \theta)|_{X\setminus D}$ corresponding to h_1 . It is adapted to $\mathcal{P}_*\mathcal{V}$ and it satisfies $h_{2|\partial X} = h_{\partial X}$. By the uniqueness in Theorem 2.8, we have $h = h_2$. \square

6. Large-scale solutions with prescribed boundary value

6.1 Harmonic metrics of regular semisimple Higgs bundles on a punctured disc

6.1.1 *General case* Let U be a neighbourhood of 0 in \mathbb{C} . Let U_0 be a relatively compact open neighbourhood of 0 in U with smooth boundary ∂U_0 . We set $U^* = U \setminus \{0\}$ and $U_0^* = U_0 \setminus \{0\}$.

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a good filtered Higgs bundle of rank r on $(U, 0)$ such that $(V, \theta) := (\mathcal{V}, \theta)|_{U^*}$ is regular semisimple. Let $h_{\partial U_0}$ be a Hermitian metric of $V|_{\partial U_0}$. According to Theorem 2.8, for any $t > 0$, there exists a unique harmonic metric h_t of $(V, t\theta)|_{U_0^*}$ such that $h_{t|\partial U_0} = h_{\partial U_0}$ and that $\mathcal{P}_*^{h_t}(V) = \mathcal{P}_*\mathcal{V}$. Note that $\det(h_t) = \det(h_1)$ for any $t > 0$.

PROPOSITION 6.1. *Let $t(i)$ be any sequence of positive numbers such that $t(i) \rightarrow \infty$. Then, there exists a subsequence $t'(j)$ such that the following hold:*

- $t'(j) \rightarrow \infty$;
- the sequence $h_{t'(j)}$ is convergent to a harmonic metric on any relatively compact open subset of U_0^* in the C^∞ -sense.

The limit h_∞ is a decoupled harmonic metric of (V, θ) such that $\mathcal{P}^{h_\infty}(V) = \mathcal{V}$ and that $\det(h_\infty) = \det(h_1)$.

Proof. By taking the pull back via a ramified covering map φ_ℓ as in §5.1.1, it is enough to consider the case where there exist meromorphic functions $\gamma(1), \dots, \gamma(r)$ on $(U, 0)$ and a decomposition

$$(\mathcal{V}, \theta) = \bigoplus_{i=1}^r (\mathcal{V}_i, \gamma(i) dz).$$

Let v_i be a frame of \mathcal{V}_i on U such that v_i is a section of $\mathcal{P}_{<0}\mathcal{V}$.

LEMMA 6.2. *There exists a constant $C > 0$ such that $h_t(v_i, v_i) \leq C$ for any $t > 0$.*

Proof. It is enough to consider the case where $\gamma(i) = 0$. We have $\theta(v_i) = 0$. Then, we have $-\partial_z\partial_{\bar{z}}|v_i|_{h_t}^2 \leq 0$ on U_0^* (see a preliminary Weitzenböck formula in [Sim90, Proof of Lemma 4.1]). Because v_i is a section of $\mathcal{P}_{<0}\mathcal{V}$, $|v_i|_{h_t}^2$ is bounded for each t . Hence, $|v_i|_{h_t}^2$ is subharmonic on U_0 . By the maximum principle, we obtain $|v_i|_{h_t}^2 \leq \max_{\partial U_0} |v_i|_{h_t}^2 = \max_{\partial U_0} |v_i|_{h_1}^2$. \square

Let $\mathcal{V}^\vee = \text{Hom}_{\mathcal{O}_U}(\mathcal{V}, \mathcal{O}_U(*0))$ denote the dual of \mathcal{V} . We have the induced filtered bundle $\mathcal{P}_*(\mathcal{V}^\vee)$ on \mathcal{V}^\vee . We set $(V^\vee, \theta^\vee) = (\mathcal{V}^\vee, \theta^\vee)|_{U^*}$. The induced harmonic metric h_t^\vee of $(V^\vee, t\theta^\vee)$ is adapted to $\mathcal{P}_*(\mathcal{V}^\vee)$.

There exists the induced decomposition $\mathcal{V}^\vee = \bigoplus_{i=1}^r \mathcal{V}_i^\vee$. Let v_i^\vee denote the section of \mathcal{V}_i^\vee such that $v_i^\vee(v_i) = 1$. There exists $m(i) \in \mathbb{Z}_{>0}$ such that $z^{m(i)}v_i^\vee$ is a section of $\mathcal{P}_{<0}(\mathcal{V}^\vee)$. By Lemma 6.2, we obtain the following lemma.

LEMMA 6.3. *There exists $C > 0$ such that $|z|^{2m(i)}h_t^\vee(v_i^\vee, v_i^\vee) \leq C$ for any $t > 0$.*

Let s_t be the automorphism of $V|_{U_0^*}$ determined by $h_t = h_1 \cdot s_t$. Let K be any relatively compact open subset of U_0^* . By Lemmas 6.2 and 6.3, there exist $C_{K,1} > 0$ such that the following holds for any $t > 0$:

$$|s_t|_{h_1} + |s_t^{-1}|_{h_1} \leq C_{K,1}. \tag{43}$$

By a variant of Simpson’s main estimate (see [Moc16, Proposition 2.3]), there exist $t_{K,1}, C_{K,2}, C_{K,3} > 0$ such that the following holds for any $t > t_{K,1}$ and for any local sections $u_{\beta(i)}$ and $u_{\beta(j)}$ of $V_{\beta(i)}$ and $V_{\beta(j)}$ on K ($i \neq j$):

$$|h_t(u_{\beta(i)}, u_{\beta(j)})| \leq C_{K,2} \exp(-C_{K,3}t) |u_{\beta(i)}|_{h_t} \cdot |u_{\beta(j)}|_{h_t}. \tag{44}$$

There also exist $t_{K,2}, C_{K,4}, C_{K,5} > 0$ such that the following holds on K for any $t > t_{K,2}$ (see [Moc16, Theorem 2.9]):

$$|R(h_t)|_{h_1} \leq C_{K,4} \exp(-C_{K,5}t). \tag{45}$$

By (43) and (45), it is standard to obtain the existence of a convergent subsequence $h_{t'(j)}$. By (44) and (45), the limit is a decoupled harmonic metric. By Lemma 6.2, we obtain that $h_\infty(v_i, v_i) \leq C$. Hence, v_i are sections of $\mathcal{P}^{h_\infty}(V)$. This implies that $\mathcal{V} \subset \mathcal{P}^{h_\infty}(V)$. Because both \mathcal{V} and $\mathcal{P}^{h_\infty}(V)$ are locally free $\mathcal{O}_U(*0)$ -modules, we obtain that $\mathcal{V} = \mathcal{P}^{h_\infty}(V)$. \square

PROPOSITION 6.4. *Let h_∞ denote the limit of a convergent subsequence in Proposition 6.1. Suppose the following condition holds.*

- For every $z_0 \in \partial U_0$, the eigen decomposition of θ at z_0 is orthogonal with respect to $h_{\partial U_0}$.

Then, $h_\infty|_{\partial U_0} = h_{\partial U_0}$.

Proof. Let U_1 be a relatively compact open neighbourhood of 0 in U_0 with smooth boundary ∂U_1 . Because h_∞ is a decoupled harmonic metric, the following condition is satisfied.

- For every $z_1 \in \partial U_1$, the eigen decomposition of θ at z_1 is orthogonal with respect to h_∞ .

We set $A = U_0 \setminus \overline{U_1}$. By Proposition 5.37, there exists a decoupled harmonic metric $h^{(1)}$ of $(V, \bar{\partial}_V, \theta)|_A$ such that $h^{(1)}|_{\partial U_0} = h_{\partial U_0}$ and $h^{(1)}|_{\partial U_1} = h_\infty|_{\partial U_1}$. We note that $h^{(1)}$ is a harmonic metric of $(V, \bar{\partial}_V, t\theta)|_A$ for any $t > 0$. We also note that $\det(h^{(1)}) = \det(h_1)|_A$ because $\det(h^{(1)})|_{\partial A} = \det(h_1)|_{\partial A}$.

Let s_t be determined by $h_t = h^{(1)}s_t$ on A . We have $-\partial_z \partial_{\bar{z}} \text{Tr}(s_t) \leq 0$. We have $s_{t'(j)} \rightarrow \text{id}$ on ∂U_1 and $s_{t'(j)} = \text{id}$ on ∂U_0 . Hence, we obtain $|\text{Tr}(s_{t'(j)} - \text{id})| \rightarrow 0$ as $t'(j) \rightarrow \infty$. This implies the claim of the proposition. \square

6.1.2 *The irreducible case* Suppose that the spectral curve is irreducible, i.e. $\Sigma_{V,\theta}$ is connected. We obtain the decomposable filtered bundle $\mathcal{P}_*^*(\mathcal{V})$ determined by $\det(\mathcal{P}_*\mathcal{V})$ as in Proposition 5.13, which is not necessarily equal to $\mathcal{P}_*(\mathcal{V})$.

LEMMA 6.5. *Let h_∞ denote the limit of a convergent subsequence in Proposition 6.1. Then, we have $\mathcal{P}_*^{h_\infty}(V) = \mathcal{P}_*^*(\mathcal{V})$.*

Proof. We have $\mathcal{P}^{h_\infty}V = \mathcal{V}$. Because h_∞ is a decoupled harmonic metric, $\mathcal{P}_*^{h_\infty}(\mathcal{V})$ is decomposable. Because $\det(h_\infty) = \det(h_1)$, we obtain $\det(\mathcal{P}_*^{h_\infty}V) = \det(\mathcal{P}_*\mathcal{V})$. Then, the claim follows from the uniqueness of $\mathcal{P}_*^*(\mathcal{V})$. \square

Let h_0 be any decoupled harmonic metric of $(V, \bar{\partial}_V, \theta)$ such that $\mathcal{P}^{h_0}(V) = \mathcal{V}$ and that $\det(h_0)$ is adapted to $\det(\mathcal{P}_*\mathcal{V})$. By the argument in the proof of Lemma 6.5, we can prove $\mathcal{P}_*^{h_0}(V) = \mathcal{P}_*^*(\mathcal{V})$. Let h_t ($t > 0$) be the harmonic metrics of $(V, \bar{\partial}_V, t\theta)$ adapted to $\mathcal{P}_*\mathcal{V}$ such that $h_t|_{\partial U_0} = h_0|_{\partial U_0}$.

PROPOSITION 6.6. *The sequence h_t is convergent to h_0 as $t \rightarrow \infty$ in the C^∞ -sense on any relatively compact open subset of U_0^* .*

Proof. Let t_i be any subsequence such that $t_i \rightarrow \infty$ and that h_{t_i} is convergent. Let h_∞ denote the limit. By Proposition 6.4, we have $h_\infty|_{\partial U_0} = h_0|_{\partial U_0}$. We also have $\mathcal{P}_*^{h_\infty}(V) = \mathcal{P}_*^*(\mathcal{V}) = \mathcal{P}_*^{h_0}(V)$. Hence, we obtain $h_\infty = h_0$. This implies that h_t is convergent to h_0 as $t \rightarrow \infty$. \square

6.1.3 *Symmetric case* We do not assume that the spectral curve is irreducible. Instead, suppose that there exists a perfect pairing C of $(\mathcal{P}_*\mathcal{V}, \theta)$. There uniquely exists a decoupled harmonic metric h^C of (V, θ) which is compatible with C . As in Lemma 5.23, we have $\mathcal{P}_*^{h^C}(V) = \mathcal{P}_*^C\mathcal{V}$.

Suppose that $h_{\partial U_0}$ is compatible with $C|_{\partial U_0}$. Then, h_t ($t > 0$) are compatible with C by Corollary 2.9. Let s_t be determined by $h_t = h^C s_t$. We note that $\det(h_t) = \det(h_1) = \det(h^C)$ by the compatibility with C . The following proposition is a special case of Corollary 3.5.

PROPOSITION 6.7. *If $h_{\partial U_0}$ is compatible with $C|_{\partial U_0}$, the sequence h_t is convergent to h^C in the C^∞ -sense on any relatively compact subset K of U_0^* . Moreover, there exists $t(K) > 0$ such that the following holds for any $\ell \geq 0$.*

- There exists $C(K, \ell)$ and $\epsilon(K, \ell)$ such that the norms of $s_t - \text{id}$ ($t \geq t(K)$) and their derivatives up to order ℓ are dominated by $C(K, \ell) \exp(-\epsilon(K, \ell)t)$.

Let us also consider the case where $h_{\partial U_0}$ is not necessarily compatible with $C|_{\partial U_0}$, but $\det(h_{\partial U_0})$ is compatible with $\det(C)|_{\partial U_0}$. Because $\det(h_t)$ are compatible with $\det(C)$ on U_0 , we obtain $\det(h_t) = \det(h_1) = \det(h^C)$.

PROPOSITION 6.8. *Let $h_{t(i)}$ be a convergent subsequence, and let h_∞ denote the limit as in Proposition 6.1. Then, $\mathcal{P}_*^{h_\infty}(V) = \mathcal{P}_*^C(\mathcal{V})$.*

Proof. Let h'_t ($t > 0$) be harmonic metrics of $(V, t\theta)$ which are compatible with C , such that $\det(h'_t) = \det(h_1)$. We have already proved that the sequence h'_t is convergent to h^C . We have $\det(h'_t) = \det(h_t)$. Let s_t be the automorphism determined by $h_t = h'_t s_t$. Let s_∞ be determined by $h_\infty = h^C s_\infty$. The sequence s_t is convergent to s_∞ . Because $\det(s_t) = 1$, we have $\det(s_\infty) = 1$. Because $\text{Tr}(s_t)$ is subharmonic on U_0 , we obtain that $\max_{U_0} \text{Tr}(s_t) = \max_{\partial U_0} \text{Tr}(s_t) = \max_{\partial U_0} \text{Tr}(s_1)$. We obtain that $\text{Tr}(s_\infty)$ is bounded. Then, s_∞ and s_∞^{-1} are bounded, and we obtain $\mathcal{P}^{h_\infty}(V) = \mathcal{P}_*^C(\mathcal{V})$. \square

Suppose that for every $z_0 \in \partial U_0$ the eigen decomposition of θ is orthogonal with respect to $h_{\partial U_0}$. There exists a decoupled harmonic metric \tilde{h} of (V, θ) such that $\tilde{h}|_{\partial U_0} = h_{\partial U_0}$ and $\mathcal{P}_*^{\tilde{h}}(V) = \mathcal{P}_*^C(\mathcal{V})$.

COROLLARY 6.9. *The sequence h_t is convergent to \tilde{h} .*

6.2 Local symmetrizability of Higgs bundles

Let U be a simply connected open subset in \mathbb{C} . Let D be a finite subset of U . Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle on U such that $(V, \theta) = (E, \theta)|_{U \setminus D}$ is regular semisimple. Let $\pi : \Sigma_{E, \theta} \rightarrow U$ denote the projection. Let $\rho : \tilde{\Sigma}_{E, \theta} \rightarrow \Sigma_{E, \theta}$ denote the normalization of $\Sigma_{E, \theta}$. We set $\tilde{D} = (\pi \circ \rho)^{-1}(D)$. We assume the following condition.

- There exists a line bundle L on $\tilde{\Sigma}_{E, \theta}$ with an isomorphism $(\pi \circ \rho)_* L \simeq E$. Moreover, the Higgs field θ of E is induced by the \mathcal{O}_{T^*U} -action on $\rho_* L$.

For any $P \in D$, let U_P be a simply connected neighbourhood of P in U such that $U_P \cap D = \{P\}$. We set $U_P^* = U_P \setminus \{P\}$. There exists the decomposition

$$(V, \theta)|_{U_P^*} = \bigoplus_{k \in S(P)} (V_P^{[k]}, \theta_P^{[k]}), \tag{46}$$

such that the spectral curves of $(V_P^{[k]}, \theta_P^{[k]})$ are connected. Because $E \simeq (\pi \circ \rho)_* L$, (46) extends to the decomposition

$$(E, \theta)|_{U_P^*} = \bigoplus_{i \in S(P)} (E_P^{[k]}, \theta_P^{[k]}).$$

Let h be a decoupled harmonic metric of (V, θ) . The decomposition (46) is orthogonal with respect to h . Let $h_P^{[k]}$ denote the restriction of h to $V_P^{[k]}$. We consider the following condition.

CONDITION 6.10. $\det(h_P^{[k]})$ induces a flat metric of $\det(E_P^{[k]})$, and $\mathcal{P}^h V = E(*D)$ holds.

We shall prove the following proposition in §6.2.2 after the preliminary in §6.2.1.

PROPOSITION 6.11. Suppose that Condition 6.10 is satisfied at each $P \in D$. Moreover, we assume that each connected component of $\tilde{\Sigma}_{E, \theta}$ is simply connected. Then, the following claims hold.

- There exists a non-degenerate symmetric pairing C of (E, θ) such that $C|_{U \setminus D}$ is compatible with h .
- Let C' be a non-degenerate symmetric pairing of (V, θ) which is compatible with h . Then, C' induces a non-degenerate symmetric pairing of E .

Remark 6.12. If $\Sigma_{E, \theta}$ is a simply connected complex submanifold of T^*U , we can apply Proposition 6.11 to (E, θ) .

6.2.1 *Special case* Let us study the case that $D = \{0\}$, and that $\Sigma_{V, \theta}$ is connected. We set $\mathcal{V} = E(*0)$. We use the notation in §5.1.1. By choosing an r -th root of $(\pi \circ \rho)^*(z)$ on $\tilde{\Sigma}_{E, \theta}$, we obtain a holomorphic isomorphism $\psi : \tilde{\Sigma}_{E, \theta} \rightarrow U^{(r)}$ such that $\varphi_r \circ \psi = \pi \circ \rho$. There exists the decomposition (41) on $U^{(r)}$. There exists the natural isomorphism $\psi_*(L)(*0) \simeq \mathcal{V}_{\beta(1)}$. Let $E_{\beta(1)} \subset \mathcal{V}_{\beta(1)}$ denote the image of L . We have $\varphi_{r*}(E_{\beta(1)}) = E$.

Let $C_{\beta(1)} : \mathcal{V}_{\beta(1)} \otimes \mathcal{V}_{\beta(1)} \rightarrow \mathcal{O}_{U^{(r)}}(*0)$ be a non-degenerate symmetric pairing. There exists the morphism $\text{tr} : \varphi_{r*} \mathcal{O}_{U^{(r)}}(*0) \rightarrow \mathcal{O}_U(*0)$ as in §2.2.2. We obtain the induced symmetric pairing $\Psi(C_{\beta(1)}) = \text{tr} \circ \varphi_{r*}(C_{\beta(1)})$ of $\mathcal{V} = \varphi_{r*}(\mathcal{V}_{\beta(1)})$. There exists an integer k such that $C_{\beta(1)}(E_{\beta(1)} \otimes E_{\beta(1)}) = \mathcal{O}_{U^{(r)}}(k\{0\})$.

LEMMA 6.13. The pairing $\Psi(C_{\beta(1)})$ induces a symmetric pairing of E if and only if $k \leq r - 1$. The induced pairing is non-degenerate if and only if $k = r - 1$.

Proof. There exists a frame v of $E_{\beta(1)}$ such that $C_{\beta(1)}(v, v) = \zeta^{-k}$. The tuple $v, \zeta v, \dots, \zeta^{r-1}v$ induces a frame of E . Note that $\text{tr}(\zeta^j) = 0$ unless $j \in r\mathbb{Z}$. It is easy to see that

$$\text{tr}(C_{\beta(1)}(\zeta^i v, \zeta^j v)) = \text{tr}(\zeta^{i+j-k}) \quad (0 \leq i, j \leq r - 1)$$

are holomorphic at 0 if and only if $k \leq r - 1$, and that the induced pairing is non-degenerate at 0 if and only if $k = r - 1$. □

Let $C_{0,\beta(1)}$ be a non-degenerate symmetric pairing of $\mathcal{V}_{\beta(1)}$ such that $C_{0,\beta(1)}(E_{\beta(1)} \otimes E_{\beta(1)}) = \mathcal{O}_{U^{(r)}}((r - 1)\{0\})$. We set $C_0 = \Psi(C_{0,\beta(1)})$ which is a non-degenerate symmetric pairing of (E, θ) . Let h_0 be a decoupled harmonic metric of (V, θ) compatible with C_0 . We note that $\det(h_0)$ is compatible with $\det(C_0)$, and hence it induces a Hermitian metric of $\det(E)$.

Let h_1 be any decoupled harmonic metric of (V, θ) such that $\mathcal{P}^{h_1}(V) = \mathcal{V}$ and that $\det(h_1) = \det(h_0)$. According to Corollary 5.22, there exists a holomorphic function γ_1 on $U^{(r)}$ such that (i) $\varphi_r^*(h_1)|_{V_{\beta(1)}} = \exp(2\text{Re}\gamma_1)\varphi_r^*(h_0)|_{V_{\beta(1)}}$ and (ii) $\sum_{\sigma \in \text{Gal}(r)} \sigma^* \gamma_1 = 0$. We set

$$C_{1,\beta(1)} = \exp(2\gamma_1)C_{0,\beta(1)}.$$

It is a non-degenerate symmetric pairing of $\mathcal{V}_{\beta(1)}$ satisfying $C_{1,\beta(1)}(E_{\beta(1)} \otimes E_{\beta(1)}) = \mathcal{O}_{U^{(r)}}((r - 1)\{0\})$. We obtain a non-degenerate symmetric pairing $C_1 = \Psi(C_{1,\beta(1)})$ of (E, θ) such that $C_1|_{U^*}$ is compatible with h_1 .

Let h be any decoupled harmonic metric of (V, θ) such that $\mathcal{P}^h(V) = \mathcal{V}$ and that $\det(h)$ induces a flat metric of $\det(E)$. There exists a holomorphic function γ_2 on U such that $\det(h) = \exp(2r\text{Re}(\gamma_2)) \det(h_1)$. Then, $C = \exp(2\gamma_2)C_1$ is compatible with h , and it induces a non-degenerate symmetric pairing of E .

LEMMA 6.14. Let C' be a non-degenerate symmetric pairing of (V, θ) compatible with h . Then, C' induces a non-degenerate symmetric pairing of E .

Proof. There exist non-degenerate symmetric pairings $C_{\beta(1)}$ and $C'_{\beta(1)}$ of $\mathcal{V}_{\beta(1)}$ such that $\Psi(C_{\beta(1)}) = C$ and $\Psi(C'_{\beta(1)}) = C'$, respectively. Because both $C_{\beta(1)}$ and $C'_{\beta(1)}$ are compatible with $\varphi_r^*(h)|_{V_{\beta(1)}}$, there exists a constant α such that $|\alpha| = 1$ and $C'_{\beta(1)} = \alpha C_{\beta(1)}$. Hence, $C'_{\beta(1)}(E_{\beta(1)} \otimes E_{\beta(1)}) = \mathcal{O}_{U^{(r)}}((r - 1)\{0\})$, and hence C' induces a non-degenerate symmetric pairing of E . □

6.2.2 *Proof of Proposition 6.11* It is enough to consider the case where $\Sigma_{V,\theta}$ is connected, which implies that $\tilde{\Sigma}_{E,\theta}$ is connected. Let h_L denote the flat metric of $L|_{\Sigma_{V,\theta}}$ corresponding to the decoupled harmonic metric h . Let P be any point of D . By Proposition 5.25, there exists a non-degenerate symmetric pairing of $V|_{U_P^*}$ which is compatible with $h|_{U_P^*}$. There exists a non-degenerate symmetric pairing of L on $(\pi \circ \rho)^{-1}(U_P^*)$ which is compatible with h_L . Hence, the monodromy of the Chern connection of h_L around any point of \tilde{D} is 1 or -1 . Because $\tilde{\Sigma}_{E,\theta}$ is simply connected, Lemma 2.7 implies that there exists a non-degenerate symmetric pairing C_L of $L|_{\Sigma_{V,\theta}}$ compatible with h_L . It induces a non-degenerate symmetric pairing C of (V, θ) compatible with h . By Lemma 6.14, C induces a non-degenerate symmetric pairing of E . Thus, we obtain the first claim of Proposition 6.11. The second claim also follows from Lemma 6.14. □

6.3 A uniform estimate in the symmetric case

6.3.1 *Setting* For $R > 0$, we set $B(R) = \{z \in \mathbb{C} \mid |z| < R\}$. Let $\mathcal{S} \subset \mathbb{C}^n$ be a connected open subset with a base point x_0 . Let \mathcal{Z}_i ($i = 1, 2$) be an open subset of $\mathcal{S} \times \mathbb{C}_{z_i}$. For simplicity, we assume that $\mathcal{Z}_2 = \mathcal{S} \times B(2)$. Let $p_i : \mathcal{Z}_i \rightarrow \mathcal{S}$ denote the projections. We set $T^*(\mathcal{Z}_2/\mathcal{S}) = \mathcal{S} \times T^*B(2)$. Let $\pi_2 : T^*(\mathcal{Z}_2/\mathcal{S}) \rightarrow \mathcal{Z}_2$ denote the projection. Let $\Phi_0 : \mathcal{Z}_1 \rightarrow T^*(\mathcal{Z}_2/\mathcal{S})$ be a holomorphic map such that $p_1 = \pi_2 \circ \Phi_0$. We set $\Phi_1 := \pi_2 \circ \Phi_0 : \mathcal{Z}_1 \rightarrow \mathcal{Z}_2$. We assume the following conditions:

- Φ_1 is proper and finite;
- there exists a complex analytic closed hypersurface $\mathcal{D} \subset \mathcal{S} \times B(R_1) \subset \mathcal{Z}_2$ for some $0 < R_1 < 1$ such that (i) the induced map $\mathcal{Z}_1 \setminus \Phi_1^{-1}(\mathcal{D}) \rightarrow \mathcal{Z}_2 \setminus \mathcal{D}$ is a covering map, (ii) Φ_0 induces an injection $\mathcal{Z}_1 \setminus \Phi_1^{-1}(\mathcal{D}) \rightarrow T^*(\mathcal{Z}_2 \setminus \mathcal{D})$ and (iii) $\mathcal{D} \cap (\{x_0\} \times \mathbb{C}) = \{(x_0, 0)\}$.

We set $r := |\Phi_1^{-1}(P)|$ for any $P \in \mathcal{Z}_2 \setminus \mathcal{D}$. We also set $\tilde{\mathcal{D}} = \Phi_1^{-1}(\mathcal{D})$.

LEMMA 6.15. The sheaf $\mathcal{E} = \Phi_{1*}(\mathcal{O}_{\mathcal{Z}_1})$ is a locally free $\mathcal{O}_{\mathcal{Z}_2}$ -module of rank r .

Proof. By a change of local holomorphic coordinate system on \mathcal{Z}_1 , it is enough to consider the case where $\Phi_1^*(z_2)$ is expressed as a Weierstrass polynomial. Then, it is reduced to [GR84, Chapter 2, §4.2, Theorem]. □

Note that $\mathcal{E} = \pi_{2*}(\Phi_{0*}\mathcal{O}_{\mathcal{Z}_1})$ is naturally a $\pi_{2*}(\mathcal{O}_{T^*(\mathcal{Z}_2/\mathcal{S})})$ -module. Hence, we obtain the relative Higgs field $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{Z}_2/\mathcal{S}}^1$. The following lemma is clear by the construction.

LEMMA 6.16. For any $P \in \mathcal{Z}_2 \setminus \mathcal{D}$, there exist a neighbourhood \mathcal{U} of P in $\mathcal{Z}_2 \setminus \mathcal{D}$ and a decomposition

$$(\mathcal{E}, \theta)|_{\mathcal{U}} = \bigoplus_{i=1}^r (\mathcal{E}_{P,i}, \theta_{P,i}), \tag{47}$$

where $\text{rank } \mathcal{E}_{P,i} = 1$, and $\theta_{P,i} - \theta_{P,j}$ ($i \neq j$) are nowhere vanishing.

For any $x \in \mathcal{S}$, we set $\mathcal{Z}_{i,x} = \mathcal{Z}_i \cap (\{x\} \times \mathbb{C})$, $\tilde{\mathcal{D}}_x = \tilde{\mathcal{D}} \cap (\{x\} \times \mathbb{C})$ and $\mathcal{D}_x = \mathcal{D} \cap (\{x\} \times \mathbb{C})$. Note that $\mathcal{Z}_{2,x} = B(2)$ for any $x \in \mathcal{S}$. Let $\iota_x : \mathcal{Z}_{2,x} \rightarrow \mathcal{Z}_2$ denote the inclusion. We obtain the Higgs bundles $(\mathcal{E}_x, \theta_x) := \iota_x^*(\mathcal{E}, \theta)$ on $\mathcal{Z}_{2,x}$ which is regular semisimple outside \mathcal{D}_x .

6.3.2 *A uniform estimate in the symmetric case* Let h_x° ($x \in \mathcal{S}$) be decoupled harmonic metrics of $(\mathcal{E}_x, \theta_x)|_{B(2) \setminus \mathcal{D}_x}$ such that they induce a C^∞ -metric of $\mathcal{E}|_{\mathcal{Z}_2 \setminus \mathcal{D}}$. Assume the following.

CONDITION 6.17. For each $(x, P) \in \mathcal{D}$, Condition 6.10 is satisfied for $(\mathcal{E}_x, \theta_x, h_x^\circ)$ at P .

Let $h_{x,t}$ be harmonic metrics of $(\mathcal{E}_x, t\theta_x)|_{B(1)}$ such that $h_{x,t}|_{\partial B(1)} = h_x^\circ|_{\partial B(1)}$. Let $s_{x,t}$ be the automorphism of $\mathcal{E}_x|_{B(1) \setminus \mathcal{D}_x}$ determined by $h_{x,t} = h_x^\circ \cdot s_{x,t}$.

PROPOSITION 6.18. Let $R_1 < R_2 < 1$. Let \mathcal{S}' be a relatively compact open subset of \mathcal{S} . Then, there exists $t_0 > 0$ such that the following hold.

- For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C(\ell)$ and $\epsilon(\ell)$ such that

$$|(s_{x,t} - \text{id})|_{B(R_2) \setminus B(R_1)}|_{L_\ell^2} \leq C(\ell) \exp(-\epsilon(\ell)t)$$

for any $x \in \mathcal{S}'$ and any $t \geq t_0$. Here, we consider the L_ℓ^2 -norms with respect to h_x° and the standard Euclidean metric $dz_2 d\bar{z}_2$.

Proof. For $0 < R \leq 2$, we set $\mathcal{Z}_{1,x}(R) := \Phi_1^{-1}(\{x\} \times B(R)) \subset \mathcal{Z}_{1,x}$.

LEMMA 6.19. *If $R_1 < R \leq 2$, each connected component of $\mathcal{Z}_{1,x}(R)$ is diffeomorphic to a two-dimensional disc.*

Proof. Let us consider the case $R_1 < R < 2$. We set $\mathcal{Z}_1(R) := \Phi_1^{-1}(\mathcal{S} \times B(R)) \subset \mathcal{Z}$. The closure $\overline{\mathcal{Z}}_1(R)$ of $\mathcal{Z}_1(R)$ is a C^∞ -manifold with smooth boundary. The projection $\overline{\mathcal{Z}}_1(R) \rightarrow \mathcal{S}$ is submersive and proper. Each connected component of $\mathcal{Z}_{1,x_0}(R)$ is diffeomorphic to a disc. Because \mathcal{S} is connected, we obtain that each connected component of $\mathcal{Z}_{1,x}(R)$ is diffeomorphic to a disc. For $R_1 < R < 2$, there exists a diffeomorphism $\rho_R : B(R) \simeq B(2)$ whose restriction to $B(R_1)$ is the identity. We can construct a diffeomorphism $\mathcal{Z}_{1,x}(R) \simeq \mathcal{Z}_{1,x}(2)$ by lifting ρ_R . \square

LEMMA 6.20. *There exist holomorphic non-degenerate symmetric pairings C_x ($x \in \mathcal{S}$) of $(\mathcal{E}_x, \theta_x)$ such that the restrictions $C_x|_{B(1) \setminus \mathcal{D}_x}$ are compatible with h_x° and continuous with respect to x .*

Proof. Let $h_{0,x}^\circ$ denote the flat metric of $\mathcal{O}_{\mathcal{Z}_{1,x} \setminus \tilde{\mathcal{D}}_x}$ corresponding to h_x° , which are continuous with respect to x . Let $\nabla_{0,x}^\circ$ denote the Chern connection. They are flat connections, and continuous with respect to x .

By Proposition 6.11 and Lemma 6.19, for each $x \in \mathcal{S}$, there exists a holomorphic non-degenerate symmetric pairing C'_x of $(\mathcal{E}_x, \theta_x)$ such that the restriction $(C'_x)|_{B(1) \setminus \mathcal{D}_x}$ is compatible with h_x° . Let $C'_{0,x}$ denote the holomorphic non-degenerate symmetric bilinear form of $\mathcal{O}_{\mathcal{Z}_{1,x} \setminus \tilde{\mathcal{D}}_x}$ corresponding to C'_x , which is compatible with $h_{0,x}^\circ$.

Let $z_1 \in B(1) \setminus B(R_1)$. There exists a continuous family of non-degenerate symmetric pairings $C_{0,(x,z_1)}^\circ$ of the vector space $\mathcal{O}_{\mathcal{Z}_1|_{(x,z_1)}}$ which are compatible with $(h_{0,x}^\circ)|_{z_1}$. We obtain $\alpha_x \in \mathbb{C}^*$ determined by $C_{0,(x,z_1)}^\circ = \alpha_x(C'_{0,x})|_{z_1}$. We set $C_{0,x} = \alpha_x C'_{0,x}$. Because $C_{0,x}$ are $\nabla_{0,x}^\circ$ -flat, they are continuous with respect to x . Let C_x denote the non-degenerate symmetric pairing of $(\mathcal{E}_x, \theta_x)$ corresponding to $C_{0,x}$. (See Proposition 6.11.) Then, they satisfy the desired condition. \square

Because $h_{t,x}|_{\partial B(1)} = h_x^\circ|_{\partial B(1)}$ are compatible with $C_x|_{\partial B(1)}$, we obtain that $h_{t,x}$ are compatible with C_x . Then, the claim of Proposition 6.18 follows from Theorem 3.4.

We also obtain the following proposition from Theorem 3.4, as in the proof of Proposition 6.18. \square

PROPOSITION 6.21. *Let $R_1 < R_2 < 2$. Let \mathcal{S}' be a relatively compact open subset of \mathcal{S} . There exists $t_0 > 0$ such that the following holds.*

- Let $h'_{x,t}$ be any harmonic metrics of $(\mathcal{E}_x, t\theta_x)$ ($x \in \mathcal{S}'$) compatible with C_x° . Let $s'_{x,t}$ be determined by $h'_{x,t} = h_x^\circ \cdot s'_{x,t}$. Then, for any $\ell \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C(\ell)$ and $\epsilon(\ell)$ such that

$$|(s'_{x,t} - \text{id})|_{B(R_2) \setminus B(R_1)}|_{L^2_\ell} \leq C(\ell) \exp(-\epsilon(\ell)t),$$

for any $t \geq t_0$.

6.3.3 *Examples of non-degenerate symmetric pairings and decoupled harmonic metrics* We obtain a holomorphic function $G = \partial_{z_1}(\Phi_1^*(z_2))$. We have $G^{-1}(0) \subset \tilde{\mathcal{D}}$. We define the symmetric product $C_0 : \mathcal{O}_{\mathcal{Z}_1} \otimes \mathcal{O}_{\mathcal{Z}_1} \rightarrow G^{-1}\mathcal{O}_{\mathcal{Z}_1}$ by

$$C_0(a \otimes b) = G^{-1}ab.$$

We obtain the following lemma by using Lemma 6.13.

LEMMA 6.22. The pairing C_0 induces a non-degenerate symmetric pairing C_1 of \mathcal{E} , which induces a non-degenerate symmetric pairing of $(\mathcal{E}_x, \theta_x)$ for any $x \in \mathcal{S}$.

Let h_0 be the flat metric of $\mathcal{O}_{\mathbb{Z}_1 \setminus \tilde{\mathcal{D}}}$ defined as follows:

$$h_0(a, b) = |G|^{-1} a \bar{b}.$$

LEMMA 6.23. The metric h_0 induces a flat metric h_1 of $\mathcal{E}_{|\mathbb{Z}_2 \setminus \mathcal{D}}$. For each $x \in \mathcal{S}$, the induced metric $h_{1,x}$ of $(\mathcal{E}_x, \theta_x)_{|\mathbb{Z}_{2,x} \setminus \mathcal{D}_x}$ is a decoupled harmonic metric such that $\det(h_{1,x})$ induces a flat metric of $\det(\mathcal{E}_x)$ for each $x \in \mathcal{S}$.

Remark 6.24. We shall use h_0 in §7.3.

7. Large-scale solutions on compact Riemann surfaces

7.1 Convergence in the locally irreducible case

7.1.1 *Statement* Let X be a compact Riemann surface. Let $\pi : T^*X \rightarrow X$ denote the projection. For any $A \subset T^*X$, the induced map $A \rightarrow X$ is also denoted by π . Let $D \subset X$ be a finite subset.

Let $(\mathcal{P}_*\mathcal{V}, \theta)$ be a good filtered Higgs bundle of degree 0 on (X, D) . We obtain the Higgs bundle $(V, \theta) = (\mathcal{V}, \theta)_{|X \setminus D}$. We assume the following.

CONDITION 7.1. The Higgs bundle (V, θ) is a regular semisimple Higgs bundle on $X \setminus D$.

Remark 7.2. If (V, θ) is generically regular semisimple, there exists a finite subset $D' \subset X$ such that $(V', \theta')_{|X \setminus D'}$ is regular semisimple and that $D \subset D'$. We set $\mathcal{V}' = \mathcal{V}(*D')$. For each $P \in D' \setminus D$, we consider the filtered bundle $\mathcal{P}_*(\mathcal{V}'_P)$ over \mathcal{V}'_P defined by $\mathcal{P}_a \mathcal{V}'_P = \mathcal{V}_P([a]P)$, where $[a] = \max\{n \in \mathbb{Z} \mid n \leq a\}$. For harmonic metrics of (V, θ) adapted to $\mathcal{P}_*\mathcal{V}$, it is enough to study harmonic metrics of (V', θ') adapted to $\mathcal{P}_*\mathcal{V}'$.

For any $P \in D$, there exist a neighbourhood X_P of P in X and a decomposition of the meromorphic Higgs bundle

$$(V, \theta)_{|X_P} = \bigoplus_{i \in S(P)} (\mathcal{V}_{P,i}, \theta_{P,i}), \tag{48}$$

such that the spectral curves of $(\mathcal{V}_{P,i}, \theta_{P,i})_{|X_P \setminus \{P\}}$ are connected.

CONDITION 7.3. We assume the following conditions:

- the spectral curve $\Sigma_{V, \theta}$ is connected;
- for any $P \in D$, the decomposition (48) is compatible with the filtered bundle $\mathcal{P}_*(\mathcal{V}_P)$ over \mathcal{V}_P , i.e. $\mathcal{P}_*(\mathcal{V}_P) = \bigoplus_{i \in S(P)} \mathcal{P}_*((\mathcal{V}_{P,i})_P)$.

For each $P \in D$, we obtain the filtered bundle $\mathcal{P}_*(\mathcal{V}_P) = \bigoplus_{i \in S(P)} \mathcal{P}_*((\mathcal{V}_{P,i})_P)$ over \mathcal{V}_P determined by the filtered bundles $\det(\mathcal{P}_*\mathcal{V}_{P,i})$ as in Proposition 5.13. By patching $\mathcal{P}_*(\mathcal{V}_P)$ ($P \in D$) with \mathcal{V} , we obtain a decomposable filtered Higgs bundle $(\mathcal{P}_*(\mathcal{V}), \theta)$.

LEMMA 7.4. The filtered Higgs bundle $(\mathcal{P}_*(\mathcal{V}), \theta)$ is stable of degree 0. As a result, there exists a decoupled harmonic metric h_∞ of (V, θ) adapted to $\mathcal{P}_*(\mathcal{V})$.

Proof. Because $\Sigma_{V,\theta}$ is connected, there does not exist a non-trivial Higgs subbundle of (V, θ) . Hence, $(\mathcal{P}_*\mathcal{V}, \theta)$ is stable. Because $\det(\mathcal{P}_*\mathcal{V}) = \det(\mathcal{P}_*\mathcal{V})$, we obtain $\deg(\mathcal{P}_*\mathcal{V}) = 0$. The second claim follows from Proposition 5.34. \square

Note that $\det(h_\infty)$ is a flat metric of $\det(V)$ adapted to $\det(\mathcal{P}_*\mathcal{V}) = \det(\mathcal{P}_*\mathcal{V})$. Because $\Sigma_{V,\theta}$ is connected, $(\mathcal{P}_*\mathcal{V}, \theta)$ is stable of degree 0 as in Lemma 7.4. Hence, for any $t > 0$, there exists a harmonic metric h_t of $(V, t\theta)$ which is adapted to $\mathcal{P}_*\mathcal{V}$ such that $\det(h_t) = \det(h_\infty)$.

THEOREM 7.5. *On any relatively compact open subset $K \subset X \setminus D$, the sequence h_t is convergent to h_∞ in the C^∞ -sense.*

7.1.2 The case of locally and globally irreducible Higgs bundles We state Theorem 7.5 in a special case for clarification (see also Remark 7.2). Let $(E, \bar{\partial}_E, \theta)$ be a generically regular semisimple Higgs bundle of degree 0 on X . Let $\Sigma_{E,\theta}$ denote the spectral curve. There exists the finite subset $D(E, \theta) \subset X$ such that the following holds:

- $P \in D(E, \theta)$ if and only if $|T_P^*X \cap \Sigma_{E,\theta}| < r$.

We impose the following condition.

CONDITION 7.6.

- The spectral curve $\Sigma_{E,\theta}$ is irreducible, i.e. $\Sigma_{E,\theta} \setminus \pi^{-1}(D(E, \theta))$ is connected.
- For any $P \in D(E, \theta)$, there exist a neighbourhood X_P of P in X and a decomposition

$$(E, \theta)|_{X_P} = \bigoplus_{i \in S(P)} (E_{P,i}, \theta_{P,i}), \tag{49}$$

such that the spectral curves $\Sigma_{E_{P,i}, \theta_{P,i}}$ are irreducible.

We set $D = D(E, \theta)$. Let $\mathcal{P}_*^{(0)}(E(*D)_P)$ be the filtered bundle over $E(*D)_P$ defined by $\mathcal{P}_a^{(0)}(E(*D)_P) = E_P([a]P)$, where $[a] = \max\{n \in \mathbb{Z} \mid n \leq a\}$. Because there exists the decomposition

$$\mathcal{P}_*^{(0)}(E(*D)_P) = \bigoplus_{i \in S(P)} \mathcal{P}_*^{(0)}(E_{P,i}(*D)_P),$$

induced by (49), we obtain the filtered bundle $\mathcal{P}_*^{(0)}(E(*D)_P)$ determined by $\det(\mathcal{P}_*^{(0)}E_{P,i}(*D)_P)$ as in Proposition 5.13. By patching them with $(E(*D), \theta)$, we obtain a filtered bundle $\mathcal{P}_*\mathcal{V}$ over $\mathcal{V} = E(*D)$. The filtered Higgs bundle $(\mathcal{P}_*^{(0)}(E(*D)), \theta)$ is decomposable.

As in Lemma 7.4, there exists a decoupled harmonic metric h_∞ of $(E, \theta)|_{X \setminus D}$ such that h_∞ is adapted to $\mathcal{P}_*\mathcal{V}$. For any $t > 0$, there exists a unique harmonic metric h_t of $(E, t\theta)$ such that $\det(h_t) = \det(h_\infty)$. As a special case of Theorem 7.5, we obtain the following.

COROLLARY 7.7. *On any relatively compact open subset $K \subset X \setminus D$, the sequence h_t is convergent to h_∞ in the C^∞ -sense.*

Remark 7.8. The second condition in Condition 7.6 is satisfied if $\Sigma_{E,\theta}$ is locally irreducible.

7.1.3 Proof of Theorem 7.5 Let $P \in D$. We set $X_P^* = X_P \setminus \{P\}$. We set $V_{P,i} = \mathcal{V}_{P,i}|_{X_P^*}$ and $r(P, i) = \text{rank } V_{P,i}$. Let z_P be a holomorphic coordinate of X_P by which $X_P \simeq \{z \in \mathbb{C} \mid |z| < 2\}$.

We set $(h_\infty)_{P,i} := h_{\infty|V_{P,i}}$. Let $h_{t,P,i}$ be a harmonic metric of $(V_{P,i}, t\theta_{P,i})$ such that (i) the boundary value at $|z_P|=1$ is equal to that of $(h_\infty)_{P,i}$ and (ii) $h_{t,P,i}$ is adapted to $\mathcal{P}_*\mathcal{V}_{P,i}$. We have $\det(h_{t,P,i}) = \det((h_\infty)_{P,i})$. We obtain the following lemma by Proposition 6.6.

LEMMA 7.9. *The sequence $h_{t,P,i}$ is convergent to $(h_\infty)_{P,i}$ as $t \rightarrow \infty$ in the C^∞ -sense on any relatively compact open subset of X_P^* .*

We regard X_P as an open subset of \mathbb{C} by the coordinate z_P . Let $\varphi_{P,r(P,i)} : \mathbb{C} \rightarrow \mathbb{C}$ be defined by $\varphi_{P,r(P,i)}(\zeta_{P,i}) = \zeta_{P,i}^{r(P,i)}$. We set $X_P^{(r(P,i))} = \varphi_{P,r(P,i)}^{-1}(X_P)$ and $X_P^{(r(P,i))*} = \varphi_{P,r(P,i)}^{-1}(X_P^*)$. The induced maps $X_P^{(r(P,i))} \rightarrow X_P$ and $X_P^{(r(P,i))*} \rightarrow X_P^*$ are also denoted by $\varphi_{P,r(P,i)}$.

We define a Hermitian product $h_{t,P,i}^{(r(P,i))}$ of $\varphi_{P,r(P,i)}^*(V_{P,i})|_{X_P^{(r(P,i))*}}$ as follows. We have the decomposition

$$\varphi_{P,r(P,i)}^*(V_{P,i}, \theta_{P,i})|_{X_P^{(r(P,i))*}} = \bigoplus_{p=1}^{r(P,i)} (V_{P,i,\beta(p)}, \beta(p) d\zeta_{P,i}),$$

where $\beta(p)$ are meromorphic functions on $X_P^{(r(P,i))}$. Let $v_{\beta(1)}$ be a holomorphic frame of $V_{P,i,\beta(1)}$. We obtain a frame $v_{\sigma^*\beta(1)} = \sigma^*(v_{\beta(1)})$ of $V_{P,i,\sigma^*\beta(1)}$. Let $\chi(\zeta_{P,i})$ be an $\mathbb{R}_{\geq 0}$ -valued function such that (i) $\chi(\zeta_{P,i})$ depends only on $|\zeta_{P,i}|$ and (ii) $\chi(\zeta_{P,i}) = 1$ ($|\zeta_{P,i}| \leq 1/2$), $\chi(\zeta_{P,i}) = 0$ ($|\zeta_{P,i}| \geq 2/3$). For $p \neq q$, we put

$$h_{t,P,i}^{(r(P,i))}(v_{\beta(p)}, v_{\beta(q)}) = \chi(\zeta_{P,i}) \varphi_{P,r(P,i)}^*(h_{t,P,i})(v_{\beta(p)}, v_{\beta(q)}).$$

We define $h_{t,P,i}^{(r(P,i))}(v_{\beta(p)}, v_{\beta(p)})$ by

$$\begin{aligned} \log h_{t,P,i}^{(r(P,i))}(v_{\beta(p)}, v_{\beta(p)}) &= \chi(\zeta_{P,i}) \log \varphi_{P,r(P,i)}^*(h_{t,P,i})(v_{\beta(p)}, v_{\beta(p)}) \\ &\quad + (1 - \chi(\zeta_{P,i})) \log \varphi_{P,r(P,i)}^*((h_\infty)_{P,i}^{(r(P,i))})(v_{\beta(p)}, v_{\beta(p)}). \end{aligned} \tag{50}$$

Then, $h_{t,P,i}^{(r(P,i))}$ is $\text{Gal}(r(P,i))$ -invariant, and we have $h_{t,P,i}^{(r(P,i))} = \varphi_{P,r(P,i)}^{-1}(h_{t,P,i})$ on $\{0 < |\zeta_{P,i}| < 1/4\}$ and $h_{t,P,i}^{(r(P,i))} = \varphi_{P,r(P,i)}^{-1}((h_\infty)_{P,i})$ on $\{4/5 < |\zeta_{P,i}|\}$. There exists a Hermitian metric $\tilde{h}_{t,P,i}$ of $V_{P,i}$ such that $\varphi_{P,r(P,i)}^{-1}(\tilde{h}_{t,P,i}) = h_{t,P,i}^{(r(P,i))}$ on $X_P^{(r(P,i))*}$. We obtain a Hermitian metric

$$\tilde{h}_{t,P} = \bigoplus_{i \in S(P)} \tilde{h}_{t,P,i},$$

of $V|_{X_P^*}$. By patching $\tilde{h}_{t,P}$ and h_∞ , we obtain Hermitian metrics \tilde{h}'_t of V . We obtain the C^∞ -function α_t on $X \setminus D$ determined by $\det(\tilde{h}'_t) = e^{\alpha_t} \det(h_\infty)$. We set $\tilde{h}_t = e^{-\alpha_t/r} \tilde{h}'_t$. By the construction, the following lemma is clear.

LEMMA 7.10. *There exists t_0 such that \tilde{h}_t is positive definite for any $t \geq t_0$. Moreover, the following holds.*

- The sequence \tilde{h}_t is convergent to h_∞ in the C^∞ -sense on any relatively compact open subset of $X \setminus D$. The support of $R(\tilde{h}_t) + [t\theta, (t\theta)^\dagger_{\tilde{h}_t}]$ is contained in $\{(\frac{1}{4})^{\text{rank}(E)} \leq |z_P| \leq \frac{4}{5}\}$ for $P \in D$. In particular,

$$\int_X \left| R(\tilde{h}_t) + [t\theta, (t\theta)^\dagger_{\tilde{h}_t}] \right|_{\tilde{h}_t, g_X} \rightarrow 0, \tag{51}$$

as $t \rightarrow \infty$.

Let g_X be a Kähler metric of X . Let s_t denote the automorphism of V determined by $h_t = \tilde{h}_t s_t$. We have $\det(s_t) = 1$. According to [Sim88, Lemma 3.1], we obtain the following on $X \setminus D$:

$$\Delta_{g_X} \text{Tr}(s_t) = \text{Tr} \left((R(\tilde{h}_t) + [t\theta, (t\theta)^\dagger]_{\tilde{h}_t}) s_t \right) - |\bar{\partial}(s_t) s_t^{-1/2}|_{\tilde{h}_t, g_X}^2 - |[t\theta, s_t] s^{-1/2}|_{\tilde{h}_t, g_X}^2. \tag{52}$$

Note that $\bigoplus_{i \in S(P)} h_{t,P,i}$ and $h_t|_{X_P^*}$ are mutually bounded for any $P \in D$. Hence, $\text{Tr}(s_t)$ is bounded. We also note the following vanishing (see Lemma [Moc21, Lemma 4.7]):

$$\int_X \Delta_X \text{Tr}(s_t) \text{dvol}_{g_X} = 0. \tag{53}$$

We set $b_t = \sup_{X \setminus D} \text{Tr}(s_t)$. Note that $b_t \geq \text{rank}(E)$, and $b_t = \text{rank}(E)$ if and only if $s_t = \text{id}_E$. We set $u_t = b_t^{-1} \cdot s_t$. There exists $C > 0$, which is independent of t such that $|u_t|_{\tilde{h}_t} \leq C$. By (51), (52) and (53) we obtain

$$\int_X \left(|\bar{\partial} u_t|_{\tilde{h}_t}^2 + |[t\theta, u_t]|_{\tilde{h}_t}^2 \right) \rightarrow 0,$$

as $t \rightarrow \infty$.

Let $t(i) > 0$ be a sequence such that $t(i) \rightarrow \infty$ as $i \rightarrow \infty$. By going to a subsequence, $u_{t(i)}$ is weakly convergent in L^2_1 locally on $X \setminus D$. In particular, it is convergent in L^q for any $q \geq 1$ locally on $X \setminus D$. Let u_∞ denote the limit which satisfies $\bar{\partial} u_\infty = [\theta, u_\infty] = 0$.

LEMMA 7.11. We have $u_\infty \neq 0$.

Proof. Note that $\sup_X \text{Tr}(u_{t(i)}) = 1$ for any i . Let $0 < \epsilon < 1$. Let $P_i \in X$ be points such that $\text{Tr}(u_{t(i)})(P_i) \geq \epsilon$. By going to a subsequence, we may assume that the sequence is convergent to a point P_∞ . Let us consider the case where

$$P_\infty \notin \bigcup_{P \in D} \{|z_P| \leq 4/5\} =: W.$$

Let (X_{P_∞}, z) be a holomorphic coordinate neighbourhood around P_∞ , which does not intersect with W . Because $F(\tilde{h}_t) = 0$ on X_{P_∞} , we obtain $\Delta_{g_X} \text{Tr}(u_t) \leq 0$. By the mean value property of the subharmonic functions, there exists $C > 0$ such that

$$C\epsilon \leq \int_{X_{P_\infty}} \text{Tr}(u_{t(i)}).$$

Because $u_{t(i)}$ is convergent to u_∞ in L^p for any $p \geq 1$ on X_{P_∞} , we obtain that $u_\infty \neq 0$.

Let us consider the case where $P_\infty \in \{|z_P| < 4/5\}$ for some $P \in D$. Let (X_P, z_P) be a holomorphic coordinate neighbourhood around P as above. By [Sim88, Lemma 3.1], we have

$$\Delta_{g_X} \log \text{Tr}(u_{t(i)}) \leq \left| R(\tilde{h}_{t(i)}) + [t\theta, (t\theta)^\dagger]_{\tilde{h}_{t(i)}} \right|_{\tilde{h}_{t(i)}, g_X}.$$

There exist C^∞ -functions α_i on X_P such that (i) $\Delta_{g_X} \alpha_i = \left| R(\tilde{h}_{t(i)}) + [t\theta, (t\theta)^\dagger]_{\tilde{h}_{t(i)}} \right|_{\tilde{h}_{t(i)}, g_X}$, (ii) $\alpha_i|_{\partial X_P} = 0$ and (iii) there exists $C > 0$ such that $|\alpha_i| \leq C$ for any i . Because $\log \text{Tr}(u_{t(i)}) - \alpha_i$ is a subharmonic function on X_P , the maximum principle allows us to obtain

$$\log \epsilon - C \leq \max_{P \in \partial X_P} \{\log \text{Tr}(u_{t(i)}) - \alpha_i\} = \max_{P \in \partial X_P} \{\log \text{Tr}(u_{t(i)})\}.$$

Hence, there exists a sequence $P'_i \in \partial X_P$ such that $\text{Tr}(u_{t(i)})(P'_i) \geq \epsilon e^{-C}$. By going to a subsequence, we may assume that the sequence P'_i is convergent to $P'_\infty \in X \setminus W$. Then, we can apply the result in the first part of this proof. \square

Recall that $u_\infty \neq 0$ is an endomorphism of (V, θ) such that $\bar{\partial}u_\infty = [\theta, u_\infty] = 0$. At each point of $X \setminus D$, an eigenspace of θ is also an eigenspace of u_∞ . Because each $u_{t(i)}$ is self-adjoint with respect to \tilde{h}_t , u_∞ is self-adjoint with respect to h_∞ . We obtain $\partial_{h_\infty} u_\infty = 0$. Hence, the eigenvalues of u_∞ are constant. Because $\tilde{h}_t(u_{t(i)}v, v) \geq 0$ for any local section v of V , we obtain $h_\infty(u_\infty v, v) \geq 0$, which implies that the eigenvalues of u_∞ are non-negative. We also note that $\Sigma_{V,\theta}$ is connected. Hence, u_∞ is a positive constant multiplication. This implies that the sequence b_t is bounded, and that the subsequence $s_{t(i)}$ is convergent to a positive constant multiplication. Because $\det(s_t) = 1$, the limit is the identity. Because this is independent of the choice of a subsequence, we obtain the desired convergence. \square

7.2 Order of convergence in a smooth case

7.2.1 Rough statement Let us study the order of the convergence in the situation of §7.1.2 assuming the following stronger condition.

CONDITION 7.12. Let $\rho: \tilde{\Sigma}_{E,\theta} \rightarrow \Sigma_{E,\theta}$ be the normalization. There exists a line bundle L on $\tilde{\Sigma}_{E,\theta}$ with an isomorphism $(\pi \circ \rho)_* L \simeq E$ such that θ is induced by the \mathcal{O}_{T^*X} -action on $\rho_* L$.

Let $g(\tilde{\Sigma}_{E,\theta})$ and $g(X)$ denote the genus of $\tilde{\Sigma}_{E,\theta}$ and X , respectively. Then, we have $\deg(L) = g(\tilde{\Sigma}_{E,\theta}) - rg(X) + r - 1$.

Remark 7.13. If Condition 7.12 is satisfied, Condition 7.6 is also satisfied. Condition 7.12 is satisfied if $\Sigma_{E,\theta}$ is smooth and connected.

We set $(V, \theta) = (E, \theta)|_{X \setminus D}$. Let $s(h_\infty, h_t)$ be the automorphism of V determined by $h_t = h_\infty \cdot s(h_\infty, h_t)$. Let g_X be a Kähler metric of X .

THEOREM 7.14. For any relatively compact open subset K of $X \setminus D$ and a non-negative integer ℓ , there exist positive constants $C(K, \ell)$ and $\epsilon(K, \ell)$ such that the L^2_ℓ -norms of $s(h_\infty, h_t) - \text{id}_E$ on K with respect to h_∞, g_X and the Chern connection of h_∞ are dominated by $C(K, \ell)e^{-\epsilon(K, \ell)t}$.

7.2.2 Refined statement We shall prove a refined statement. For that purpose, we refine the construction of \tilde{h}_t in the proof of Theorem 7.5. Let $P \in D$ and $i \in S(P)$.

LEMMA 7.15. The metric $\det((h_\infty)_{P,i})$ induces a flat metric of $\det(E_{P,i})$.

Proof. The lemma follows from the condition that $\det((h_\infty)_{P,i})$ is adapted to $\det \mathcal{P}_*(E_{P,i}(*D)_P) = \det \mathcal{P}_*^{(0)}(E_{P,i}(*D)_P)$. \square

According to Proposition 6.11, there exists a non-degenerate symmetric pairing $C_{P,i}$ of $(E_{P,i}, \theta_{P,i})$ such that $C_{P,i}|_{X_P^*}$ is compatible with $(h_\infty)_{P,i}$. For $t > 0$, there exists a harmonic metric $h_{t,P,i}$ of $(E_{P,i}, \theta_{P,i})$ which is compatible with $C_{P,i}$ such that its boundary value at ∂X_P is equal to that of $h_\infty|_{E_{P,i}}$. We construct the metric \tilde{h}_t by using $h_{t,P,i}$ as in the proof of Theorem 7.5 (see §7.1.3). By Proposition 6.7, the following holds.

LEMMA 7.16. Let $s(h_\infty, \tilde{h}_t)$ be the automorphism of $E|_{X \setminus D}$ determined by $\tilde{h}_t = h_\infty \cdot s(h_\infty, \tilde{h}_t)$. For any relatively compact open subset K of X_P^* and for any $\ell \in \mathbb{Z}_{\geq 0}$, there exist constants $C(K, \ell), \epsilon(K, \ell), t(K) > 0$ such that the L^2_ℓ -norms of $s(h_\infty, \tilde{h}_t) - \text{id}$ on K with respect to h_∞ and g_X are dominated by $C(K, \ell) \exp(-\epsilon(K, \ell)t)$ for any $t > t(K)$.

By Lemma 7.16, we obtain

$$|R(\tilde{h}_t) + [t\theta, (t\theta)^\dagger]_{\tilde{h}_t, g_X} \leq C e^{-\epsilon t}. \tag{54}$$

for some $\epsilon, C > 0$. Moreover, by the construction, the support of (54) is contained in

$$\bigcup_{P \in D} \{(1/4)^{\text{rank}(E)} \leq |z_P| \leq 4/5\}.$$

Let s_t be the automorphism of E determined by $h_t = \tilde{h}_t \cdot s_t$. We obtain Theorem 7.14 from Lemma 7.16 and the following theorem.

THEOREM 7.17. *For any $\ell > 0$, there exist $C(\ell), \epsilon(\ell) > 0$ such that the L^2_ℓ -norms of $s_t - \text{id}$ on X with respect to \tilde{h}_t, g_X and the Chern connection of \tilde{h}_t are dominated by $C(\ell) \exp(-\epsilon(\ell)t)$.*

Proof. By [Sim88, Lemma 3.1] and (54), there exist $C_1, \epsilon_1 > 0$ such that

$$\int_X \left(|s_t^{-1/2} \partial_{E, \tilde{h}_t}(s_t)|_{\tilde{h}_t}^2 + |[\theta, s_t] s_t^{-1/2}|_{\tilde{h}_t}^2 \right) \leq C_1 \exp(-\epsilon_1 t).$$

By Corollary 7.7, $|s_t|_{\tilde{h}_t}$ and $|s_t^{-1}|_{\tilde{h}_t}$ are uniformly bounded. There exist $C_2, \epsilon_2 > 0$ such that

$$\int_X \left(|\partial_{E, \tilde{h}_t}(s_t)|_{\tilde{h}_t}^2 + |[\theta, s_t]|_{\tilde{h}_t}^2 \right) \leq C_2 \exp(-\epsilon_2 t). \tag{55}$$

Let K be a relatively compact open subset of $X \setminus D$. By the variant of Simpson’s main estimate ([Moc16, Theorem 2.9]) and Lemma 7.16, there exist $C_3(K), \epsilon_3(K) > 0$ such that the following holds on K :

$$|\bar{\partial}_E(s_t^{-1} \partial_{E, \tilde{h}_t}(s_t))|_{\tilde{h}_t}^2 \leq C_3(K) \exp(-\epsilon_3(K)t).$$

Together with (55), we obtain that there exist $C_4(K), \epsilon_4(K) > 0$ such that the following holds on K :

$$|\partial_{E, \tilde{h}_t}(s_t)|_{\tilde{h}_t} \leq C_4(K) \exp(-\epsilon_4(K)t). \tag{56}$$

Because s_t is self-adjoint with respect to \tilde{h}_t , we obtain the following on K :

$$|\bar{\partial}(s_t)|_{\tilde{h}_t} \leq C_4(K) \exp(-\epsilon_4(K)t). \tag{57}$$

LEMMA 7.18. *There exist $C(K), \epsilon(K) > 0$ such that the following holds on K :*

$$|s_t - \text{id}|_{\tilde{h}_t} \leq C(K) \exp(-\epsilon(K)t).$$

Proof. Let P be any point of $X \setminus D$. Let X_P be a simply connected neighbourhood of P in $X \setminus D$. There exists a decomposition into Higgs bundles of rank 1:

$$(E, \theta)|_{X_P} = \bigoplus_{i=1}^{\text{rank}(E)} (E_{P,i}, \theta_{P,i}).$$

We obtain the decomposition $s_t = \sum (s_t)_{j,i}$, where $(s_t)_{j,i} : E_{P,i} \rightarrow E_{P,j}$. By [Moc16, Proposition 2.3], there exist $C_5(P), \epsilon_5(P) > 0$ such that the following for $i \neq j$ on X_P :

$$|(s_t)_{j,i}|_{\tilde{h}_t} \leq C_5(P) \exp(-\epsilon_5(P)t). \tag{58}$$

By (56) and (57), there exist $C_6(P), \epsilon_6(P) > 0$ such that

$$|d(s_t)_{i,i}| \leq C_6(P) \exp(-\epsilon_6(P)t).$$

Hence, there exist $C_7(P), \epsilon_7(P) > 0$ such that the following holds for any $P_1, P_2 \in X_P$:

$$|(s_t)_{i,i}(P_1) - (s_t)_{i,i}(P_2)| \leq C_7(P) \exp(-\epsilon_7(P)t).$$

Let $i \neq j$. There exists a loop γ in $X \setminus D$ such that the monodromy of $\Sigma_{E,\theta}$ along γ exchanges E_i and E_j . By taking a finite covering of γ by relatively compact open subsets and by applying the above consideration, we obtain that there exist $C_8(P), \epsilon_8(P) > 0$ such that the following holds for any $P_1 \in X_P$:

$$|(s_t)_{i,i}(P_1) - (s_t)_{j,j}(P_1)| \leq C_8(P) \exp(-\epsilon_8(P)t). \tag{59}$$

By (58), there exist $C_9(P), \epsilon_9(P) > 0$ such that the following holds on X_P :

$$\left| \prod_{i=1}^{\text{rank}(E)} (s_t)_{i,i} - 1 \right| \leq C_9(P) \exp(-\epsilon_9(P)t). \tag{60}$$

By (59) and (60), there exist $C_{10}(P), \epsilon_{10}(P) > 0$ such that

$$|(s_t)_{i,i} - 1| \leq C_{10}(P) \exp(-\epsilon_{10}(P)t).$$

Then, we obtain the claim of Lemma 7.18. □

We obtain the estimate of $|s_t - \text{id}|_{\tilde{h}_t}$ around D by using Theorem 4.3. We can also obtain the estimate for the higher derivatives by using Theorem 4.3. □

7.3 A family case

7.3.1 Setting Let \mathcal{S} be a connected complex manifold. Let \mathcal{Y} be a complex manifold with a proper smooth morphism $p_1 : \mathcal{Y} \rightarrow \mathcal{S}$. Let $p_2 : \mathcal{S} \times X \rightarrow \mathcal{S}$ and $\pi_2 : \mathcal{S} \times T^*X \rightarrow \mathcal{S} \times X$ denote the projections. Let $\Phi_0 : \mathcal{Y} \rightarrow \mathcal{S} \times T^*X$ be a holomorphic map such that $p_1 = p_2 \circ \pi_2 \circ \Phi_0$. We set $\Phi_1 = \pi_2 \circ \Phi_0$. We assume the following conditions:

- each fiber of p_1 is connected and one-dimensional;
- Φ_1 is proper and finite;
- there exists a closed complex analytic hypersurface $\mathcal{D} \subset \mathcal{S} \times X$ such that (i) \mathcal{D} is finite over \mathcal{S} , (ii) the induced map $\mathcal{Y} \setminus \Phi_1^{-1}(\mathcal{D}) \rightarrow (\mathcal{S} \times X) \setminus \mathcal{D}$ is a covering map and (iii) Φ_0 induces an injection $\mathcal{Y} \setminus \Phi_1^{-1}(\mathcal{D}) \rightarrow \mathcal{S} \times T^*X$.

We set $r := |\Phi_1^{-1}(P)|$ for any $P \in (\mathcal{S} \times X) \setminus \mathcal{D}$. We set $\tilde{\mathcal{D}} := \Phi_1^{-1}(\mathcal{D})$. For any $x \in \mathcal{S}$, we set $\mathcal{Y}_x := p_1^{-1}(x)$, $\tilde{\mathcal{D}}_x := \mathcal{Y}_x \cap \tilde{\mathcal{D}}$ and $\mathcal{D}_x := p_2^{-1}(x) \cap \mathcal{D}$. Let $g(X)$ denote the genus of X . Let \tilde{g} denote the genus of \mathcal{Y}_x , which is independent of $x \in \mathcal{S}$.

Let \mathcal{L} be a line bundle on \mathcal{Y} such that

$$\text{deg}(\mathcal{L}|_{\mathcal{Y}_x}) = \tilde{g} - rg(X) + r - 1.$$

We obtain the locally free $\mathcal{O}_{\mathcal{S} \times X}$ -module $\mathcal{E} = \Phi_{1*}\mathcal{L}$. It is equipped with the relative Higgs field

$$\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{\mathcal{S} \times X/\mathcal{S}}^1,$$

induced by the $\mathcal{O}_{\mathcal{S} \times T^*X}$ -action on $\Phi_{0*}\mathcal{L}$. For any $x \in \mathcal{S}$, let $(\mathcal{E}_x, \theta_x)$ be the induced Higgs bundle on $X \simeq \{x\} \times X$. We obtain the following lemma by the construction.

LEMMA 7.19. *Each $(\mathcal{E}_x, \theta_x)$ is stable of degree 0.*

7.3.2 Statement We obtain the holomorphic line bundle $\det(\mathcal{E})$ on $\mathcal{S} \times X$. There exists a C^∞ -Hermitian metric $h_{\det(\mathcal{E})}$ of $\det(\mathcal{E})$ such that $h_{\det(\mathcal{E}),x} := h_{\det(\mathcal{E})|_{\{x\} \times X}}$ is flat for any $x \in \mathcal{S}$.

We have the decomposable filtered Higgs bundle $(\mathcal{P}_*^* \mathcal{E}_x, \theta_x)$ on (X, \mathcal{D}_x) . Let $h_{\infty,x}$ be the decoupled harmonic metric of $(\mathcal{E}_x, \theta_x)|_{X \setminus \mathcal{D}_x}$ such that $\det(h_{\infty,x}) = h_{\det(\mathcal{E}),x}$.

LEMMA 7.20. The metrics $h_{\infty,x}$ ($x \in \mathcal{S}$) induce a C^∞ -metric of $\mathcal{E}|_{(\mathcal{S} \times X) \setminus \mathcal{D}}$.

Proof. It is enough to study locally around any point $x_0 \in \mathcal{S}$. By using the examples in §6.3.3, we can construct a C^∞ -Hermitian metric h_0 of $\mathcal{L}|_{\mathcal{Y} \setminus \tilde{\mathcal{D}}}$ such that (i) h_0 is flat around $\tilde{\mathcal{D}}$ and (ii) $h_0|_{\mathcal{Y}_x \setminus \tilde{\mathcal{D}}_x}$ is adapted to $\mathcal{P}_*^*(\mathcal{L}|_{\mathcal{Y}_x})$. By using Lemma 7.23 below, we can construct a C^∞ -function f on \mathcal{Y} such that $h_{1,x} = e^f h_{0,x}$ ($x \in \mathcal{S}$) is a family of flat metrics $\mathcal{L}|_{\mathcal{Y}_x \setminus \tilde{\mathcal{D}}_x}$. It induces a family of decoupled harmonic metrics $h_{2,x}$ of $(\mathcal{E}_x, \theta_x)|_{X \setminus \mathcal{D}_x}$ such that they give a C^∞ -Hermitian metric h_2 of $\mathcal{E}|_{(\mathcal{S} \times X) \setminus \mathcal{D}}$. Note that $\det(h_{2,x})$ induces a flat metric of $\det(\mathcal{E}_x, \theta_x)$. For each $x \in \mathcal{S}$, because both $\det(h_{2,x})$ and $h_{\det(\mathcal{E}),x}$ are flat metrics of $\det(\mathcal{E}_x)$, we obtain that $\alpha_x > 0$ determined by $\det(h_{2,x}) = \alpha_x h_{\det(\mathcal{E}),x}$. Because $\det(h_{2,x})$ ($x \in \mathcal{S}$) give a C^∞ -metric of $\det(\mathcal{E})|_{(\mathcal{S} \times X) \setminus \mathcal{D}}$, we obtain that α_x ($x \in \mathcal{S}$) give a C^∞ -function on \mathcal{S} . Because $h_{\infty,x} = e^{-\alpha_x/r} h_{2,x}$, we obtain that $h_{\infty,x}$ induces a C^∞ -metric of $\mathcal{E}|_{(\mathcal{S} \times X) \setminus \mathcal{D}}$. \square

Let $h_{t,x}$ be a harmonic metric of $(\mathcal{E}_x, t\theta_x)$ such that $\det(h_{t,x}) = h_{\det(\mathcal{E}),x}$. Let $(V_x, \theta_x) := (\mathcal{E}_x, \theta_x)|_{X \setminus \mathcal{D}_x}$. We obtain the automorphism $s(h_{\infty,x}, h_{t,x})$ of V_x determined by $h_{t,x} = h_{\infty,x} \cdot s(h_{\infty,x}, h_{t,x})$.

THEOREM 7.21. Let $x_0 \in \mathcal{S}$. Let K be any relatively compact open subset in $X \setminus \mathcal{D}_{x_0}$. Let \mathcal{S}_0 be a neighbourhood of x_0 such that $\mathcal{S}_0 \times K$ is relatively compact in $(\mathcal{S} \times X) \setminus \mathcal{D}$. For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C(\ell, K)$ and $\epsilon(\ell, K)$ such that the L_ℓ^2 -norms of $s(h_{\infty,x}, h_{t,x}) - \text{id}$ ($x \in \mathcal{S}_0, t \geq 1$) on K with respect to $h_{\infty,x}, g_X$ and the Chern connection of $h_{\infty,x}$ are dominated by $C(\ell, K) \exp(-\epsilon(\ell, K)t)$.

7.3.3 Refined statement Let $x_0 \in \mathcal{S}$. For any $P \in \mathcal{D}_{x_0}$, let (U_P, z_P) be a simply connected holomorphic coordinate neighbourhood of P in X such that $U_P \cap \mathcal{D}_{x_0} = \{P\}$ and that z_P induces $(U_P, P) \simeq (B(2), 0)$. Moreover, we assume that z_P induces a holomorphic isomorphism between neighbourhoods of the closures of U_P and $B(2)$. Let $\mathcal{S}_{1,P}$ be a relatively compact open neighbourhood of x_0 in \mathcal{S} such that

$$\mathcal{D} \cap (\mathcal{S}_{1,P} \times U_P) \subset \mathcal{S}_{1,P} \times \{|z_P| \leq (1/4)^{\text{rank } E}\}.$$

Let \mathcal{S}_1 be a connected open neighbourhood of x_0 in $\bigcap_{P \in \mathcal{D}_{x_0}} \mathcal{S}_{1,P}$.

For $P \in \mathcal{D}_{x_0}$ and $x \in \mathcal{S}_1$, let $h_{t,P,x}$ be the harmonic metric of $(\mathcal{E}_x, \theta_x)|_{\{|z_P| < 1\}}$ such that $h_{t,P,x}|_{\{|z_P|=1\}} = h_{\infty,x}|_{\{|z_P|=1\}}$. We note that Condition 6.17 is satisfied for $h_{\infty,x}|_{U_P}$ by Lemma 7.15, and we can apply Proposition 6.18 to $h_{t,P,x}$. We construct Hermitian metrics $\tilde{h}_{t,x}$ of \mathcal{E}_x ($x \in \mathcal{S}_1$) from $h_{\infty,x}$ and $h_{t,P,x}$ ($P \in \mathcal{D}_{x_0}$) as in §7.1.3. Let $s(\tilde{h}_{t,x}, h_{t,x})$ be the automorphism of \mathcal{E}_x determined by $h_{t,x} = \tilde{h}_{t,x} \cdot s(\tilde{h}_{t,x}, h_{t,x})$. By using Proposition 6.18, we obtain the following theorem in the same way as Theorem 7.14, which implies Theorem 7.21.

THEOREM 7.22. For any $\ell \in \mathbb{Z}_{\geq 0}$, there exist positive constants $C(\ell)$ and $\epsilon(\ell)$ such that the L_ℓ^2 -norms of

$$s(\tilde{h}_{t,x}, h_{t,x}) - \text{id} \quad (x \in \mathcal{S}_1, t \geq 1),$$

with respect to $\tilde{h}_{t,x}, g_X$ and $\tilde{h}_{t,x}$ are dominated by $C(\ell) \exp(-\epsilon(\ell)t)$.

7.3.4 *Appendix* Let M be a compact oriented C^∞ -manifold. Let S be a C^∞ -manifold. Let $g_{S \times M}$ be a Riemannian metric of $S \times M$. For each $x \in S$, we set $M_x := \{x\} \times M$. Let g_x and Δ_x denote the induced Riemannian metric and the associated Laplacian of M_x .

LEMMA 7.23. *Let f_1 be a C^∞ -function on $S \times M$ such that $\int_{M_x} f_1 \, d\text{vol}_{g_x} = 0$. Let f_2 be a function on $S \times M$ determined by the conditions $\Delta_x(f_{2|M_x}) = f_{1|M_x}$ and $\int_{M_x} f_{2|M_x} \, d\text{vol}_{g_x} = 0$. Then, f_2 is C^∞ .*

Proof. We explain only a sketch of a proof. For any $x \in S$, let $f_{i,x} := f_{i|M_x}$. Let S_0 be a relatively compact open subset in S . There exists a uniform lower bound of the first non-zero eigenvalue of the operators Δ_x ($x \in S_0$) (see [Lil2, Theorem 5.7]). There exists $C_0 > 0$ such that $\|f_{1,x}\|_{L^2} \leq C_0$ ($x \in S_0$). By $\Delta_x(f_{1,x}) = f_{2,x}$, for any $\ell \in \mathbb{Z}_{\geq 0}$ there exists $C_1(\ell) > 0$ such that $\|f_{1,x}\|_{L^2_\ell} \leq C_1(\ell)$ for any $x \in S_0$. Let $x(i) \in S_0$ be a sequence convergent to $x(\infty) \in S_0$. There exists a subsequence $x'(j)$ convergent to $x(\infty)$ such that the sequence $f_{1,x'(j)}$ is weakly convergent in L^2_ℓ for any $\ell \in \mathbb{Z}_{\geq 0}$. The limit f_∞ satisfies $\Delta(f_\infty) = f_{2,x(\infty)}$ and $\int_{M_{x(\infty)}} f_\infty \, d\text{vol}_{g_{x(\infty)}} = 0$. We obtain $f_\infty = f_{1,x(\infty)}$. Hence, $f_{1,x}$ and their derivatives in the M -direction are continuous with respect to $x \in S$.

Let S_1 be a relatively compact open subset of S equipped with a real coordinate system (x_1, \dots, x_n) . Let $[\partial_j, \Delta_x]$ be the differential operator on $S_1 \times M$ defined by $[\partial_j, \Delta_x](f) = \partial_j(\Delta_x(f)) - \Delta_x(\partial_j f)$. It does not contain a derivative in the S_1 -direction. Note that $[\partial_j, \Delta_x](f_{1,x})$ and their derivative in the M -direction are continuous with respect to $x \in S_1$. Let $f_{1,x}^{(j)}$ be the solution of the conditions $\Delta_x(f_{1,x}^{(j)}) = \partial_j f_{2,x} - [\partial_j, \Delta_x]f_{1,x}$ and $\int_{M_x} f_{1,x}^{(j)} \, d\text{vol}_{g_x} = 0$. Choose $y = (y_1, \dots, y_n) \in S_1$. We define functions $F_x^{(j)}$ on M_x by $F_x^{(j)} = (x_j - y_j)^{-1}(f_{1,x} - f_{1,y})$ if $x_j \neq y_j$, and $F_x^{(j)} = f_{1,x}^{(j)}$ if $x_j = y_j$. They satisfy $\Delta_x(F_x^{(j)}) = (x_j - y_j)^{-1}(f_{2,x} - f_{2,y} - (\Delta_x - \Delta_y)f_{1,y})$ if $x_j \neq y_j$ and $\Delta_x(F_x^{(j)}) = \partial_j f_{2,x} - [\partial_j, \Delta_x]f_{1,x}$ if $x_j = y_j$. Then, by an argument in the previous paragraph, we can prove that $F_x^{(j)}$ and their derivatives in the M -direction are continuous with respect to x . This implies that $f_{1,x}$ is C^1 -with respect to x and that $\partial_j f_{1,x} = f_{1,x}^{(j)}$. By a similar argument, we can prove that $f_{1,x}$ and their derivatives in the M -direction are C^∞ with respect to x . □

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CONFLICTS OF INTEREST

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