Variations of the Vecten configurations

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We present some less known variations of the the Vecten configuration and give purely geometric proofs for them. It is unlikely that these variations (and even proofs?) are new, probably just well-hidden in the literature. If a reader happens to know references for the variations discussed (or other geometric proofs), please let the authors know. At [1] the reader can find a dynamic webpage on our topic.

Theorem 1: Let $\triangle ABC$ be an arbitrary triangle with outwardly erected squares. Then the lines connecting the midpoints G, H, I of the line segments UT, QV, RS with the centres D, E, F of the remote squares are concurrent at point P (Figure 1).

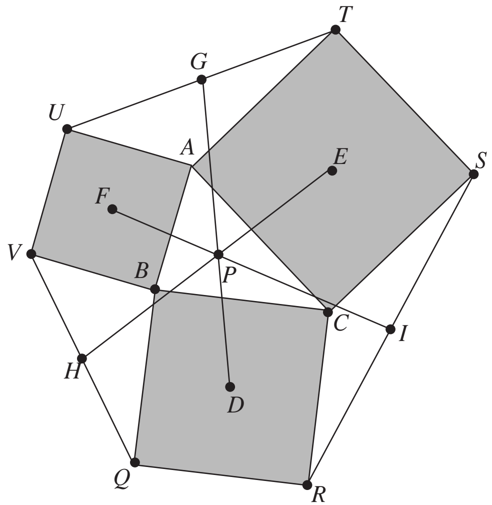

FIGURE 1

Theorem 2: In Figure 2, the line segments GD, HE and IF are equal and perpendicular to HI, IG and GH, respectively.

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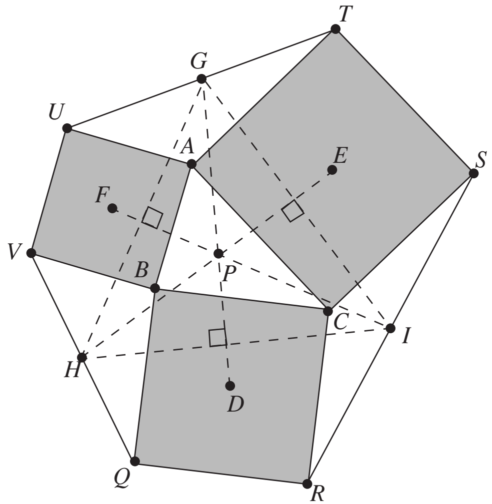

FIGURE 2

In the following we will give a geometric proof of Theorems 1 and 2. We will use the theorem of Finsler-Hadwiger and the Vecten configuration in its standard form.

- Theorem of Finsler-Hadwiger: Let the squares $ABVU$ and $ACST$ share the common vertex A. Then the midpoints Z and G of the segments BC and UT together with the centres F and E of the original squares form another square (Figure 3).

For a short proof see e.g. [2] or [3, pp. 125f]. The theorem can also be seen as an immediate consequence of the *fundamental theorem of similarity* (for a proof see e.g. [2] or [4]).

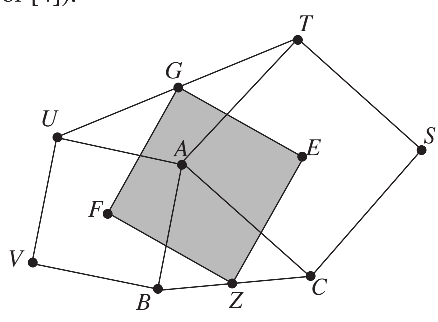

FIGURE 3

- It is well known from the *standard Vecten configuration* that the line connecting the centres of two adjacent squares is perpendicular and equal to the line connecting the common vertex of these squares with the centre of the third square (Figure 4, where $FE = AD$ and $FE \perp AD$).

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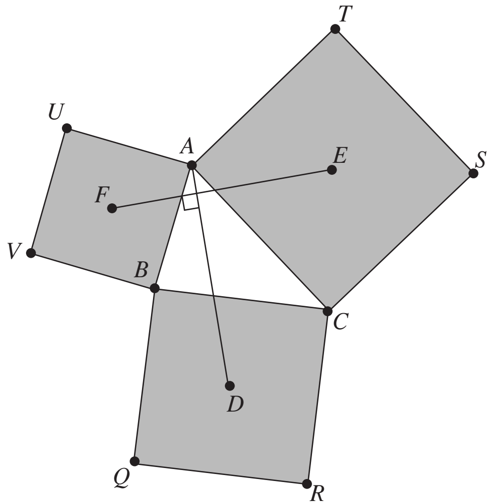

FIGURE 4

Proof of Theorems 1 and 2: In Figure 5 we construct ΔDEF and erect three further squares outwardly on its sides EF, FD and DE with centres G, H, I. This establishes a standard Vecten configuration which enables us to conclude that the line segments GD, HE and IF are concurrent at P, the Vecten point of ΔDEF. From the theorem of Finsler-Hadwiger we know

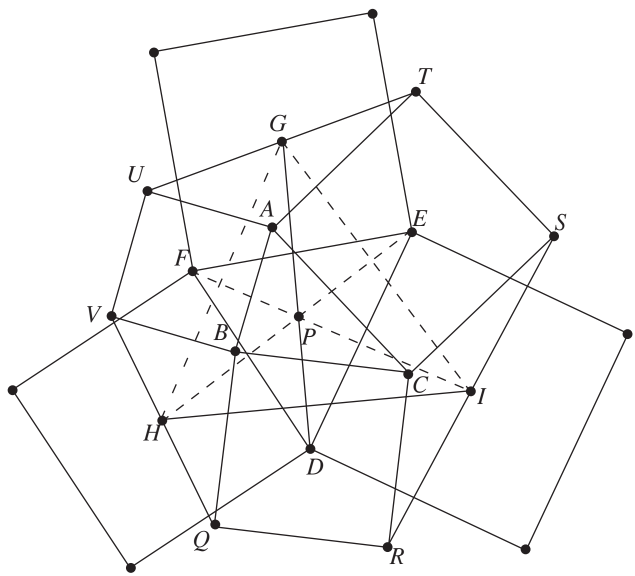

FIGURE 5

that G, H, I are the midpoints of the segments UT, VQ and RS (e.g. for G:

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note that $GF = GE$ and $GF \perp GE$; thus GFE must be half a Finsler-Hadwiger square and G the midpoint of UT). This proves both Theorems 1 and 2.

From a generalisation of van Aubel's theorem to (directly) similar rectangles erected on the sides of a quadrilateral proved in [4] and [5], we obtain the following specialisation where squares are used, as shown in Figure 6: $FD = LD$, $FE \perp LD$, $FE \cap LD = O$ and $KI = HJ$, $KI \perp HJ$, $KI \cap HJ = O$. Furthermore, as also shown in [4] and [5] for similar rectangles, lines FE and LD respectively bisect angles KOH and KOJ; hence for squares, the four lines meeting at O make angles of 45°.

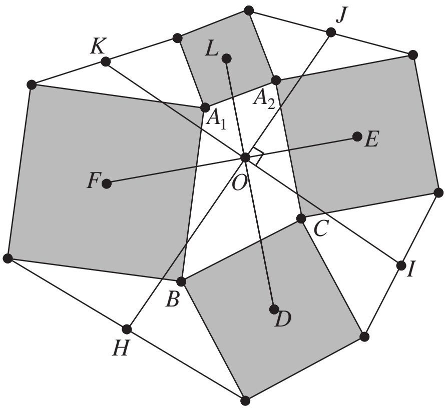

FIGURE 6

From this, by letting $A_1 = A_2 = L$, it immediately follows for triangles, with the usual Vecten configuration that $FE = LD$, $FE \perp LD$, $FE \cap LD = O$. Further (what may not be so well known in the Vecten context) $KI = HJ$, $KI \perp HJ$, $KI \cap HJ = O$, where K, J are the midpoints of sides of erected squares as shown in Figure 7. In addition, the four lines FE, LD, KI and HJ make 45° angles with each other.

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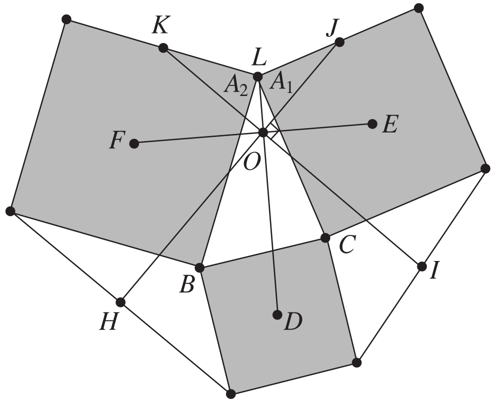

FIGURE 7

Let us formulate these facts more precisely on their own as a theorem.

Theorem 3: Let ΔABC be a triangle with outwardly erected squares on its sides and let $G, H, J, K, B_1, B_2, C_1, C_2$ be midpoints of segments as shown in Figure 8. Then:

(i) there are three pairs of equal and perpendicular line segments: KI and HJ, GC_1 and HC_2, IB_1 and GB_2, and

(ii) $HC_2 \cap IB_1 = M_{BC}$, $KI \cap GC_1 = M_{CA}$, $HJ \cap GB_2 = M_{AB}$, where M_{BC}, M_{CA} and M_{AB} are the midpoints of BC, CA and AB, respectively.

Proof: (i) follows immediately from [4] and [5]. For (ii) we show that $HC_2 \cap IB_1 = M_{BC}$.

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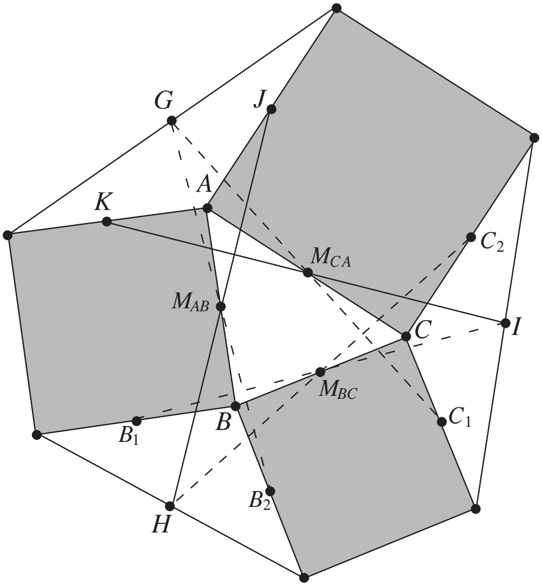

FIGURE 8

In the Vecten configuration it is well known (see e.g. [6]) that the median of a flank triangle (e.g. HB) is perpendicular to and half as long as the opposite side of the initial triangle (e.g. AC). Thus $HB = CC_2$ and $HB \,/\!/\, CC_2$, from which we can conclude that HCC_2B is a parallelogram and the diagonal HC_2 bisects the other diagonal BC. Likewise IB_1 also bisects BC. Proofs of the other two intersections are analogous.

Finally, at the end of the paper, we come back to Figure 7 and highlight an interesting additional fact. We draw a new figure (Figure 9a) and formulate a corresponding theorem.

Theorem 4: The midpoints X and Y of AD and FE together with the midpoints M_{AB} and M_{CA} form a square (Figure 9a). Furthermore, this square is concyclic with O, the joint intersection of AD and FE as well as KI and HJ.

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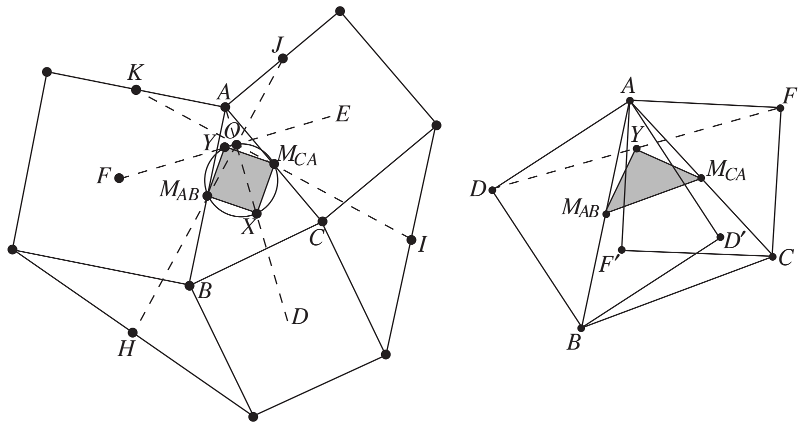

(a) (b)

FIGURE 9

Proof: This follows from a more general result in [7]. But we will give a short proof. For the first part of the theorem (square $M_{AB}XM_{CA}Y$) we will use Finsler-Hadwiger (Figure 9b) and show that $M_{AB}YM_{CA}$ is a diagonally-halved square, and similarly $M_{AB}XM_{CA}$. Then it is clear that $M_{AB}XM_{CA}Y$ is a square.

Let D' be the point D reflected in AB, and F' be the point F reflected in AC. Then two squares arise, namely $ADBD'$ and $AF'CF$, and, following Finsler-Hadwiger, $M_{AB}YM_{CA}$ is a diagonally halved square. Since $\angle M_{AB}YM_{CA} = 90° = \angle M_{AB}OM_{CA}$, and since these two angles are subtended by the same chord (diameter) $M_{AB}M_{CA}$, it follows that the square is concyclic with O.

Of course, in a similar way there exist circles associated with the other vertices B and C of ΔABC, so there are three circles of this kind (Pellegrinetti circles) associated with the Vecten configuration.

It is left to the reader to further explore Vecten variations when similar rectangles, similar rhombi, or similar parallelograms, are constructed on the sides of a triangle (compare [4] or [5]).

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