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COMPLETE DIAGONALS OF LATIN SQUARES

BY
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ABSTRACT. J. Marica and J. Schönhein [4], using a theorem of M. Hall, Jr. [3], see below, proved that if any $n-1$ arbitrarily chosen elements of the diagonal of an $n \times n$ array are prescribed, it is possible to complete the array to form an $n \times n$ latin square. This result answers affirmatively a special case of a conjecture of T. Evans [2], to the effect that an $n \times n$ incomplete latin square with at most $n-1$ places occupied can be completed to an $n \times n$ latin square. When the complete diagonal is prescribed, it is easy to see that a counterexample is provided by the case that one letter appears $n-1$ times on the diagonal and a second letter appears once. In the present paper, we prove that except in this case the completion to a full latin square is always possible. Completion to a symmetric latin square is also discussed.

1. **Introduction.** An $r \times s$ *latin rectangle* on n distinct letters is an $r \times s$ array of the n letters such that no letter appears twice in the same row or in the same column. An $n \times n$ *latin square* is an $n \times n$ latin rectangle on n letters. By the *diagonal frequency*, call it σ_i, of the ith letter we mean the number of the times that this ith letter appears in the diagonal. Since $n = \sigma_1 + \cdots + \sigma_n$, the n diagonal frequencies σ_i, after cancellation of the zero summands, determine a partition of n which will be called the partition corresponding to the given set of diagonal elements. When the entries in a latin square are renamed by permuting the letters, the result is a latin square with new σ's that are a permutation of the original σ's and therefore determine the same partition. Secondly, if we interchange the ith row and the jth row, and then interchange the ith column and the jth column, we get a new latin square having the same diagonal elements but with different order. Transformations of these two kinds will here be called *elementary*. Consequently, the partition corresponding to the diagonal of a given latin square remains unchanged under elementary transformations.

The following theorem of M. Hall [2] is needed in our proof:

THEOREM. *For a given set of n elements $b_1, \ldots, b_n$, not necessarily distinct, in an abelian group G of order n, a permutation*

$$\begin{pmatrix} a_1, \ldots, a_n \\ c_1, \ldots, c_n \end{pmatrix}$$

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of the elements of G such that $c_i - a_i = b_i$, $i = 1, \ldots, n$, *exists if and only if* $b_1 + \cdots + b_n = 0$.

By the ith cell of the jth *right diagonal* of an $n \times n$ array we mean the cell $(i, i+j-1) \pmod n$. Let D_n denote the $n \times n$ array such that the ith element of the jth right diagonal is given by $i + 2j - 2 \pmod n$ when $1 \le j \le (n+1)/2$, and by $i + 2j - 1 - 2[(n+1)/2] \pmod n$ when $(n+1)/2 < j \le n$. If n is odd, then D_n is a latin square. If n is even, then by rejecting the first right diagonal we get an incomplete latin square, say D_n^*, and by rejecting the nth right diagonal we get another incomplete latin square, say D_n^{**}. In D_n every right diagonal meets each row and each column just once, and every letter appears in each right diagonal once. For this reason, we can choose a set of r right diagonals of D_n, with the restriction that if n is even the set does not contain both the first and the nth right diagonal, in such a way as to form an $r \times n$ latin rectangle and an $n \times r$ latin rectangle, to be denoted by $R_{r \times n}(i_1, \ldots, i_r)$ and $C_{n \times r}(i_1, \ldots, i_r)$ respectively, or R and C for short.

2. **The main theorem.** In this paper we study the completion of an $n \times n$ incomplete latin square with the whole diagonal occupied and other cells unoccupied. The completion of latin squares of this kind is not always possible. For example, if the first $n-1$ cells of the diagonal are occupied by a same letter α and the last cell is a second letter β, i.e. the partition of diagonal frequency is $n = 1 + (n-1)$, then it is impossible to put an α in any cell of the last row to form a latin square. This is the only exception the following theorems says.

THEOREM. *If the whole diagonal of an $n \times n$ array is prescribed, it is always possible to complete the array to form an $n \times n$ latin square except in the case that $n-1$ terms of one letter and one term of another letter appear in the diagonal.*

Proof. Rewrite the partition of n corresponding to the prescribed diagonal in the form

$$n = x_1 + x_1 + x_2 + x_2 + \cdots + x_r + x_r + y_1 + y_2 + \cdots + y_s, \tag{1}$$

where $1 \le x_1 \le x_2 \le \cdots \le x_r$ and $1 \le y_1 < y_2 < \cdots < y_s$. And not $r = 0$, $s = 2$. $y_1 = 1$. The abelian group Z_n can be decomposed into pairs consisting of element and its inverse, i.e.

$$Z_n = \begin{cases} \{\bar{1}, -\bar{1}\} \cup \cdots \cup \{[(\bar{n} - \bar{1})/\bar{2}], -[(\bar{n} - \bar{1})/\bar{2}]\} \cup \{[\bar{n}/\bar{2}]\} & \text{if } n \text{ even}, \\ \{\bar{1}, -\bar{1}\} \cup \cdots \cup \{[(\bar{n} - \bar{1})/\bar{2}], -[(\bar{n} - \bar{1})/\bar{2}]\} \cup \{\bar{0}\} & \text{if } n \text{ odd}. \end{cases}$$

If s is odd, say $s = 2k+1$, then

$$n = 2 \sum x_i + \sum y_j \ge 2r + (k+1)(k+2) \ge 2r + 4k + 1,$$

so that

$$[(n-1)/2] \ge r + 2k. \tag{2}$$

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The $2k$ elements $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_k, -\bar{y}_{k+1}, -\bar{y}_{k+2}, \ldots, -\bar{y}_{2k}$ are in distinct pairs of the above decomposition, otherwise there are some $y_u = y_v$, which is impossible, or $y_u + y_v = n$, which implies $r = 0$ and $s = 2$, but s is odd now. By (2) there exist at least r other pairs $\{\bar{z}_i, -\bar{z}_i\}$, $i = 1, \ldots, r$ such that $\bar{y}$'s, $-\bar{y}$'s, $\bar{z}$'s, $-\bar{z}$'s and $\bar{0}$ are all distinct. Choose $\bar{z}_i$ and $-\bar{z}_i$ each x_i times, $\bar{y}_i$ each y_{k+1} times, $-\bar{y}_{k+i}$ each y_i times, and $\bar{0}$ y_{2k+1} times. These n elements have the same corresponding partition as in (1); therefore we can identify them with the prescribed diagonal b_i. Also, their sum is

$$\sum_{i=1}^{k} x_i(\bar{z}_i + -\bar{z}_i) + \sum_{i=1}^{k} y_{k+i}\bar{y}_i + \sum_{i=1}^{k} y_i - \bar{y}_{k+i} + y_{2k+1}\bar{0} = \bar{0}.$$

So by the theorem of M. Hall there exists a permutation

$$\begin{pmatrix} a_1, \ldots, a_n \\ c_1, \ldots, c_n \end{pmatrix}$$

of Z_n such that $c_i - a_i = b_i$, $i = 1, \ldots, n$. The array $(b_{ij})_{n\times n}$ with $b_{ij} = c_i - a_j$ is then a latin square with $b_{ii} = b_i$, so that that the given requirement is satisfied.

If s is even, say $s = 2k$, then (2) holds and $\bar{y}_1, \ldots, \bar{y}_k, -\bar{y}_{k+1}, \ldots, -\bar{y}_{2k}$ are in distinct pairs of the above decomposition, with two exceptions: first, if the partition of n corresponding to the diagonal is $n = 1 + \cdots + 1 + u$ with t ones, t odd and $u = 2$ or 3, and second, if $n = t + u$ with $2 \le t < u$. Apart from these exceptions, the same argument, omitting the y_{2k+1} times choice of $\bar{0}$, can be used to prove the theorem.

In the special case $n = 6 = 1 + 1 + 1 + 3$, if for the b's we choose $\bar{0}$, $\bar{1}$, $\bar{5}$ each once and $\bar{2}$ three times in Z_6, the theorem is proved. For the general case $n = 1 + \cdots + 1 + u$, we have $t > u$. Let X be a $u \times u$ latin square on the u elements $t+1, \ldots, t+u = n$, whose diagonal elements are all $t+1$. If all entries in the jth right diagonal of D_t are replaced by $j+u$, $j = t-u+1, \ldots, t$, we get a latin rectangle Y. Pick out the $(t-u+1)$th, $\ldots$, tth right diagonals of D_t to form $R_{u\times t}(t-u+1, \ldots, t)$ and $C_{t\times u}(t-u+1, \ldots, t)$. Then

$$\begin{bmatrix} X & R \\ C & Y \end{bmatrix}$$

is a latin square in which the partition of n corresponding to the diagonal is $n = 1 + \cdots + 1 + u$.

Now consider the second exception, namely the case $n = t + u$ with $2 \le t < u = n - t$. Let X be a $t \times t$ latin square on the t elements $u+1, \ldots, u+t = n$, whose diagonal elements are all $u+2$. If all entries in the jth right diagonal of D_u are replaced by $u+j$, $j = 1, \ldots, t$, we get a latin rectangle Y. Abbreviate $R_{t\times u}(1, \ldots, u)$ and $C_{u\times t}(1, \ldots, u)$ to R and C respectively. Then

$$\begin{bmatrix} X & R \\ C & Y \end{bmatrix}$$

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is a latin square in which the partition of n corresponding to the diagonal is $n = t + u$.

3. **Symmetric latin squares.** A latin square is called *symmetric* if each (i, j)-entry is equal to the (j, i)-entry. Here we have the following result.

THEOREM. *For the completion of a prescribed diagonal to a symmetric $n \times n$ latin square it is necessary and sufficient that the diagonal contains each letter exactly once for odd n, and an even number of times, including zero, for even n.*

Proof. In an $n \times n$ symmetric latin square the ith letter will appear $n - \sigma_i$ times off the diagonal, so that $n - \sigma_i$ is even. If n is odd, then each σ_1 is also odd, which implies that $\sigma_i = 1$, i.e. each of the n letters appears once in the diagonal. If n is even, then each σ_i is even, i.e. each letter appears an even number of times in the diagonal.

If n is odd, the $n \times n$ latin square in which the (i, j)-entry is $i + j \pmod n$ is symmetric and every letter appears once in the diagonal. If n is even, equal to a sum of even numbers d_i, $i = 1, \ldots, n$, say $n = \sum_{i=1}^{n} d_i$, we wish to prove that there exists an $n \times n$ symmetric latin square with diagonal frequencies $\sigma_i = d_i$, $i = 1, \ldots, n$.

The case for $n = 2$ is obvious. Suppose the theorem holds for all even $n' < n$.

Set $n = 2m$. Without loss of generality, we can assume $n = \sum_{i=1}^{m} d_i$.

If m is even, we can find even numbers x_i and y_i such that

$$d_i = x_i + y_i, \quad i = 1, \ldots, m, \quad \text{and} \quad m = \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i.$$

By the induction hypothesis, we can then construct two $m \times m$ symmetric latin squares X and Y on $1, \ldots, m$, whose diagonal frequencies are given by the x's and y's respectively. Let A be an arbitrarily $m \times m$ symmetric latin square on $m + 1, \ldots, 2m$. Then

$$\begin{bmatrix} X & A \\ A & Y \end{bmatrix}$$

is an $n \times n$ symmetric latin square, as required.

If m is odd, there is at least one d, say d_1, of the form $4p - 2$. We can find even numbers x_i and y_i such that.

$$d_i = x_i + y_i, \qquad i = 2, \ldots, m, \qquad x_1 = y_1 = 2p, \qquad x_{m+1} = y_{m+1} = 0,$$

and

$$m + 1 = \sum_{i=1}^{m+1} x_i = \sum_{i=1}^{m+1} y_i.$$

Construct two $(m+1) \times (m+1)$ symmetric latin squares X and Y on $1, \ldots, m$, $m + 1$, whose diagonal frequencies are given by the x's and y's respectively. By interchanging rows and columns, i.e. using the transformations of second kind, we can arrange that the jth element of the first row of X and Y is j, $j=1,\ldots,m+1$. Deleting the first row and the first column of X and Y we get two $m\times m$ symmetric latin rectangles U and V on $1,\ldots,m,\ m+1$, whose ith row does not contain the letter $i+1$. Let A be an $m\times m$ latin square on $m+1,\ldots,2m$, whose diagonal consists of the element $m+1$ repeated m times. Let B be the latin rectangle obtained by replacing each (i, i)-entry of A by the letter $i+1$. Then

$$\begin{bmatrix} U & B \\ C & V \end{bmatrix}$$

is a required $n\times n$ latin square, where C is the transpose of B.

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