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The Waring Problem with the Ramanujan τ-Function, II

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Abstract. Let $\tau(n)$ be the Ramanujan τ-function. We prove that for any integer N with $|N| \geq 2$ the diophantine equation

$$\sum_{i=1}^{148000} \tau(n_i) = N$$

has a solution in positive integers $n_1, n_2, \ldots, n_{148000}$ satisfying the condition

$$\max_{1 \leq i \leq 148000} n_i \ll |N|^{2/11} e^{-c \log |N| / \log \log |N|},$$

for some absolute constant $c > 0$.

1 Introduction

The Ramanujan function $\tau(n)$ is defined by the expansion

$$X \prod_{n=1}^{\infty} (1 - X^n)^{24} = \sum_{n=1}^{\infty} \tau(n) X^n.$$

It possesses many remarkable properties of an arithmetical nature. It is known that:

- $\tau(n)$ is an integer-valued multiplicative function, that is, $\tau(nm) = \tau(n)\tau(m)$ if $\gcd(n, m) = 1$;
- for any integer $\alpha \geq 0$ and prime q, $\tau(q^{\alpha+2}) = \tau(q^{\alpha+1})\tau(q) - q^{11}\tau(q^{\alpha})$, in particular, $\tau(q^2) = \tau^2(q) - q^{11}$, and
- $|\tau(q)| \leq 2q^{11/2}$ for any prime q and $|\tau(n)| \leq d(n)n^{11/2}$ for any integer $n > 0$, where $d(n)$ is the number of divisors of n. In particular, there exists a positive absolute constant c such that $|\tau(n)| \leq n^{11/2} e^{c \log n / \log \log n}$ for any integer $n \geq 3$. This has been proved by Deligne [3].

There are many formulas that connect $\tau(n)$ with the function $\sigma_s(n) = \sum_{d|n} d^s$ and are useful for numerical computations of $\tau(n)$. It is known, for example, that

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{k=1}^{n-1}\sigma_5(k)\sigma_5(n-k)$$

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and that

$$\tau(n) = n^4\sigma_0(n) - 24\sum_{k=1}^{n-1}(35k^4 - 52k^3n + 18k^2n^2)\sigma_0(k)\sigma_0(n-k).$$

These properties of $\tau(n)$ can be found in [1, 3, 6, 7, 9, 10]. Various properties of $\tau(n)$ modulo a prime number p can be found in [10].

Based on the deep sum-product estimate of Bourgain, Katz and Tao [2], Shparlinski [11] proved that the values of $\tau(n)$, $n \le p^4$, form a finite additive basis modulo p, *i.e.,* there exists an absolute integer constant $s \ge 1$ such that any residue class modulo p is representable in the form $\tau(n_1)+\cdots+\tau(n_s) \pmod p$ with some positive integers $n_1, \dots, n_s \le p^4$.

In [4] we established that the set of values of $\tau(n)$ forms a finite additive basis for the *set of integers*. We proved that for any integer N the diophantine equation $\sum_{i=1}^{74000} \tau(n_i) = N$ has a solution in positive integers $n_1, n_2, \dots, n_{74000}$ satisfying the condition

$$\max_{1\le i\le 74000} n_i \ll |N|^{2/11} + 1$$

(here and throughout the paper, the implied constants in Vinogradov's symbol "$\ll$" are absolute). From Deligne's result it follows that the constant $2/11$ in the exponent of $|N|$ is the best possible in the sense that it cannot be substituted by a smaller constant. Nevertheless, there still arises a question whether a similar result holds with the variables n_i being of the size $\le |N|^{2/11}\Delta(N)$, where $\Delta(N) \to 0$ as $|N| \to \infty$. In this paper we give the following answer to this question.

Theorem 1 *There exists an absolute constant $c > 0$ such that for any integer N with $|N| \ge 2$ the diophantine equation $\sum_{i=1}^{148000} \tau(n_i) = N$ has a solution in positive integers $n_1, n_2, \dots, n_{148000}$ satisfying the condition*

$$\max_{1\le i\le 148000} n_i \ll |N|^{2/11} e^{-c\log|N|/\log\log|N|}.$$

In view of Deligne's result, Theorem 1 reflects the best possible bound for the size of the variables n_i, apart from the value of the constant c.

Regarding the number of summands 148000, it can be several times reduced using, in particular, the present state of art concerning the Waring–Goldbach problem. However, we do not consider such a reduction to be essential and we do not pursue this issue in the present paper.

2 Auxiliary Statements

Theorem 1 will be deduced from the following proposition.

Proposition 2 *Any integer L with a sufficiently large modulus $|L|$ can be represented in the form*

$$L = \sum_{i=1}^{74000} \tau(n_i)$$

with some positive integers $n_1, \dots, n_{74000}$ having the property that the interval $(\log\log|L|, \log^2|L|)$ *is free of prime divisors of n_i and*

$$\max_{1\le i\le 74000} n_i \ll |L|^{2/11}.$$

Proposition 2 extends [4, Theorem 1]; in that theorem we proved a similar result without any restrictions on prime divisors of n_i.

The proof of Proposition 2 is based on the following consequence of the classical result of Hua [5].

Lemma 3 *Let s_0 be a fixed integer ≥ 2049 and let J denote the number of solutions of the Waring–Goldbach equation*

$$\sum_{i=1}^{s_0} q_i^{11} = N$$

in primes $q_1, \dots, q_{s_0}$ with $q_i > (\log N)^3$ for all $1 \le i \le s_0$. There exist positive constants $c_1 = c_1(s_0)$ and $c_2 = c_2(s_0)$ such that for any sufficiently large integer N with $N \equiv s_0 \pmod 2$, the following bounds hold:

$$c_1 \frac{N^{s_0/11-1}}{(\log N)^{s_0}} \le J \le c_2 \frac{N^{s_0/11-1}}{(\log N)^{s_0}}.$$

For the proof of Proposition 2 we refer the reader to [4], since it follows exactly the same lines as the proof of [4, Theorem 1]; the main difference is that in the proof of that theorem instead of Lemma 3 we used its analog where $q_i > (\log N)^3$ was replaced by $q_i > 105$. Consequently, the set Ω that we defined in the proof of that theorem should now be defined as the set of all prime numbers q with $\log^3 M < q \le M^{1/11}$.

The following lemma, which is a consequence of a more general result of M. Ram Murty [8], forms the main ingredient in the deduction of Theorem 1 from Proposition 2.

Lemma 4 *For a positive density of primes p, we have $|\tau(p)| > 1.4p^{11/2}$.*

3 Proof of Theorem 1

Observe that it is sufficient to prove Theorem 1 for large values of N, that is for all N with $|N| > N_0$, where N_0 is some absolute positive integer constant. For the values of N with $|N| \le N_0$ Theorem 1 is a consequence of the aforementioned result from our work [4].

Let N be an integer with a sufficiently large modulus $|N|$. According to Lemma 4, there exists an absolute constant $C > 100$ such that the interval

$$\Big[\frac{\log|N|}{C^2}, \frac{\log|N|}{C}\Big]$$

contains $\gg \log|N|/\log\log|N|$ primes p with $|\tau(p)| > 1.4p^{11/2}$. Define T to be the product of all these primes, except maybe one of them (to make $\tau(T)$ positive). Since $\tau(n)$ is multiplicative, we have

$$\tau(T)/T^{11/2} > e^{c_1 \log|N|/\log\log|N|} \tag{1}$$

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for some absolute constant $c_1 > 0$. Therefore, for some large absolute positive constant C_1, we have

$$\big((\log|N|)/C^2\big)^{(\log|N|/(C_1\log\log|N|))} < T < e^{2\log|N|/C}.$$

From this we deduce that

$$|N|^{c_2} < T < |N|^{0.02} \tag{2}$$

with the additional property that any prime divisor $q|T$ satisfies

$$c_3\log|N| < q < \log|N|,$$

where c_2, c_3 are some absolute positive constants.

Having T defined this way, we let L be an integer such that

$$N = \tau(T)L + u, \quad \tau(T) \le u \le 2\tau(T). \tag{3}$$

Deligne's estimate combined with (1) and (2) yields $|N|^{11c_2/2} \le \tau(T) \le |N|^{1/5}$. Clearly, if $|N|$ is large, so are u and $|L|$. According to Proposition 2, we have a representation of u in the form

$$u = \sum_{i=1}^{74000}\tau(k_i), \quad \max_i k_i \ll u^{2/11}. \tag{4}$$

Since $|L|$ is large, we can use Proposition 2 again to get $L = \sum_{i=1}^{74000}\tau(n_i)$, where the positive integers $n_1, \dots, n_{74000}$ are free of prime divisors from the interval $(\log\log|L|, \log^2|L|)$ and

$$\max_{1\le i\le 74000} n_i \ll |L|^{2/11}.$$

In particular, since $|N|^{4/5} \ll |L| \le |N|$, the numbers n_i are free of prime divisors from the interval $(\log\log|N|, \log|N|)$. On the other hand, all the prime divisors of T belong to the interval $c_3\log|N| < q < \log|N|$. Therefore, by the multiplicative property of $\tau(n)$ we deduce

$$\tau(T)L = \sum_{i=1}^{74000}\tau(Tn_i). \tag{5}$$

From (1) and (3),

$$T \le \tau(T)^{2/11}e^{-(2c_1/11)\log|N|/\log\log|N|} \ll \left(\frac{|N|}{|L|}\right)^{2/11} e^{-(2c_1/11)\log|N|/\log\log|N|}.$$

Therefore,

$$Tn_i \ll T|L|^{2/11} \le |N|^{2/11}e^{-c\log|N|/\log\log|N|}$$

for some absolute positive constant c. Combining this with (3), (4) and (5), we derive the result. ■

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4 Remark

Let f be a fixed non-zero cusp form of a given weight k of the modular group such that f is a normalized eigenform for all the Hecke operators. Let the function $f(z)$ have the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

in the upper-half plane. In the particular case $k = 12$, the coefficients $a(n)$ coincide with $\tau(n)$. In the general case, the coefficients $a(n)$ satisfy some multiplicative relations similar to those satisfied by $\tau(n)$. A suitable modification of our arguments together with results from [8] imply that if $a(n)$ are integers, then for some positive integer ℓ and some constant $c > 0$, any integer N with $|N| \geq 2$ is representable in the form $\sum_{i=1}^{\ell} a(n_i) = N$ for some $n_1, \ldots, n_\ell$ with

$$\max_{1 \leq i \leq \ell} n_i \ll |N|^{2/(k-1)} e^{-c \log |N| / \log\log |N|}.$$

References

[1] T. M. Apostol, *Modular functions and Dirichlet series in number theory.* Second ed., Graduate Texts in Mathematics 41, Springer-Verlag, New York, 1990.

[2] J. Bourgain, N. Katz, and T. Tao, *A sum-product estimate in finite fields and their applications.* Geom. Funct. Anal. **14**(2004), no. 1, 27–57.

[3] P. Deligne, *La conjecture de Weil. I.* Inst. Hautes Études Sci. Publ. Math. **43**(1974), 273–307.

[4] M. Z. Garaev, V. C. Garcia, and S. V. Konyagin, *Waring problem with the Ramanujan τ-function.* Izvestiya: Mathematics **72**(2008), no. 1, 35–46.

[5] L.-K. Hua, *The additive prime number theory.* Trav. Inst. Math. Stekloff **22**(1947).

[6] H. Iwaniec, *Topics in clasical automorphic forms.* Graduate Studies in Mathematics 17, American Mathematical Society, Providence, Rhode Island, 1997.

[7] N. Koblitz, *Introduction to elliptic curves and modular forms* Second ed., Springer-Verlag, New York, 1993.

[8] M. R. Murty, *Oscillations of Fourier coefficients of modular forms.* Math. Ann. **262**(1983), no. 4, 431–446.

[9] D. Niebur, *A formula for Ramanujan's τ-function.* Illinois J. Math. **19**(1975), 448–449.

[10] J.-P. Serre, *Congruences et formes modulaires.* Lecture Notes in Math. 317, Springer, Berlin, 1973, 319–338.

[11] I. E. Shparlinski, *On the value set of the Ramanujan function.* Arch. Math. **85**(2005), no. 6, 508–513.

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