

## CONTINUOUS INVERSE SHADOWING AND HYPERBOLICITY

KEONHEE LEE

We study the concepts of continuous shadowing and continuous inverse shadowing with respect to various classes of admissible pseudo orbits, and characterise hyperbolicity and structural stability using the notion of continuous inverse shadowing.

### 1. INTRODUCTION

Simulating the behaviour of a dynamical system we often encounter the following problems.

Does the orbit displayed on the computer screen really correspond to some true orbit?

Can every true orbit be recovered, at least with a given accuracy?

The first problem is in fact a question about the shadowing property of the system, while the second corresponds to the property known as inverse shadowing. Shadowing, or the pseudo orbit tracing property, was first established for systems generated by hyperbolic diffeomorphisms and later for those generated by hyperbolic homeomorphisms. It says that any  $\delta$ -pseudo orbit can be uniformly approximated by a true orbit with a given accuracy if  $\delta > 0$  is sufficiently small.

Inverse shadowing was established by Corless and Pilyugin [1] and also studied as a part of the concept of bishadowing by Diamond, Kloeden, Kozyakin and Pokrovskii [2, 3]. Kloeden and Ombach redefined this property using the  $\delta$ -method (see [6]). Generally speaking, a dynamical system is inverse shadowing with respect to a class of methods if any true orbit can be uniformly approximated with given accuracy by a  $\delta$ -pseudo orbit generated by a method from the chosen class if  $\delta > 0$  is sufficiently small. An appropriate choice of the class of admissible pseudo orbits is crucial here (see [1, 4, 6]).

Recently Kloden, Ombach and Pokrovskii [7] introduced the notion of continuous shadowing of a dynamical systems to discuss the Shadowing Lemma of semi-hyperbolic

---

Received 15th November, 2001

This work was supported by grant No. R01-2000-00005 from the Korea Science and Engineering Foundation.

I would like to thank Professor Phil Diamond for his warm hospitality and useful discussions during my visit to University of Queensland in Australia from August, 2000 to January, 2002.

---

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/03 \$A2.00+0.00.

systems. Very recently Diamond, Han and Lee [4] characterised the notion of stability in terms of shadowing, and expressed hyperbolicity using the concept of inverse shadowing.

In Section 2, we present the concepts of continuous shadowing with respect to various classes of admissible pseudo orbits, and study the relationships among them. It is then shown that a dynamical system which is continuous shadowing is topologically stable.

In Section 3, we introduce the notions of continuous inverse shadowing with respect to various classes of admissible pseudo orbits, and characterise hyperbolicity and structural stability in terms of continuous inverse shadowing. More precisely, we show that a dynamical system is hyperbolic if and only if it is expansive and  $\mathcal{T}_d$ -continuous inverse shadowing. We prove that the  $C^1$  interior of the set of all dynamical systems with the  $\mathcal{T}_d$ - (continuous) inverse shadowing property is characterised as the set of structurally stable systems.

## 2. CONTINUOUS SHADOWING

Let  $X$  be a compact metric space with a metric  $d$ , and let  $Z(X)$  denote the space of homeomorphisms on  $X$  with the  $C_0$ -metric  $d_0$ . A homeomorphism  $f \in Z(X)$  will be identified with the dynamical system it generates by iteration.

A  $\delta$ -pseudo orbit of  $f \in Z(X)$  is a sequence of points  $\xi = \{x_k \in X : k \in \mathbb{Z}\}$  such that  $d(f(x_k), x_{k+1}) < \delta$  for all  $k \in \mathbb{Z}$ . A  $\delta$ -pseudo orbit  $\xi = \{x_k\}$  is said to be  $\varepsilon$ -shadowed by a point  $x \in X$  (or an orbit  $\{f^k(x) : k \in \mathbb{Z}\}$ ) if  $d(f^k(x), x_k) < \varepsilon$  for all  $k \in \mathbb{Z}$ . Say that  $f \in Z(X)$  is shadowing (or has the pseudo orbit tracing property) if given  $\varepsilon > 0$  there exists  $\delta > 0$  such that any  $\delta$ -pseudo orbit of  $f$  is  $\varepsilon$ -shadowed by a point in  $X$ .

Let  $X^{\mathbb{Z}}$  be the compact metric space of all two sided sequences  $\xi = \{x_k : k \in \mathbb{Z}\}$  in  $X$ , endowed with the product topology. For a constant  $\delta > 0$  and  $f \in Z(X)$ , let  $\Phi_f(\delta)$  denote the set of all  $\delta$ -pseudo orbits of  $f$ .

A mapping  $\varphi : X \rightarrow \Phi_f(\delta) \subset X^{\mathbb{Z}}$  satisfying  $\varphi(x)_0 = x$ ,  $x \in X$ , is said to be a  $\delta$ -method for  $f$ . For convenience, write  $\varphi(x)$  for  $\{\varphi(x)_k\}_{k \in \mathbb{Z}}$ . Say that  $\varphi$  is a continuous  $\delta$ -method for  $f$  if  $\varphi$  is continuous. The set of all  $\delta$ -methods [respectively continuous  $\delta$ -methods] for  $f$  will be denoted by  $\mathcal{T}_0(f, \delta)$  [respectively  $\mathcal{T}_c(f, \delta)$ ]. Every  $g \in Z(X)$  with  $d_0(f, g) < \delta$  induces a continuous  $\delta$ -method  $\varphi_g : X \rightarrow X^{\mathbb{Z}}$  for  $f$  by defining  $\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}$ . Let  $\mathcal{T}_h(f, \delta)$  denote the set of all continuous  $\delta$ -methods  $\varphi_g$  for  $f$  which are induced by  $g \in Z(X)$  with  $d_0(f, g) < \delta$ . To introduce the notions of continuous shadowing with respect to various classes of admissible pseudo orbits we define  $\mathcal{P}_\alpha(f, \delta)$  by

$$\mathcal{P}_\alpha(f, \delta) = \bigcup_{\varphi \in \mathcal{T}_\alpha(f, \delta)} \varphi(X),$$

for  $\alpha = 0, c, h$ . Clearly we have

$$\mathcal{P}_h(f, \delta) \subset \mathcal{P}_c(f, \delta) \subset \mathcal{P}_0(f, \delta) = \Phi_f(\delta)$$

DEFINITION 2.1: We say  $f \in Z(X)$  is  $\mathcal{T}_\alpha$ -shadowing for  $\alpha = 0, c, h$ , if for any  $\varepsilon > 0$  there is a  $\delta > 0$  and a map  $r : \mathcal{P}_\alpha(f, \delta) \rightarrow X$  such that

$$d(f^k(r(\mathbf{x})), x_k) < \varepsilon,$$

for all  $\mathbf{x} = \{x_k : k \in \mathbb{Z}\} \in \mathcal{P}_\alpha(f, \delta)$  and all  $k \in \mathbb{Z}$ .

We say that  $f$  is  $\mathcal{T}_\alpha$ -continuous shadowing ( $\mathcal{T}_\alpha$ -CS) if the map  $r$  is continuous.

Recently Diamond, Han and Lee [4] showed that the shadowing property for  $f \in Z(X)$  is independent of the choice of the class of admissible pseudo orbits if the space  $X$  is a compact manifold, that is, the concepts of  $\mathcal{T}_0, \mathcal{T}_c, \mathcal{T}_h$ -shadowing and the pseudo orbit tracing property are then equivalent.

Shadowing is a weaker property than  $\mathcal{T}_h$ -continuous shadowing when  $X$  is a manifold. In fact, Yano [14] constructed an example of a homeomorphism  $f \in Z(S^1)$  which is shadowing but is not topologically stable; and so it is not  $\mathcal{T}_h$ -continuous shadowing by Theorem 2.5 below. Clearly we have the following relations among the above notions of continuous shadowing.

$$\mathcal{T}_0 - \text{CS} \Rightarrow \mathcal{T}_c - \text{CS} \Rightarrow \mathcal{T}_h - \text{CS}.$$

In the following theorem we show that they are equivalent if the space  $X$  is a compact manifold with  $\dim \geq 2$ .

**THEOREM 2.2.** *Let  $M$  be a compact manifold with  $\dim \geq 2$ . Then  $f \in Z(M)$  is  $\mathcal{T}_0$ -continuous shadowing if and only if it is  $\mathcal{T}_h$ -continuous shadowing.*

PROOF: Suppose  $f \in Z(M)$  is  $\mathcal{T}_h$ -continuous shadowing, and let  $\varepsilon > 0$  be arbitrary. Then we can choose  $\delta_1 > 0$  and continuous map  $r : \mathcal{P}_h(f, \delta_1) \rightarrow M$  such that

$$d(f^k(r(\mathbf{y})), y_k) < \varepsilon,$$

for  $\mathbf{y} = \{y_k : k \in \mathbb{Z}\} \in \mathcal{P}_h(f, \delta_1)$  and all  $k \in \mathbb{Z}$ . Put  $\delta = \delta_1/4$ . Let  $\xi = \{x_k : k \in \mathbb{Z}\}$  be a  $\delta$ -pseudo orbit of  $f$ , that is,  $\xi \in \mathcal{P}_0(f, \delta)$ . For each  $m \in \mathbb{N}$ , we let

$$\xi_m = \{x_{-m}, \dots, x_{-1}, x_0, x_1, \dots, x_m\}.$$

Then we can select a set  $\xi' = \{y_{-m}, \dots, y_{-1}, y_0 = x_0, y_1, \dots, y_m\}$  of points in  $M$  such that

- (1)  $d(x_k, y_k) < 1/m, \quad -m \leq k \leq m;$
- (2)  $d(f(y_k), y_{k+1}) < 2\delta, \quad -m \leq k \leq m-1;$
- (3)  $y_i \neq y_j, \quad -m \leq i < j \leq m.$

From [10, Lemma 13], there is a homeomorphism  $\phi_m \in Z(M)$  satisfying

$$d_0(\phi_m, 1_M) < \delta_1 \quad \text{and} \quad \phi_m(f(y_k)) = y_{k+1}$$

for  $-m \leq k \leq m - 1$ . Put  $g_m = \phi_m \circ f$ . Then we have

$$d_0(f, g_m) < \delta_1 \text{ and } d(g_m^k(x_0), x_k) < \frac{1}{m}$$

for  $-m \leq k \leq m$ . Let  $O(g_m, x_0)$  be the orbit of  $g_m$  through  $x_0$ , and let  $r(O(g_m, x_0)) = w_m$  for each  $m \in \mathbb{N}$ . Then the sequence  $\{w_m\}$  converges to a point in  $M$ . Hence we define a map  $H : \mathcal{P}_0(f, \delta) \rightarrow M$  by

$$H(\xi) = \lim_{m \rightarrow \infty} r(O(g_m, x_0)),$$

for  $\xi = \{x_k : k \in \mathbb{Z}\} \in \mathcal{P}_0(f, \delta)$ . Then  $H$  is a well defined continuous map satisfying

$$d(f^k(H(\xi)), x_k) < \varepsilon,$$

for all  $k \in \mathbb{Z}$ . In fact, let  $\xi' = \{x'_k : k \in \mathbb{Z}\}$  be another point in  $\mathcal{P}_0(f, \delta)$  which is sufficiently close to  $\xi$  under the metric of  $M^{\mathbb{Z}}$ . By the same techniques as above, we can construct a sequence  $\{g'_m : m \in \mathbb{N}\}$  in  $Z(M)$  such that

$$d_0(f, g'_m) < \delta_1 \text{ and } d(g'^k_m(x'_0), x'_k) < \frac{1}{m}$$

for  $-m \leq k \leq m$ . For each  $m \in \mathbb{N}$ , we let  $r(O(g'_m, x'_0)) = w'_m$ . Since two point  $\xi$  and  $\xi'$  in  $M^{\mathbb{Z}}$  are sufficiently close, we can see that the points  $O(g_m, x_0)$  and  $O(g'_m, x'_0)$  in  $\mathcal{P}_h(f, \delta)$  are close if  $m$  is sufficiently large. Then the points  $w_m$  and  $w'_m$  in  $M$  are close since the map  $r$  is continuous. This implies that the map  $H$  is continuous, and so completes the proof. □

When we study the theory of global stability of diffeomorphisms the notion of shadowing is sometimes too strong as can be seen from Theorem 3.3 below. In the following we give another type of shadowing. For our purpose, let  $M$  be a compact smooth manifold with  $\dim M = n$  and let  $\text{Diff}(M)$  denote the space of  $C^1$ -diffeomorphisms on  $M$  with the  $C^1$  metric  $d_1$ .

As before, for any  $\delta > 0$  and  $f \in \text{Diff}(M)$ , every  $g \in \text{Diff}(M)$  with  $d_1(f, g) < \delta$  induces a continuous  $\delta$ -method  $\varphi_g : M \rightarrow M^{\mathbb{Z}}$  for  $f$  by defining  $\varphi_g(x) = \{g^k(x) : k \in \mathbb{Z}\}$ . Let  $\mathcal{T}_\delta(f, \delta)$  denote the set of all continuous  $\delta$ -methods  $\varphi_g$  for  $f$  which are induced by  $g \in \text{Diff}(M)$  with  $d_1(f, g) < \delta$ . Put

$$\mathcal{P}_\delta(f, \delta) = \bigcup_{\varphi \in \mathcal{T}_\delta(f, \delta)} \varphi(M).$$

DEFINITION 2.3: We say  $f \in \text{Diff}(M)$  is  $\mathcal{T}_\delta$ -shadowing if for any  $\varepsilon > 0$  there is  $\delta > 0$  and a map  $r : \mathcal{P}_\delta(f, \delta) \rightarrow M$  such that

$$d(f^k(r(x)), x_k) < \varepsilon,$$

for all  $\mathbf{x} = \{x_k : k \in \mathbb{Z}\} \in \mathcal{P}_d(f, \delta)$  and all  $k \in \mathbb{Z}$ .

We say that  $f$  is  $\mathcal{T}_d$ -continuous shadowing if the map  $r$  is continuous.

If  $f$  is  $\mathcal{T}_h$ -shadowing then it is  $\mathcal{T}_d$ -shadowing. The following example shows that the converse is untrue in general.

EXAMPLE 2.4. To get a desired diffeomorphism on  $\mathbb{R}$ , we let  $a_0 = 0$  and  $a_n = \sum_{k=1}^n 1/k$  for  $n = 1, \dots$ . Define a map  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} x + (-1)^n n(x - a_{n-1})(x - a_n) & \text{if } x \in [a_{n-1}, a_n], \\ x + (-1)^{n+1} n(x + a_{n-1})(x + a_n) & \text{if } x \in [-a_n, -a_{n-1}] \end{cases}$$

where  $n = 1, 2, \dots$ . Then  $f$  is a  $C^1$  diffeomorphism which is  $\mathcal{T}_d$ -shadowing but is not  $\mathcal{T}_h$ -shadowing. In fact, for any  $\delta > 0$ , choose an odd number  $n \in \mathbb{N}$  with  $1/n < \delta$ . Define a map  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(x) = \begin{cases} f(x) & \text{if } x \leq a_{n-1} + \frac{1}{2n}, \\ x + \frac{1}{4n} & \text{if } x \geq a_{n-1} + \frac{1}{2n}. \end{cases}$$

Then  $g$  is a  $C^1$  diffeomorphism with  $d_0(f, g) < \delta$  and  $d_1(f, g) \geq 1$ . If we let  $y = a_{n-1} + 1/2n$  then we have

$$d(f^k(x), g^k(y)) \rightarrow \infty \text{ as } k \rightarrow \infty,$$

for any  $x \in \mathbb{R}$ . This means that  $f$  is not  $\mathcal{T}_h$ -shadowing. Moreover we get

$$f'(-a_{2n-1}) = f'(a_{2n-1}) = 0 \text{ and } f'(-a_{2n}) = f'(a_{2n}) = 2$$

for each  $n \in \mathbb{N}$ . If  $g$  is another  $C^1$  diffeomorphism on  $\mathbb{R}$  with  $d_1(f, g) < 1/4$ , then  $g$  have fixed points  $z_n \in (a_{n-1}, a_{n+1})$  and  $w_n \in (-a_{n+1}, -a_{n-1})$  for each  $n \in \mathbb{N}$ . Consequently we see that  $f$  is  $\mathcal{T}_d$ -shadowing.

**THEOREM 2.5.** *If  $f \in Z(M)$  is  $\mathcal{T}_h$ -continuous shadowing then it is topologically stable.*

PROOF: Suppose  $f \in Z(M)$  is  $\mathcal{T}_h$ -continuous shadowing, and let  $\varepsilon > 0$  be arbitrary. Then we can choose  $\delta > 0$  and continuous map  $r : \mathcal{P}_h(f, \delta) \rightarrow M$  such that

$$d(f^k(r(\mathbf{x})), x_k) < \varepsilon,$$

for  $\mathbf{x} = \{x_k\} \in \mathcal{P}_h(f, \delta)$  and all  $k \in \mathbb{Z}$ . Choose  $g \in Z(M)$  satisfying  $d_0(f, g) < \delta$ , and let  $O(g) = \{O(g, x) : x \in M\}$ . Note that  $r(O(g, x)) \neq r(O(g, g(x)))$  in general. Let  $\alpha : O(g) \subset \mathcal{P}_h(f, \delta) \rightarrow M$  be a continuous choice function such that  $\alpha(O(g, x)) \in O(g, x)$  and  $\alpha(O(g, x)) = \alpha(O(g, y))$  if  $y \in O(g, x)$  for each  $x \in M$ . Define a map  $H : M \rightarrow M$  by

$$H(x) = f^{-n} \circ r|_{O(g)} \circ O_g \circ \alpha \circ O_g(x), \quad x \in M,$$

where  $n$  is the integer satisfying  $g^n(x) = \alpha(O(g, x))$ , and  $O_g : M \rightarrow O(g)$  is the orbit map given by  $O_g(x) = \{g^k(x) : k \in \mathbb{Z}\}$  for  $x \in M$ . Then  $H$  is a continuous map satisfying

$$d_0(H, 1_M) < \varepsilon \text{ and } H \circ g = f \circ H.$$

In fact, for any  $x \in M$  there exists  $m \in \mathbb{Z}$  with

$$g^m(x) = \alpha(O(g, x)) \equiv \bar{x}.$$

Then we get

$$d(H(x), x) = d\left(f^{-m}(\tau(O(g, \bar{x}))), g^{-m}(\bar{x})\right) < \varepsilon.$$

Moreover if we let  $\tau(O(g, \bar{x})) = \bar{y}$  then we have

$$H(g(x)) = f^{-(m-1)}(\bar{y}) = f(H(x)).$$

Since the map  $H$  is continuous, it is surjective for small  $\varepsilon > 0$ . This means that  $H$  is a topological semi-conjugacy between  $f$  and  $g$ , and so completes the proof. □

### 3. CONTINUOUS INVERSE SHADOWING

Hyperbolicity is a principal object of interest in the global qualitative theory of dynamical systems. Many attempts have been made to express the concept of hyperbolicity in topological terms. Notions of shadowing, coordinates, expansiveness and the like have been to be useful for this purpose (see [3, 4, 9, 11, 12]).

In this section we introduce the notions of continuous inverse shadowing with respect to various classes of admissible pseudo orbits, and characterise hyperbolicity and structural stability using the concepts of continuous inverse shadowing. When discussing inverse shadowing, an appropriate choice of the class of admissible pseudo orbits is crucial (see [1, 4, 6]).

**DEFINITION 3.1:** We say  $f \in \text{Diff}(M)$  is  $\mathcal{T}_\alpha$ -inverse shadowing for  $\alpha = 0, c, h, d$ , if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any  $\delta$ -method  $\varphi \in \mathcal{T}_\alpha(f, \delta)$  there is a map  $s : M \rightarrow M$  satisfying

$$d\left(f^k(x), \varphi(s(x))_k\right) < \varepsilon,$$

for  $x \in M$  and all  $n \in \mathbb{Z}$ .

We say that  $f$  is  $\mathcal{T}_\alpha$ -continuous inverse shadowing ( $\mathcal{T}_\alpha$ -IS) if the map  $s$  is continuous.

Clearly we have the following relations among the various notions of inverse shadowing.

$$\mathcal{T}_0 - \text{IS} \Rightarrow \mathcal{T}_c - \text{IS} \Rightarrow \mathcal{T}_h - \text{IS} \Rightarrow \mathcal{T}_d - \text{IS}.$$

In [8], Lewowicz introduced a notion of persistency for a homeomorphism  $f \in Z(X)$ ; and he showed that a pseudo Anosov diffeomorphism on a compact surface is expansive and persistent, but is not shadowing. Note that  $\mathcal{T}_h$ -inverse shadowing here is equivalent to persistency in [8, 13]; and  $\mathcal{T}_d$ -inverse shadowing is a weaker property than  $\mathcal{T}_h$ -inverse shadowing as we can see in Example 2.4.

**THEOREM 3.2.** *If  $f \in \text{Diff}(M)$  is  $\mathcal{T}_\alpha$ -continuous inverse shadowing then it is  $\mathcal{T}_\alpha$ -shadowing,  $\alpha = 0, c, h, d$ .*

PROOF: Suppose  $f$  is  $\mathcal{T}_c$ -continuous inverse shadowing, and let  $\varepsilon > 0$  be arbitrary. Then we can choose  $\delta > 0$  such that for any  $\delta$ -method  $\varphi \in \mathcal{T}_c(f, \delta)$  there exists a continuous map  $s : M \rightarrow M$  satisfying

$$d(f^k(x), \varphi(s(x)))_k < \varepsilon,$$

for  $x \in M$  and all  $n \in \mathbb{Z}$ . If  $k = 0$  then we get  $d(x, s(x)) < \varepsilon$  for  $x \in M$ . Since the map  $s$  is continuous, it is surjective for sufficiently small  $\varepsilon > 0$ . To show that  $f$  is  $\mathcal{T}_c$ -shadowing, we define a map  $r : \mathcal{P}_c(f, \delta) \rightarrow M$  as follows. For any  $\mathbf{y} = \{y_k\} \in \mathcal{P}_c(f, \delta)$ , there exist  $y \in M$  and  $\varphi \in \mathcal{T}_c(f, \delta)$  satisfying  $\mathbf{y} = \varphi(y)$ . Note that the point  $y$  is unique since  $y_0 = \varphi(y)_0 = y$ . Choose  $x \in M$  with  $s(x) = y$ , and define  $r(\mathbf{y}) = x$ . Then  $r$  is a desired map. In fact, we have

$$d(f^k(r(\mathbf{y})), y_k) = d(f^k(x), \varphi(s(x)))_k < \varepsilon,$$

for  $\mathbf{y} = \{y_k\}$  and all  $k \in \mathbb{Z}$ .

Similarly we can show that if  $f$  is  $\mathcal{T}_\alpha$ -continuous inverse shadowing then it is  $\mathcal{T}_\alpha$ -shadowing,  $\alpha = 0, h, d$ .  $\square$

We say that  $f \in \text{Diff}(M)$  is *structurally stable* if there is a  $C^1$  neighbourhood  $\mathcal{U}$  of  $f$  such that every  $g \in \mathcal{U}$  is conjugate to  $f$ . The diffeomorphism  $f$  is said to be *hyperbolic* in a closed invariant set  $\Lambda$  if there is a continuous splitting of the tangent bundle,  $TM|_\Lambda = E^s \oplus E^u$ , and there are constants  $C > 0, 0 < \lambda < 1$ , such that  $\|Df^n|_{E^s}\| < C\lambda^n$  and  $\|Df^{-n}|_{E^u}\| < C\lambda^n$  for any  $n > 0$  and  $x \in \Lambda$ . We say that  $f$  is *Anosov* if it is hyperbolic on  $M$ . The *stable* [respectively *unstable*] *manifold* of  $x \in \Lambda$  is the set of points  $p \in M$  such that  $d(f^k(x), f^k(p))$  tends to 0 as  $k$  tends to  $\infty$  [respectively  $-\infty$ ]. The diffeomorphism  $f$  satisfies *Axiom A* if its periodic points are dense in the set of nonwandering points  $\Omega$ , and  $f$  is hyperbolic on  $\Omega$ . We say that  $f$  satisfies the *strong transversality condition* if all stable and unstable manifolds intersect transversally.

It is well known that  $f \in \text{Diff}(M)$  is Anosov if and only if it is expansive and structurally stable (see [11]).

It is clear that if  $f$  is structurally stable then it is shadowing and  $\mathcal{T}_d$ -continuous inverse shadowing. But the converse does not hold in general. Moreover we know that an expansive diffeomorphism which is  $\mathcal{T}_d$ -inverse shadowing need not be shadowing. In fact,

pseudo Anosov diffeomorphism on a compact surface is an example of diffeomorphism which is expansive and  $\mathcal{T}_d$ -inverse shadowing but it is not shadowing. In the following theorem we show that an expansive diffeomorphism which is  $\mathcal{T}_d$ -continuous inverse shadowing is structurally stable.  $\square$

**THEOREM 3.3.**  *$f \in \text{Diff}(M)$  is Anosov if and only if it is expansive and  $\mathcal{T}_d$ -continuous inverse shadowing.*

**PROOF:** Suppose  $f$  is expansive and  $\mathcal{T}_d$ -continuous inverse shadowing. To prove that  $f$  is Anosov, it suffices to show that  $f$  is structurally stable. Let  $e > 0$  be an expansive constant of  $f$  and  $\varepsilon$  a constant with  $\varepsilon < e/2$ . Choose  $\delta > 0$  corresponding to the constant  $\varepsilon$  in the definition of  $\mathcal{T}_d$ -continuous inverse shadowing. Let  $g \in \text{Diff}(M)$  be such that  $d_1(f, g) < \delta$ . Then there exist a continuous map  $k : M \rightarrow M$  such that  $d(f^n(x), g^n(k(x))) < \varepsilon$ , for  $x \in M$  and all  $k \in \mathbb{Z}$ . Let  $\alpha : O(f) \rightarrow M$  be a continuous choice function satisfying  $\alpha(O(f, x)) \in O(f, x)$  and  $\alpha(O(f, x)) = \alpha(O(f, y))$  if  $y \in O(f, x)$ . Define a map  $h : M \rightarrow M$  by

$$h(x) = g^n \circ k \circ f^{-n}(x),$$

where  $n$  is the integer satisfying  $f^{-n}(x) = \alpha(O(f, x))$ . Then  $h$  is a conjugacy between  $f$  and  $g$ . In fact, for any  $x_o \in M$  we let  $\alpha(O(f, x_o)) = \bar{x}_0$ . Then there exists  $m \in \mathbb{Z}$  with  $x_o = f^m(\bar{x}_0)$ . And we get

$$h(f(x_o)) = g^{m+1}(k(\bar{x}_0)) = g(h(x_o)).$$

Moreover we have

$$d(h(x_o), x_o) = d(h(f^m(\bar{x}_0)), f^m(\bar{x}_0)) = d(g^m(k(\bar{x}_0)), f^m(\bar{x}_0)) < \varepsilon.$$

Since the map  $h$  is continuous, it is surjective for small  $\varepsilon$ . To show that  $h$  is injective, we suppose  $h(x) = h(y)$  for  $x, y \in M$ . Since  $h \circ f = g \circ h$ , we obtain

$$d(f^n(x), f^n(y)) \leq d(f^n(x), h(f^n(x))) + d(h(f^n(y)), f^n(y)) < e,$$

for all  $n \in \mathbb{Z}$ . By the expansivity of  $f$ , we get  $x = y$ .

Conversely, if  $f$  is Anosov then it is easy to show that  $f$  is  $\mathcal{T}_d$ -continuous inverse shadowing.  $\square$

Now we characterise structural stability using the concept of  $\mathcal{T}_d$ -(continuous) inverse shadowing. More precisely we shall show that the  $C^1$  interior of the set of all  $f \in \text{Diff}(M)$  which are  $\mathcal{T}_d$ -(continuous) inverse shadowing is the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

In this direction, it was proved in [9] and [13] respectively that the  $C^1$  interior of the set of all  $f \in \text{Diff}(M)$  having topological stability and the  $C^1$  interior of the set of

all  $f \in \text{Diff}(M)$  having persistency were characterised as the set of all diffeomorphisms satisfying Axiom A and the strong transversality condition.

Note that persistency in [8, 13] is a weaker property than topological stability, and the  $\mathcal{T}_d$ -inverse shadowing is a weaker property than persistency, that is, topological stability  $\Rightarrow$  persistency  $\Rightarrow \mathcal{T}_d$ -inverse shadowing.

**LEMMA 3.4.** *Suppose  $f, g \in \text{Diff}(M)$  are conjugate. If  $f$  is  $\mathcal{T}_\alpha$ -inverse shadowing,  $\alpha = 0, c, h, d$ , if and only if  $g$  is  $\mathcal{T}_\alpha$ -inverse shadowing.*

The proof is straightforward and is omitted

**LEMMA 3.5.**  *$f \in \text{Diff}(M)$  is  $\mathcal{T}_\alpha$ -inverse shadowing [respectively  $\mathcal{T}_\alpha$ -continuous inverse shadowing],  $\alpha = 0, c, h, d$ , if and only if  $f^k$  is  $\mathcal{T}_\alpha$ -inverse shadowing [respectively  $\mathcal{T}_\alpha$ -continuous inverse shadowing] for some positive integer  $k$ .*

**PROOF:** Assume that  $f^k$  is  $\mathcal{T}_c$ -inverse shadowing, and let  $\varepsilon > 0$  be arbitrary. Choose  $0 < \varepsilon_1 < \varepsilon/2$  such that  $d(x, y) < \varepsilon_1$  implies  $d(f(x), f(y)) < \varepsilon/2$ . Moreover we can choose  $0 < \varepsilon_k < \varepsilon_{k-1} < \dots < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$  such that  $d(x, y) < \varepsilon_i$  implies  $d(f(x), f(y)) < \varepsilon_{i-1}$ , where  $i = 2, 3, \dots, k$ . Choose  $\delta > 0$  corresponding to the constant  $\varepsilon_k$  in the definition of  $\mathcal{T}_c$ -inverse shadowing of  $f^k$ .

Choose  $\alpha > 0$  satisfying  $k\alpha < \min\{\delta, \varepsilon/2\}$ . Then there is  $\beta > 0$  such that  $d(x, y) < \beta$  implies  $d(f^i(x), f^i(y)) < \alpha$ , where  $i = 0, 1, \dots, k$ . Let  $\varphi \in \mathcal{T}_c(f, \beta)$ . We define a map  $\psi : M \rightarrow M^{\mathbb{Z}}$  by

$$\psi(x)_n = \varphi(x)_{nk},$$

for  $x \in M$  and all  $n \in \mathbb{Z}$ . Then  $\psi$  is a continuous  $\delta$ -method for  $f^k$ . In fact, we have

$$\begin{aligned} d(f^k(\psi(x)_n), \psi(x)_{n+1}) &= d(f^k(\varphi(x)_{nk}), \varphi(x)_{(n+1)k}) \\ &\leq d(\varphi(x)_{nk+k}, f(\varphi(x)_{nk+(k-1)})) \\ &\quad + d(f(\varphi(x)_{nk+(k-1)}), f^2(\varphi(x)_{nk+(k-2)})) \\ &\quad + \dots \\ &\quad + d(f^{k-1}(\varphi(x)_{nk+1}), f^k(\varphi(x)_{nk})) < k\alpha < \delta, \end{aligned}$$

for all  $n \in \mathbb{Z}$ . Since  $f^k$  is  $\mathcal{T}_c$ -inverse shadowing, there exists a map  $r : M \rightarrow M$  such that

$$d((f^k)^n(x), \psi(r(x))_n) < \varepsilon_k,$$

for  $x \in M$  and all  $n \in \mathbb{Z}$ . Then we have

$$\begin{aligned} d(f^i(f^{nk}(x)), \varphi(r(x))_{nk+i}) &\leq d(f^i(f^{nk}(x)), f^i(\varphi(r(x))_{nk})) \\ &\quad + d(f^i(\varphi(r(x))_{nk}), f^{i-1}(\varphi(r(x))_{nk+1})) \\ &\quad + d(f^{i-1}(\varphi(r(x))_{nk+1}), f^{i-2}(\varphi(r(x))_{nk+2})) \end{aligned}$$

$$\begin{aligned}
 &+ \dots\dots\dots \\
 &+ d\left(f^2\left(\varphi(r(x))_{nk+i-2}\right), f\left(\varphi(r(x))_{nk+i-1}\right)\right) \\
 &+ d\left(f\left(\varphi(r(x))_{nk+i-1}\right), \varphi(r(x))_{nk+i}\right) \\
 &< \varepsilon_{k-i} + i\alpha < \varepsilon_{k-i} + k\alpha < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,
 \end{aligned}$$

for any  $0 \leq i \leq k - 1$ . This means that  $f$  is  $\mathcal{T}_c$ -shadowing.

Similarly we can show that if  $f^k$  is  $\mathcal{T}_\alpha$ -inverse shadowing then  $f$  is  $\mathcal{T}_\alpha$ -inverse shadowing for  $\alpha = 0, h, d$ . □

Denote by  $CIS_d(M)$  [respectively  $IS_d(M)$ ] the set of all diffeomorphisms  $f \in \text{Diff}(M)$  which are  $\mathcal{T}_d$ -continuous inverse shadowing [respectively  $\mathcal{T}_d$ -inverse shadowing]; and denote by  $CIS_d(M)^\circ$  [respectively  $IS_d(M)^\circ$ ] the  $C^1$  interior of the set  $CIS_d(M)$  [respectively  $IS_d(M)$ ] in  $\text{Diff}(M)$ .

**THEOREM 3.6.** *The set  $IS_d(M)^\circ$  is characterised as the set of all structurally stable diffeomorphisms on  $M$ . In particular,  $CIS_d(M)^\circ = IS_d(M)^\circ$ .*

**PROOF:** First we show that every  $f \in IS_d(M)^\circ$  satisfies Axiom A with no cycles. For this we let  $\mathcal{F}(M)$  be the set of all  $f \in \text{Diff}(M)$  having a  $C^1$  neighbourhood  $\mathcal{U} \subset \text{Diff}(M)$  such that every periodic point of  $g \in \mathcal{U}$  is hyperbolic. Then it is enough to show that  $IS_d(M)^\circ \subset \mathcal{F}(M)$  by the results of Hayashi (see [5]). To show that  $IS_d(M)^\circ \subset \mathcal{F}(M)$ , we cannot use the proof of [13, Proposition A] directly, since we cannot guarantee that the map  $\tilde{g}$  constructed in the proof satisfies  $d_1(\tilde{g}, g) < \tilde{\delta}$ .

Suppose  $f \notin \mathcal{F}(M)$ , and let  $\tau > 0$  be arbitrary. Then we can choose  $g \in \text{Diff}(M)$  such that  $d_1(f, g) < \tau/3$  and  $g$  has a non-hyperbolic periodic point  $p$ . By applying Lemma 3.5, if necessary, we may assume that  $p$  is a non-hyperbolic fixed point of  $g$ . Then the linear map  $Dg_p : T_pM \rightarrow T_pM$  has an eigenvalue  $\lambda$  with  $|\lambda| = 1$ . Suppose  $\lambda = 1$ . Choose  $h \in \text{Diff}(M)$  such that

- (1)  $d_1(g, h) < \tau/3$  and  $p$  is a fixed point of  $h$ ;
- (2)  $\lambda$  is an eigenvalues of  $Dh_p$ ;
- (3) for any other eigenvalues  $\mu_1, \dots, \mu_{n-1}$  of  $Dh_p$ ,  $|\mu_i| \neq 1$  for  $i = 1, \dots, n - 1$ .

We find a local coordinate neighbourhood  $(V, \varphi)$  of  $p$ ,  $\varphi = (y_1, \dots, y_n)$ , and a diffeomorphism  $\tilde{f}$  such that

- (1)  $d_1(h, \tilde{f}) < \tau/3$  and  $\varphi(p) = 0 \in \mathbb{R}^n$ ; and
- (2)  $\tilde{f}$  coincides with  $Dh_p$  on  $V$  with respect to  $\varphi$ .

Let  $L$  be the one dimensional subspace of  $Dh_p$  corresponding to  $\lambda$ , and assume that  $L = \{y \in \mathbb{R}^n : y_2 = \dots = y_n = 0\}$ . Then each point of the set  $L \cap V$  is fixed under  $\tilde{f}$ , but the orbit  $O(\tilde{f}, x)$  of  $\tilde{f}$  through  $x \in V/L$  is not contained in  $V$ .

Now we show that  $\tilde{f}$  is not  $\mathcal{T}_d$ -inverse shadowing. To prove this we may apply Lemma 3.4, if necessary. Choose  $\varepsilon' > 0$  satisfying  $(-2\varepsilon', 2\varepsilon')^n \subset V$ . For any  $0 < \varepsilon < \varepsilon'$ , select  $\delta > 0$  and diffeomorphism  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

- (1)  $H(x) = x + \delta$  for  $x \in [-\epsilon, \epsilon] \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-1}$ ;
- (2)  $H(x) = x$  for  $x \notin [-\epsilon - \delta, \epsilon + \delta] \times [-\delta, \delta]^{n-1}$ ;
- (3)  $\|DH - I\| < 2\delta$  and  $\|D\tilde{f}\|2\delta < \epsilon$ ,

where  $I$  denotes the identity map. Put  $F = H \circ \tilde{f}$ . Then we have

$$d_1(\tilde{f}, F) \leq \|D\tilde{f}\|2\delta < \epsilon.$$

Choose an integer  $N$  satisfying  $\epsilon < N\delta$ . Then we get

$$d(\tilde{f}^N(p), F^N(x)) \geq \epsilon \text{ or } d(\tilde{f}^{-N}(p), F^{-N}(x)) \geq \epsilon$$

for any  $x \in L \cap V$ . If  $x \notin L \cap V$  then  $O(F, x) \cap (M/V) \neq \emptyset$ . This means that  $F^n(x) \in M/V$  for some  $n \in \mathbb{Z}$ , and so  $d(\tilde{f}^n(p), F^n(x)) \geq \epsilon$ . Consequently we showed that for any  $r > 0$  there exists  $\tilde{f} \in \text{Diff}(M)$  such that  $d_1(f, \tilde{f}) < r$  and  $\tilde{f} \notin IS_d(M)$ . This means that  $f \notin IS_d(M)^\circ$ .

Next we show that  $f \in IS_d(M)^\circ$  satisfies the strong transversality condition. To show this, we can adopt the proof of [13, Proposition B] directly. In fact, in the proof of [13, Lemma 2], we can choose  $\psi_\delta \in \text{Diff}(M)$  satisfying

- (1)  $\psi_\delta|_{(B_{\nu_3}(g^l(x)))^c} = id$ ,
- (2)  $d_1(\psi_\delta, id) < \delta$ ,
- (3)  $\psi_\delta\left(\text{exp}_p(E'_{(\nu_3)/2}(g^l(x)))\right) \cap W_{\epsilon_1}^s(p, g') = \emptyset$ .

Then we have  $\tilde{g} = g' \circ \psi_\delta \in \text{Diff}(M)$  and  $d_1(\tilde{g}, g') < \delta$ . Hence we can show that  $f$  satisfies the strong transversality condition by the same technique mentioned in [13, Proposition B].

Conversely, it is easy to show that if  $f$  is structurally stable then  $f$  is  $\mathcal{T}_d$ -continuous inverse shadowing.

Consequently the following inclusions complete the proof;

$$CIS_d(M)^\circ \subset IS_d(M)^\circ \subset SS(M) \subset CIS_d(M)^\circ,$$

where  $SS(M)$  is the set of all structurally stable diffeomorphisms on  $M$ . □

### REFERENCES

- [1] R. Corless and S. Pilyugin, 'Approximate and real trajectories for generic dynamical systems', *J. Math. Anal. Appl.* **189** (1995), 409–423.
- [2] P. Diamond, P. Kloeden, V.Kozyakin and A. Pokrovskii, 'Computer robustness of semi-hyperbolic mappings', *Random Comput. Dynamics* **3** (1995), 53–70.
- [3] P. Diamond, P. Kloeden, V.Kozyakin and A. Pokrovskii, 'Expansivity of semi-hyperbolic Lipschitz mappings', *Bull. Austral. Math. Soc.* **51** (1995), 301–308.
- [4] P. Diamond, K. Lee and Y. Han, 'Bishadowing and hyperbolicity', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* (2002) (to appear).

- [5] S. Hayashi, 'Diffeomorphisms in  $\mathcal{F}^1(M)$  satisfy Axiom A', *Ergodic Theory Dynamical Systems* **12** (1992), 233–253.
- [6] P. Kloeden and J. Ombach, 'Hyperbolic homeomorphisms and bishadowing', *Ann. Polon. Math.* **65** (1997), 171–177.
- [7] P. Kloeden, J. Ombach and A. Pokrovskii, 'Continuous and inverse shadowing', *Funct. Differ. Equ.* **6** (1999), 137–153.
- [8] J. Lewowicz, 'Persistence in expansive systems', *Ergodic Theory Dynamical Systems* **3** (1983), 567–578.
- [9] K. Moriyasu, 'The topological stability of diffeomorphisms', *Nagoya Math. J.* **123** (1991), 91–102.
- [10] Z. Nitecki and M. Shub, 'Filtrations, decompositions and explosions', *Amer. J. Math.* **97** (1975), 1029–1047.
- [11] J. Ombach, 'Shadowing, expansiveness and hyperbolic homeomorphisms', *J. Austral. Math. Soc.* **61** (1996), 57–72.
- [12] S. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Math. **1706** (Springer-Verlag, Berlin, 1999).
- [13] K. Sakai, 'Diffeomorphisms with persistency', *Proc. Amer. Math. Soc.* **124** (1996), 2249–2254.
- [14] K. Yano, 'Topologically stable homeomorphisms of the circle', *Nagoya Math. J.* **79** (1980), 145–149.

Department of Mathematics  
Chungnam National University  
Daejeon, 305-764  
Korea  
e-mail: khlee@math.cnu.ac.kr