Canad. Math. Bull. Vol. **54** (2), 2011 pp. 288–296
doi:10.4153/CMB-2011-013-1
© Canadian Mathematical Society 2011

The Resultant of Chebyshev Polynomials

David P. Jacobs, Mohamed O. Rayes, and Vilmar Trevisan

Abstract. Let T_n denote the n-th Chebyshev polynomial of the first kind, and let U_n denote the n-th Chebyshev polynomial of the second kind. We give an explicit formula for the resultant $\operatorname{res}(T_m, T_n)$. Similarly, we give a formula for $\operatorname{res}(U_m, U_n)$.

1 Introduction

The resultant of two polynomials is, in general, a complicated formula involving its coefficients, and there exist few polynomial families for which simple closed formulas for their resultants are known. The best example of such a formula is due to Apostol [1] who obtained the elegant formula

$$\operatorname{res}(\Phi_m, \Phi_n) = \begin{cases} p^{\varphi(m)} & \text{if } \frac{m}{n} \text{ is a power of a prime } p, \\ 1 & \text{otherwise,} \end{cases}$$

where Φ_n is the cyclotomic polynomial

$$\Phi_n = \prod_{\substack{k=1 \\ (k,n)=1}}^{n} (x - e^{2\pi i k/n}).$$

In a similar vein, we wish to study the resultants $\operatorname{res}(T_m, T_n)$ and $\operatorname{res}(U_m, U_n)$, where T_n and U_n denote, respectively, the n-th Chebyshev polynomials of the first and second kind. Assume that m and n are natural numbers, not both zero. We show that if $m = gm_1$, $n = gn_1$, and $g = \gcd(m, n)$, then

$$\operatorname{res}(T_m, T_n) = \begin{cases} 0 & \text{if } n_1 m_1 \text{ is odd,} \\ (-1)^{\frac{mn}{2}} 2^{(m-1)(n-1)+g-1} & \text{otherwise.} \end{cases} \tag{1.1}$$

We also show that

$$\operatorname{res}(U_m, U_n) = \begin{cases} 0 & \text{if } \gcd(m+1, n+1) \neq 1, \\ (-1)^{\frac{mn}{2}} 2^{mn} & \text{otherwise.} \end{cases} \tag{1.2}$$

Received by the editors June 4, 2008.
Published electronically February 10, 2011.
This research was supported by CNPq Grant 478290/04-7, FAPERGS Grant 06/1001.7, and the Brazilian Fulbright Commission.
AMS subject classification: **11Y11**, 68W20.
Keywords: resultant, Chebyshev polynomial.

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

The proofs of equations (1.1) and (1.2) are given in Sections 4 and 5, respectively. In Section 2 we recall the properties of Chebyshev polynomials needed for our proofs, while in Section 3 the properties of resultants are recalled.

Although our arguments use elementary properties of resultants and Chebyshev polynomials, we are not aware of any result that explicitly computes the resultant of arbitrary Chebyshev polynomials. Related results include the recent paper by Dilcher and Stolarsky [3], where parameters are introduced into cyclotomic polynomials to obtain formulas for the resultant and discriminant of certain linear forms of Chebyshev polynomials of the second kind. Some of the results in [3] were generalized and also extended to polynomials of the first kind by Gishe and Ismail in [5]. In the larger setting of ultraspherical polynomials, these linear combinations play a role studying spherical arrangements, as in the paper by Cohn and Kumar [2]. Some of the most advanced work done on discriminants and resultants of special families of polynomials was done by Roberts [7]. For simply computing the discriminants of Chebyshev polynomials, we refer the reader to Rivlin [8].

2 Properties of Chebyshev Polynomials

The Chebyshev polynomials of the first kind $T_n(x)$ may be defined by the following recurrence relation. Set $T_0(x) = 1$ and $T_1(x) = x$, then

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \dots .$$

The polynomial T_n has degree n, and for $n \geq 1$, its leading coefficient is 2^{n-1}. Alternatively, they may be defined as

$$T_n(x) = \cos(n \cdot \arccos x),$$

where $0 \leq \arccos x \leq \pi$. The roots of $T_n(x)$ are real, distinct, and lie within the interval $[-1, 1]$, and are given by the closed formula

$$\xi_k = \cos\Big(\frac{\pi}{2}\frac{2k-1}{n}\Big) \quad k = 1, \dots, n.$$

Using well-known decomposition formulas, one can easily prove [6] that for two nonnegative integers $m \geq n$, we have

$$T_m(x) = 2T_{m-n}(x)T_n(x) - T_{|m-2n|}(x). \tag{2.1}$$

The Chebyshev polynomials of the second kind are defined by setting $U_0(x) = 1$, $U_1(x) = 2x$ and using the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \dots .$$

The polynomial U_n has degree n and leading coefficient 2^n. It can also be defined by

$$U_n(x) = \frac{1}{n+1}T'_{n+1}(x) = \frac{\sin\big((n+1)\arccos x\big)}{\sin(\arccos x)}.$$

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

Its roots are also real, distinct, lie in $[-1, 1]$, and are given by

$$\eta_k = \cos\Big(\pi \frac{k}{n+1}\Big) \quad k = 1, \dots, n.$$

Given two natural numbers m and n, $n \le m$, a useful decomposition property for the U polynomials is the following ([6]):

$$U_m(x) = \begin{cases} 2T_{m-n}(x)U_n(x) - U_{2n-m}(x), & n \le m < 2n+1, \\ 2T_{m-n}(x)U_n(x), & m = 2n+1, \\ 2T_{m-n}(x)U_n(x) + U_{m-2n-2}(x), & m > 2n+1. \end{cases} \tag{2.2}$$

3 Properties of the Resultant

The resultant of two polynomials

$$\begin{aligned} f(x) &= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0 \\ g(x) &= b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0 \end{aligned} \tag{3.1}$$

with coefficients in Euclidean domain D is defined by the Sylvester determinant

$$\operatorname{res}(f,g) = \begin{vmatrix} a_m & a_{m-1} & \cdots & a_0 & & & & \\ & a_m & a_{m-1} & \cdots & a_0 & & & \\ & & & \vdots & & & & \\ & & & a_m & a_{m-1} & \cdots & & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & & & & \\ & b_n & b_{n-1} & \cdots & b_0 & & & \\ & & & & \vdots & & & \\ & & & & b_n & b_{n-1} & \cdots & b_0 \end{vmatrix}$$

in which there are n rows of f coefficients and m rows of g coefficients and the remaining entries are zero. Clearly, $\operatorname{res}(f,g)$ is an element of D. If f and g are expressed in terms of their zeros

$$f(x) = a_m \prod_{k=1}^{m}(x - \alpha_k) \quad \text{and} \quad g(x) = b_n \prod_{j=1}^{n}(x - \beta_j),$$

then the resultant can be expressed by the products

$$\operatorname{res}(f,g) = a_m^n \prod_{i=1}^{m} g(\alpha_i) = b_n^m \prod_{j=1}^{n} f(\beta_j) \tag{3.2}$$

$$\operatorname{res}(f,g) = a_m^n b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j). \tag{3.3}$$

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

Other well-known properties of the resultants are

$$\operatorname{res}(f,g) = (-1)^{mn}\operatorname{res}(g,f), \tag{3.4}$$

$$\operatorname{res}(f,qg) = \operatorname{res}(f,q)\operatorname{res}(f,g), \tag{3.5}$$

and if $a \neq 0$ is a constant, then

$$\operatorname{res}(f,a) = \operatorname{res}(a,f) = a^m. \tag{3.6}$$

The fundamental property of the resultants is the following.

Theorem 3.1 *Let f and g be polynomials over a Euclidean domain. Then $\operatorname{res}(f,g) = 0$ if and only if f and g have a common divisor of positive degree.*

These properties are well known and may be found, for example, in [9]. A more complete treatment may be found in [4]. The following lemma, with no bound on the degree of r, is a result of [3].

Lemma 3.2 *Let f and g be polynomials as in* (3.1).

(i) *If we can write $f(x) = q(x)g(x) + r(x)$, with polynomials q, r and $\delta = \deg r$, then* $\operatorname{res}(g,f) = b_n^{m-\delta}\operatorname{res}(g,r)$.

(ii) *If $\deg(qf+g) = \deg g$ for a polynomial q, then $\operatorname{res}(f,qf+g) = \operatorname{res}(f,g)$.*

4 Resultant of Chebyshev Polynomials of the First Kind

With a slight change in notation, [6, Theorem 2] may be stated as follows.

Theorem 4.1 *For positive integers m and n, where $g = \gcd(m,n)$, $m = gm_1$, and $n = gn_1$,*

$$\gcd(T_m,T_n) = \begin{cases} 1 & \text{if } m_1 \text{ or } n_1 \text{ is even,} \\ T_g(x) & \text{otherwise.} \end{cases}$$

Theorem 3.1 and Theorem 4.1 imply the following.

Corollary 4.2 *For positive integers m and n, $\operatorname{res}(T_m,T_n) \neq 0$ if and only if m_1 or n_1 is even, where $g = \gcd(m,n)$, $m = gm_1$, and $n = gn_1$.*

Corollary 4.3 *If m and n are odd, then $\operatorname{res}(T_m,T_n) = 0$.*

Lemma 4.4 *Let m, n be natural numbers. Then*

(i) $\operatorname{res}(T_m,T_n) = \operatorname{res}(T_n,T_m)$.

(ii) $\operatorname{res}(T_m,-T_n) = (-1)^m\operatorname{res}(T_m,T_n)$.

Proof If both m and n are odd, then it follows by Corollary 4.3 that $\operatorname{res}(T_m,T_n) = 0 = \operatorname{res}(T_n,T_m)$. If either m or n is even, then $\operatorname{res}(T_m,T_n) = (-1)^{mn}\operatorname{res}(T_n,T_m) = \operatorname{res}(T_n,T_m)$. For item (ii), we observe that from equations (3.5) and (3.6) we have $\operatorname{res}(T_m,-T_n) = \operatorname{res}(T_m,-1)\operatorname{res}(T_m,T_n) = (-1)^m\operatorname{res}(T_m,T_n)$. ■

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

Theorem 4.5 *Let $n \geq 1$. Then*

$$\operatorname{res}(T_1, T_n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n = 4k, \\ -1 & \text{otherwise.} \end{cases}$$

Proof If we write $T_n(x) = \sum_{i=0}^{n} a_k x^k$, then $a_n = 2^{n-1}$. By observing that $T_1(x) = x$, and $T_n(x) = 2^{n-1} \prod_{i=1}^{n}(x - \xi_i)$, where $\xi_i = \cos \frac{(2i-1)\pi}{2n}$ and by applying property (3.3), we obtain

$$\operatorname{res}(T_1, T_n) = 2^{n-1} \prod_{i=1}^{n} (0 - \xi_i) = T_n(0) = \cos(n\frac{\pi}{2}),$$

and the result follows. ∎

We now state our general formula for the resultant of two Chebyshev polynomials of the first kind.

Theorem 4.6 *Let m, n be natural numbers, not both zero, and $g = \gcd(m, n)$, $m = gm_1$, and $n = gn_1$. Then*

$$\operatorname{res}(T_m, T_n) = \begin{cases} 0 & \text{if } n_1 m_1 \text{ is odd,} \\ (-1)^{\frac{mn}{2}} 2^{(m-1)(n-1)+g-1} & \text{otherwise.} \end{cases} \tag{4.1}$$

Proof We first dispense of some simple cases. If $m = 0$, then $m_1 n_1 = 0$ is even, $g = n$, and from (3.6) we have

$$\operatorname{res}(T_m, T_n) = \operatorname{res}(1, T_n) = 1 = (-1)^{\frac{mn}{2}} 2^{(m-1)(n-1)+g-1}.$$

Similarly, if $n = 0$, the formula holds. So we may assume both m and n are positive. If $m = n$, then $m_1 n_1 = 1$, and Theorem 3.1 implies $\operatorname{res}(T_m, T_m) = 0$. Hence we may assume that $n \neq m$. Using Lemma 4.4(i), we may also assume that $m > n > 0$. We induct on m. The basis case occurs when $m = 2$ and so $n = 1$. It can be verified that

$$\operatorname{res}(T_2, T_1) = -1 = (-1)^1 2^{1(0)+1-1}.$$

Next, assume that for all $k < m$, where $n < k$, $\operatorname{res}(T_k, T_n)$ is given by (4.1). In particular, assuming $k_1 = \frac{k}{\gcd(k,n)}$ and $n_1 = \frac{n}{\gcd(k,n)}$, if $k_1 n_1$ is even, then

$$\operatorname{res}(T_k, T_n) = (-1)^{\frac{kn}{2}} 2^{(k-1)(n-1)+g-1}. \tag{4.2}$$

Now consider $\operatorname{res}(T_m, T_n)$, where $n < m$, $g = \gcd(m, n)$, $m = gm_1$ and $n = gn_1$. By Corollary 4.2, we may assume $n_1 m_1$ is even, for otherwise $\operatorname{res}(T_m, T_n)$ is zero. Now let $k = |m - 2n|$. Applying Lemma 3.2(i) to equation (2.1), where $f = T_m$, $g = T_n$, and $r = -T_k$, and then using Lemma 4.4(ii), we have

$$\operatorname{res}(T_n, T_m) = (2^{n-1})^{m-k} \operatorname{res}(T_n, -T_k) = (-1)^n 2^{(n-1)(m-k)} \operatorname{res}(T_n, T_k). \tag{4.3}$$

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

However, this is also $\operatorname{res}(T_m, T_n)$ by Lemma 4.4(i). Next, it is easy to see that $k < m$. Note that $k \neq n$, for otherwise we would have $m = n$ or $m = 3n$, and so $n_1 m_1$ would be odd. We next observe that $g = \gcd(m, n) = \gcd(k, n)$. In fact, $k = k_1 g$, where

$$k_1 = \frac{k}{g} = \frac{|m - 2n|}{g} = |m_1 - 2n_1|. \tag{4.4}$$

From (4.4), it follows that $n_1 k_1$ is even, since $m_1 n_1$ is even. Since n and k are distinct positive integers for which $n_1 k_1$ is even, we may apply the induction assumption to obtain (4.2). Replacing $\operatorname{res}(T_n, T_k)$ in (4.3) with the right side of (4.2) yields

$$\operatorname{res}(T_m, T_n) = (-1)^{\frac{mn}{2}} 2^{(m-1)(n-1)+g-1},$$

completing the induction. ■

By observing that $\gcd(n, n-1) = 1$, we have the following.

Corollary 4.7 *For any integer $n \geq 1$, we have*

$$\operatorname{res}(T_n, T_{n-1}) = (-1)^{\frac{n(n-1)}{2}} 2^{(n-1)(n-2)}.$$

Corollary 4.8 *For $n \geq 1$, we have*

$$\operatorname{res}(T_2, T_n) = \begin{cases} -2^{n-1} & \text{if } n \text{ is odd,} \\ 2^n & \text{if } n \equiv 0 \bmod 4, \\ 0 & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Proof This may be seen directly from Theorem 4.6. An alternative proof may be obtained by observing that $T_2(x) = 2x^2 - 1 = 2(x - 1/\sqrt{2})(x + 1/\sqrt{2})$ and by applying (3.2), so we can write

$$\operatorname{res}(T_2, T_n) = 2^n \prod_{i=1}^{2} T_n(\alpha_i) = 2^n T_n(\frac{1}{\sqrt{2}}) T_n(\frac{-1}{\sqrt{2}}) = 2^n \cos(n\pi/4)\cos(3n\pi/4).$$

The result now follows by direct computation. ■

5 Resultant of Chebyshev Polynomials of the Second Kind

The following result can be found in [8] and also in [6].

Theorem 5.1 *Let $m > n$ be natural numbers, $g = \gcd(m+1, n+1)$. Then*

$$\gcd(U_m, U_n) = U_{g-1}.$$

Using the interpretation of Theorem 3.1, the Theorem 5.1 immediately implies the following.

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

Corollary 5.2 *For any nonnegative integers m, n,*

$$\operatorname{res}(U_m, U_n) \neq 0 \text{ if and only if } \gcd(m+1, n+1) = 1.$$

Corollary 5.3 *If both m and n are odd, then* $\operatorname{res}(U_m, U_n) = 0$.

Lemma 5.4 *Let m, n be natural numbers. Then*

(i) $\operatorname{res}(U_m, U_n) = \operatorname{res}(U_n, U_m)$;
(ii) $\operatorname{res}(U_m, -U_n) = (-1)^m \operatorname{res}(U_m, U_n)$.

Proof If both m and n are odd, then it follows by Corollary 5.3 that $\operatorname{res}(U_m, U_n) = 0 = \operatorname{res}(U_n, U_m)$. If either m or n is even, then from (3.4) we have $\operatorname{res}(U_m, U_n) = (-1)^{mn} \operatorname{res}(U_n, U_m) = \operatorname{res}(U_n, U_m)$. The proof of (ii) is similar to that of Lemma 4.4(ii). ■

Theorem 5.5 *Let $n \geq 1$. Then*

$$\operatorname{res}(U_1, U_n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} 2^n & \text{if } n \text{ is even.} \end{cases}$$

Proof If we write $U_n(x) = \sum_{i=0}^{n} a_k x^k$, then $a_n = 2^n$. By observing that $U_1(x) = 2x$, that $U_n(x) = 2^n \prod_{i=1}^{n} (x - \eta_i)$, where $\eta_k = \cos \frac{k\pi}{n+1}$, and by applying property (3.3), we obtain

$$\operatorname{res}(U_1, U_n) = 2^n 2^n \prod_{i=1}^{n} (0 - \eta_i) = 2^n U_n(0) = 2^n \sin((n+1)\frac{\pi}{2}),$$

and the result follows. ■

We now state the general formula for the resultant of two Chebyshev polynomials of the second kind.

Theorem 5.6 *Let m, n be natural numbers, not both zero, Then*

$$\operatorname{res}(U_m, U_n) = \begin{cases} 0 & \text{if } \gcd(m+1, n+1) \neq 1, \\ (-1)^{\frac{mn}{2}} 2^{mn} & \text{otherwise.} \end{cases}$$

Proof If either $n = 0$ or $m = 0$, it is easy to see that (1.2) holds. Thus we may assume that both m and n are positive. Since $\operatorname{res}(U_m, U_n) = \operatorname{res}(U_n, U_m)$, we may assume that $m \geq n$. Note that if $m = n > 0$, then both $\operatorname{res}(U_m, U_n) = 0$ and $\gcd(m+1, n+1) \neq 1$. We induct on m. The basis case occurs when $m = n = 1$, which has already been established. So let us assume that for each value of $k < m$, the resultant $\operatorname{res}(U_k, U_t)$, is given by formula (1.2), for all $t < k$. Now consider $\operatorname{res}(U_m, U_n)$, where $0 < n < m$. We consider three cases.

Case i: $n < m < 2n + 1$. Using equation (2.2), Lemma 3.2(i) and Lemma 5.4, we have

$$\operatorname{res}(U_m, U_n) = (2^n)^{m-(2n-m)} \operatorname{res}(U_n, -U_{2n-m}) = (-1)^n 2^{n(2m-2n)} \operatorname{res}(U_n, U_{2n-m}).$$

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

Since $0 \le 2n - m < n < m$, we may apply the induction hypothesis. Note also that

$$\gcd(m+1, n+1) = \gcd(n+1, 2n-m+1),$$

so that $\operatorname{res}(U_m, U_n) = 0$ if and only if $\operatorname{res}(U_n, U_{2n-m}) = 0$. When $\operatorname{res}(U_m, U_n) \neq 0$, we obtain

$$\operatorname{res}(U_m, U_n) = (-1)^{n+\frac{n(2n-m)}{2}} 2^{n(2m-2n)+n(2n-m)} = (-1)^{n+\frac{n(2n-m)}{2}} 2^{nm}.$$

Noticing that the parity of $nm/2$ equals the parity of $n + \frac{n(2n-m)}{2}$, the result follows.

Case ii: $m = 2n+1$. Using equation (2.2), U_m is a multiple of U_n and by Corollary 5.2, it follows that $\operatorname{res}(U_m, U_n) = 0$, in accordance with the fact that $\gcd(m+1, n+1) = \gcd(2n+2, n+1) = n+1 \neq 1$.

Case iii: $m > 2n+1$. Using equation (2.2) and Lemma 3.2(i) we have

$$\operatorname{res}(U_m, U_n) = (2^n)^{m-(m-2n-2)} \operatorname{res}(U_n, U_{m-2n-2}).$$

We notice that again $\gcd(m+1, n+1) = \gcd(m-2n-2+1, n+1)$, so that we need to consider only when $\operatorname{res}(U_m, U_n) \neq 0$. Since $0 \le m-2n-2 < m$, we apply the induction hypothesis to obtain

$$\operatorname{res}(U_m, U_n) = (-1)^{\frac{n(m-2n-2)}{2}} 2^{n(2n+2)} 2^{n(m-2n-2)} = (-1)^{\frac{mn}{2}} 2^{mn}$$

Hence the result follows by induction. ∎

The following result also appears in [3, p. 379].

Corollary 5.7 (Dilcher and Stolarsky) *For integer $n \ge 1$,*

$$\operatorname{res}(U_n, U_{n-1}) = (-1)^{\frac{n(n-1)}{2}} 2^{n(n-1)}.$$

References

[1] T. M. Apostol, *Resultants of cyclotomic polynomials.* Proc. of the Amer. Math. Soc. **24**(1970), 457–462.

[2] H. Cohn and A. Kumar, *Universally optimal distribution of points on spheres.* J. Amer. Math. **20**(2007), no. 1, 99–148. doi:10.1090/S0894-0347-06-00546-7

[3] K. Dilcher and K. B. Stolarsky, *Resultants and discriminants of Chebyshev and related polynomials.* Trans. Amer. Math. Soc. **357**(2005), no. 3, 965–981. doi:10.1090/S0002-9947-04-03687-6

[4] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinksky, *Discriminants, resultants and multidimensional determinants.* Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994.

[5] J. Gishe and M. E. H. Ismail, *Resultants of Chebyshev polynomials.* Z. Anal. Anwend. **27**(2008), no. 4, 491–500. doi:10.4171/ZAA/1368

[6] M. O. Rayes, V. Trevisan, and P. S. Wang, *Factorization properties of Chebyshev polynomials.* Comput. Math. Appl. **50**(2005), no. 8–9, 1231–1240. doi:10.1016/j.camwa.2005.07.003

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press

[7] D. P. Roberts, *Discriminants of some Painlevé polynomials.* In: Number theory for the millennium, III (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 205–221.
[8] T. J. Rivlin, *The Chebyshev polynomials. From approximation theory to algebra and number theory.* Second ed., Pure and Applied Mathematics, John Wiley & Sons, New York, 1990.
[9] B. L. van der Waerden, *Modern algebra.* v. 1, Ungar, New York, 1950.

School of Computing, Clemson University, Clemson, SC 29634–0974, U.S.A.
e-mail: dpj@cs.clemson.edu

Dept. of Comp. Sci. and Eng., Southern Methodist University, U.S.A.
e-mail: mrayes@engr.smu.edu

Instituto de Matemática, UFRGS, Porto Alegre, Brazil
e-mail: trevisan@mat.ufrgs.br

https://doi.org/10.4153/CMB-2011-013-1 Published online by Cambridge University Press