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# Complementary Series for Hermitian Quaternionic Groups

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*Abstract.* Let $G$ be a hermitian quaternionic group. We determine complementary series for representations of $G$ induced from super-cuspidal representations of a Levi factor of the Siegel maximal parabolic subgroup of $G$.

## Introduction

Let $F$ be a non-archimedean field of characteristic zero. Let $F'$ be a finite dimensional division algebra over $F$ with an anti-involution $\tau$, such that the set of fixed points of $\tau$ is $F$. We have three cases:

(I) *$F' = F$ and $\tau$ is the identity map on $F$.*
(II) *$F'$ is a quadratic extension of $F$ and $\tau$ is the non-trivial element of the Galois group* $\mathrm{Gal}(F'/F)$.
(III) *$F' = D$ is the unique (up to an isomorphism) quaternion algebra, central over $F$ and $\tau$ is the usual involution, fixing the center of $D$.*

Every such algebra $F'$ defines a reductive group $G$ over $F$ as follows. Let

$$V_n = e_1F' \oplus \cdots \oplus e_nF' \oplus e_{n+1}F' \oplus \cdots \oplus e_{2n}F',$$

be a right vector space over $F'$. If we fix $\epsilon \in \{\pm 1\}$, then $(e_i, e_{2n-j+1}) = \delta_{ij}$ defines an $\epsilon$-hermitian form on $V_n$:

$$\begin{cases} (v, v') = \epsilon \cdot \tau\big((v, v')\big), & v, v' \in V_n, \\ (vx, v'x') = \tau(x)(v, v')x', & x, x' \in F'. \end{cases}$$

Let $G = G_n(F', \epsilon)$ be the group of isometries of the form $(\ ,\ )$, and let $P$ be the parabolic subgroup of $G$, which stabilizes the isotropic space

$$V_n' = e_1F' \oplus \cdots \oplus e_nF'.$$

The group $P$ has a Levi decomposition $P = MN$, where $M \cong \mathrm{Aut}_{F'}(V_n')$. We fix an isomorphism $M \cong \mathrm{GL}(n, F')$ using the above fixed basis of $V_n'$.

Let $\rho \in \mathrm{Irr}(M)$ be a unitary representation and let $s$ be a real number. Define a generalized principal series representation by

$$I(\rho, s) = \mathrm{Ind}_P^G(|\det_{F'}|^s \otimes \rho),$$

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where $\det_{F'}$ is the reduced norm of the simple algebra $M(n, F')$ of all $n \times n$ matrices with coefficients in $F'$, and $|\ |$ is the normalized absolute value of $F$ in the cases (I) and (III) and the normalized absolute value of $F'$ in the case (II). Let $\bar{P} = M\bar{N}$ be the opposite parabolic subgroup. Analogously, we define the induced representation $\bar{I}(\rho, s)$ for $\bar{P}$. For $f \in I(\rho, s)$, let

$$A(s, \rho, N, \bar{N}) f(g) = \int_{\bar{N}} f(\bar{n}g)\, d\bar{n} \quad (g \in G)$$

be the the standard intertwining operator from $I(\rho, s)$ to $\bar{I}(\rho, s)$ (meromorphicaly continued from the domain of convergence of the integral). Let $\mu(s, \rho)$ be the Plancherel measure defined by

$$A(s, \rho, \bar{N}, N) A(s, \rho, N, \bar{N}) = \mu(s, \rho)^{-1}.$$

It follows from the work of Harish-Chandra [Si] that the Plancherel measure $\mu(s, \rho)$ determines points of reducibility and complementary series of $I(\rho, s)$ if $\rho$ is supercuspidal.

In the cases (I) and (II) the group $G$ is quasi-split. Thus, if $\rho$ is supercuspidal, the reducibilities and complementary series of $I(\rho, s)$ are part of a general theory of Shahidi [Sh2] for generic representations. For more details and for a nice interpretation in terms of conjectural twisted endoscopy theory, see [Sh1] (case (I)) and [G] (case (II)). In this paper we study the remaining case (III). Then $G$ is no longer quasi-split and our induced representations do not have Whittaker models.

Let us describe the main results of this paper in more details. First, note that $G$ is an inner form of the group

$$G' = G_{2n}(F, -\epsilon) = \begin{cases} \mathrm{Sp}(4n, F) & \text{if } \epsilon = +1 \\ \mathrm{SO}(4n, F) & \text{if } \epsilon = -1. \end{cases}$$

Let $P' = M'N'$ be the Siegel maximal parabolic subgroup of $G'$ as above. Note that $M' = \mathrm{GL}(2n, F)$. Furthermore, there is a natural $1-1$ correspondence between regular elliptic conjugacy classes of $\mathrm{GL}(n, D)$ and $\mathrm{GL}(2n, F)$. For each $\pi \in \mathrm{Irr}(G)$, we write $\mathrm{ch}_\pi$ for its character, which is, by a well-known result of Harish-Chandra, a locally integrable function, locally constant on the set of all regular conjugacy classes. By [DKV], there exists a $1-1$ correspondence $\rho \leftrightarrow \rho'$ between the sets of all classes irreducible essentially square-integrable representations of $\mathrm{GL}(n, D)$ and $\mathrm{GL}(2n, F)$ characterized by

$$(-1)^n \mathrm{ch}_\rho = \mathrm{ch}_{\rho'}$$

on the set of the regular elliptic classes. In Section 2 we prove (Proposition 2.1)

$$\mu(s, \rho) = \mu(s, \rho'),$$

under certain normalizations of Haar measures on $N$ and $N'$; for more details see Section 2. Combining this with the results of [Sh1], we compute reducibility and complementary series of $I(\rho, s)$ if $\rho$ is supercuspidal. This can be found in Section 3.

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## 1 Results of DKV

In this section we describe the correspondence $\rho \leftrightarrow \rho'$, between essentially square integrable representations of $\mathrm{GL}(n, D)$ and $\mathrm{GL}(2n, F)$, in more details.

By a result of Bernstein [Ze], there exists a positive integer $k$ and a supercuspidal representation $\rho_0$ of $\mathrm{GL}(2n/k, F)$ such that $\rho'$ is the unique irreducible subrepresentation of

$$\nu^{(k-1)/2}\rho_0 \times \cdots \times \nu^{-(k-1)/2}\rho_0.$$

(Here, as usual [Ze], $\nu = |\det|_F$.) We will write $\rho' = \delta(\rho_0, k)$. Now, by [DKV, B.2.b] $\rho$ is supercuspidal if and only if the lowest common multiple of 2 and $2n/k$ is $2n$. Thus, if $n$ is even, $\rho$ is supercuspidal if and only if $\rho'$ is. If $n$ is odd, $\rho$ is supercuspidal if and only if either $\rho'$ is supercuspidal, or $\rho' = \delta(\rho_0, 2)$.

Assume now that $\rho$ is a supercuspidal representation. Define, as in [T, p. 53], the character $\nu_\rho$ of $\mathrm{GL}(n, D)$ by

$$\nu_\rho = \begin{cases} |\det_D|_F & \text{if } \rho' \text{ is supercuspidal} \\ |\det_D|_F^2 & \text{if } \rho' = \delta(\rho_0, 2). \end{cases}$$

Let $\delta(\rho, k)$ be the unique irreducible subrepresentation of

$$\nu_\rho^{(k-1)/2}\rho \times \cdots \times \nu_\rho^{-(k-1)/2}\rho.$$

By [DKV, B.2] and [T, Proposition 2.7] this representation is essentially square integrable. Furthermore, its lift to $\mathrm{GL}(2n, F)$ is given by [DKV, B.2.b]

$$\begin{cases} \delta(\rho, k)' = \delta(\rho', k) & \text{if } \rho' \text{is supercuspidal} \\ \delta(\rho, k)' = \delta(\rho_0, 2k) & \text{if } \rho' = \delta(\rho_0, 2). \end{cases} \tag{1.1}$$

We will end this section by introducing the natural involution on the set of irreducible representations of $\mathrm{GL}(n, D)$. First, for $g = (g_{ij}) \in \mathrm{GL}(n, D)$ define $\tau(g) = \big(\tau(g_{ij})\big) \in \mathrm{GL}(n, D)$. If $g^t$ denote the transpose matrix (with respect to the main diagonal), we put $g^\tau = \tau(g^t)$. The map $g \mapsto (g^\tau)^{-1}$ is a continuous involution on $\mathrm{GL}(n, D)$, for any $n$. Thus, it acts on representations by $\pi^\tau(g) = \pi\big((g^\tau)^{-1}\big)$. Now, we will prove

***Lemma 1.1*** *Let $\pi$ be an irreducible representation of* $\mathrm{GL}(n, D)$*. Let $\tilde{\pi}$ be the contragredient representation of $\pi$. Then $\pi^\tau \cong \tilde{\pi}$.*

**Proof** We will prove this result under our assumption that the characteristic of $F$ is zero. This assumption enable us to consider the characters $\chi_\pi$ and $\chi_{\pi^\tau}$ as locally integrable functions, locally constant on the set of all regular semisimple conjugacy classes. Hence, to prove the lemma, it is enough to check

$$\chi_{\pi^\tau}(g) = \chi_{\tilde{\pi}}(g),$$

for all regular semisimple elements $g \in \mathrm{GL}(n, D)$. This is equivalent to

$$\chi_\pi(g^\tau) = \chi_\pi(g).$$

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Hence, it is enough to check that $g^\tau$ and $g$ are conjugate for all regular semisimple $g$.

Note that $g^\tau$ and $g$ have the same characteristic polynomial. In particular, they are conjugate over the algebraic closure $\bar{F}$ of $F$. Let $\mathcal{A}$ be the centralizer of $g$ in $M(n, D)$. Then

$$\mathcal{A} = \bigoplus_j F_j,$$

where for any $j$, $F_j$ is an extension of $F$ (of degree $[F_j : F]$), and $\sum_j [F_j : F] = n$. Thus, the centralizer of $g$ in $\mathrm{GL}(n, D)$ is

$$\mathrm{GL}(\mathcal{A}) = \times_j F_j^\times.$$

By the Hilbert Theorem 90, the first Galois cohomology group $H^1\big(\mathrm{Gal}(\bar{F}/F), \mathrm{GL}(\mathcal{A})\big)$ is trivial. In particular, $g^\tau$ and $g$ are conjugated over $F$. ■

## 2 Plancherel Measures

In this section we will prove the equality of Plancherel measures. Abusing our notation, let $G = G_n(F', \epsilon)$ and let $P = MN$ be the Siegel maximal parabolic subgroup as in the Introduction. First, we need to normalize Haar measures on $N$ and $\bar{N}$. We shall fix a nontrivial additive character $\psi_F$ of $F$. Let $M(n, F')$ be the vector space over $F$ of $n \times n$-matrices with coefficients in $F'$. Then

$$M(n, F') = M(n, F')^+ \oplus M(n, F')^-,$$

where $M(n, F')^+$ and $M(n, F')^-$ are the sets of $\tau$-hermitian symmetric and $\tau$-hermitian skew-symmetric matrices. Then, using the basis $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$ of $V_n$ we can identify both $N$ and $\bar{N}$ with

$$\begin{cases} M(n, F')^+ & \text{if } \epsilon = -1 \\ M(n, F')^- & \text{if } \epsilon = +1. \end{cases}$$

Let $\mu_n(F')$ be the Haar measure on $M(n, F')$ self-dual with respect to $\psi_F$ and the bilinear form $\mathrm{Tr}_{F'}(xy)$, where $\mathrm{Tr}_{F'}$ is the reduced trace on $M(n, F')$. Let $\mu_n^\pm(F')$ be the self-dual Haar measure on $M(n, F')^\pm$ such that

$$\mu_n(F') = \mu_n^+(F') \cdot \mu_n^-(F').$$

Specifying $F' = D$ and $F' = F$ we obtain normalizations of the Haar measures for $G = G_n(D, \epsilon)$ and $G' = G_{2n}(F, -\epsilon)$, respectively.

***Proposition 2.1*** *Assume that $\rho$ is a square integrable representation of* $\mathrm{GL}(n, D)$*, and $\rho'$ its lift to* $\mathrm{GL}(2n, F)$*. Then, under above normalization of Haar measures on $N$ and $N'$, we have*

$$\mu(s, \rho) = \mu(s, \rho').$$

**Proof** We will prove the proposition using global means. Let $k$ be an algebraic number field. For each place $v$ of $k$ let $k_v$ denote its completion at $v$. Let $\mathbf{A}$ be the ring of adeles of $k$. We may assume that $k$ has two places $v_1$ and $v_2$, such that

$$\begin{cases} k_{v_1} \cong F \\ k_{v_2} \cong F. \end{cases}$$

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Let $\mathbf{D}$ be a quaternion algebra over $k$, ramified at $v_1$ and $v_2$ only. Let $\mathbf{G} = G_n(\mathbf{D}, \epsilon)$. It is a form of $G$ defined over $k$. Note that

$$\mathbf{G}(k_{v_1}) \cong \mathbf{G}(k_{v_2}) \cong G$$
$$\mathbf{G}(k_v) \quad \text{is split if} \quad v \notin \{v_1, v_2\}.$$

Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be the Siegel parabolic subgroup of $\mathbf{G}$. Note that $\mathbf{M} = \mathrm{GL}(n, \mathbf{D})$. Take a nontrivial additive character $\psi = \bigotimes_v \psi_v$ on $\mathbf{A}$, trivial on $k$, such that $\psi_{v_1} = \psi_F$ and $\psi_{v_2} = \psi_F$. For each place $v$, we fix the Haar measures on $\mathbf{N}(k_v)$, and $\bar{\mathbf{N}}(k_v)$ self-dual with respect to $\psi_v$, as above. In this way we have fixed Tamagawa measures (see [We, p. 113]) on $\mathbf{N}(\mathbf{A})$ and $\bar{\mathbf{N}}(\mathbf{A})$. This means that

$$\begin{cases} \mathrm{vol}\big(\mathbf{N}(k) \setminus \mathbf{N}(\mathbf{A})\big) = 1 \\ \mathrm{vol}\big(\bar{\mathbf{N}}(k) \setminus \bar{\mathbf{N}}(\mathbf{A})\big) = 1. \end{cases} \tag{2.1}$$

Let $\mathbf{G}' = G_{2n}(k, -\epsilon)$. It is a split form of $\mathbf{G}$. Let $\mathbf{P}' = \mathbf{M}'\mathbf{N}'$ be the Siegel parabolic subgroup of $\mathbf{G}'$. Note that $\mathbf{M}' = \mathrm{GL}(2n, k)$. As in the case of $\mathbf{G}$, we fix Tamagawa measures on $\mathbf{N}'(\mathbf{A})$ and $\bar{\mathbf{N}}'(\mathbf{A})$ using $\psi = \bigotimes_v \psi_v$.

Now, we will fix a unitary character $\omega$ of $\mathbf{A}^\times$, trivial on $k^\times$, such that $\omega_{v_1}$ and $\omega_{v_2}$ are equal to the central character of $\rho$.

***Lemma 2.1*** *Fix a finite place $u$, different from $v_1$ and $v_2$, and choose any supercuspidal unitary representation $\delta$ of $\mathrm{GL}(2n, k_u)$, whose central character is $\omega_u$. Then there exists an automorphic cuspidal representation $\pi' = \bigotimes_v \pi'_v$ of $\mathrm{GL}(2n, \mathbf{A}) = \mathbf{M}'(\mathbf{A})$, whose central character is $\omega$, such that*

$$\pi'_{v_1} \cong \pi'_{v_2} \cong \rho' \quad \text{and} \quad \pi'_u \cong \delta.$$

**Proof** This lemma is an application of the trace formula. For example, the proof of [F, Proposition III.3] can be adapted to this situation. We leave details to the reader. ■

The automorphic cuspidal representation, described in Lemma 2.1, can be lifted [FK, Theorem 3] to the automorphic cuspidal representation $\pi = \bigotimes_v \pi_v$ of $\mathbf{M}(\mathbf{A})$, defined as follows:

$$\pi_{v_1} \cong \pi_{v_2} \cong \rho$$
$$\pi_v \cong \pi'_v, \quad \text{for any } v, v \notin \{v_1, v_2\}.$$

Let $S$ be a finite set of places of $k$ containing $\{v_1, v_2\}$ and all places of residual characteristic 2, such that if $v \notin S$ then $\psi_v$ and $\pi_v$ are unramified. For every $v \notin S$, we denote by $f_v^s$ (resp. $\bar{f}_v^s$) the unique unramified vector in $I(\pi_v, s)$ (resp. $\bar{I}(\pi_v, s)$) normalized as in [Sh1, p. 6]. Since for $v \notin S$ our choice of Haar measures coincides with the usual one (where on each root subgroup one takes a self dual measure measure with respect to $\psi_v$), we can apply a result of Langlands (see for example [Sh1, p. 6]):

$$\begin{cases} A\big(s, \pi_v, \mathbf{N}(k_v), \bar{\mathbf{N}}(k_v)\big) f_v^s(g) = c_v(s, \pi_v) \bar{f}_v^s(g) \\ A\big(s, \pi_v, \bar{\mathbf{N}}(k_v), \mathbf{N}(k_v)\big) \bar{f}_v^s(g) = c_v(-s, \tilde{\pi}_v) f_v^s(g), \end{cases} \tag{2.2}$$

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where $c_v(s, \pi_v)$ is a quotient of certain $L$-functions. The explicit formula for $c_v(s, \pi)$ can be found in [Sh1, p. 6]. For our purpose, it is important that the Euler product

$$c_S(s, \pi) = \prod_{v \notin S} c_v(s, \pi_v)$$

converges for $\operatorname{Re}(s) >> 0$, and it continues to a meromorphic function on $\mathbb{C}$.

Take $f^s = \bigotimes_v f_v^s$ in $I(\pi, s)$ such that $f_v^s$ is the unramified vector as above, for all $v \notin S$. In view of (2.1) we can apply [MW, Theorem IV.1.10] and obtain

$$A\big(s, \pi, \bar{\mathbf{N}}(\mathbf{A}), \mathbf{N}(\mathbf{A})\big) A\big(s, \pi, \mathbf{N}(\mathbf{A}), \bar{\mathbf{N}}(\mathbf{A})\big) f^s = f^s. \tag{2.3}$$

Now, using (2.2), it follows from (2.3) that

$$\prod_{v \in S} \mu(s, \pi_v) \cdot c_S(s, \pi) \cdot c_S(-s, \tilde{\pi}) = 1. \tag{2.4}$$

Analogously, we can prove

$$\prod_{v \in S} \mu(s, \pi'_v) \cdot c_S(s, \pi') \cdot c_S(-s, \tilde{\pi}') = 1. \tag{2.5}$$

Next, if $v \notin \{v_1, v_2\}$, then $\mathbf{D}_v \cong M(2, k_v)$. This induces an isomorphism of $\mathbf{G}(k_v)$ and $\mathbf{G}'(k_v)$ restricting to isomorphisms

$$\begin{cases} \mathbf{N}(k_v) \cong \mathbf{N}'(k_v) \\ \bar{\mathbf{N}}(k_v) \cong \bar{\mathbf{N}}'(k_v). \end{cases} \tag{2.6}$$

It is easy to check that these isomorphisms preserve the self-dual measures. In particular it follows that

$$\mu(s, \pi'_v) = \mu(s, \pi_v) \tag{2.7}$$

if $v \notin \{v_1, v_2\}$. Now, (2.4), (2.5) and (2.7) imply

$$\big(\mu(s, \rho)\big)^2 = \big(\mu(s, \rho')\big)^2.$$

Since both Plancherel measures are non-negative along the imaginary axis $\operatorname{Re}(s) = 0$ [Si, Chapter 5], we obtain the proposition. ∎

## 3 Applications

Let $\rho$ be a unitarizable supercuspidal representation of $\mathrm{GL}(n, D)$. In this section we determine the reducibility points of $I(\rho, s)$, where $s$ is a real number. First, let us write $w_0$ for the non-trivial element of the group $N_G(M)/M$. Clearly, $w_0$ acts on representations of $M = \mathrm{GL}(n, D)$. More precisely, the action is given by $w_0(\rho)(g) = \rho\big((g^\tau)^{-1}\big)$. Hence, by Lemma 1.1 we have

$$w_0(\rho) \cong \tilde{\rho}. \tag{3.1}$$

Now, we have

***Proposition 3.1*** *If $\rho \ncong \tilde{\rho}$, then $I(\rho, s)$ is irreducible for all real numbers $s$. Moreover, $I(\rho, s)$ is unitarizable only for $s = 0$.*

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**Proof** It follows from [Be, Theorem 28] and (3.1) (and also from Harish-Chandra [Si]) that $\rho \cong \tilde{\rho}$ is a necessary condition for reducibility of $I(\rho, s)$. Hence $I(\rho, s)$ is irreducible, for real numbers $s$. Since, this representation is not Hermitian for $s \neq 0$, the lemma follows. ∎

In what follows we shall assume that $\rho \cong \tilde{\rho}$. Then, by a result of Silberger, there exists the unique $s_0 \geq 0$ such that $I(\rho, \pm s_0)$ reduces, and $I(\rho, s)$ is irreducible for $|s| \neq s_0$ [Si1, Lemma 1.2]. Moreover, by the general theory of Harish-Chandra [Si, Chapter 5], we have

$$\begin{cases} s_0 = 0 & \text{if and only if } \mu(s_0, \rho) \neq 0 \\ s_0 > 0 & \text{if and only if } \mu(s_0, \rho) = \infty \end{cases} \tag{3.2}$$

In the remainder of this section we will calculate $s_0$, using Proposition 2.1 and (3.2). Thus, let $\rho'$ be the corresponding square integrable representation of $\mathrm{GL}(2n, F)$. Note that $\rho'$ is also self-contragredient.

First, we shall assume that $\rho'$ is supercuspidal. Then, the work of Shahidi [Sh1] implies that there is $s_0' \in \{0, 1/2\}$, such that $I(\rho', \pm s_0')$ is reducible and $I(\rho', s)$ is irreducible for $|s| \neq s_0'$. As in [Sh1], we call $\rho'$ a representation of symplectic type if $I(\rho', 1/2)$ is reducible, and a representation of orthogonal type if $I(\rho', 0)$ is reducible. Also, [Sh1, Lemma 3.6] implies that every self-contragredient supercuspidal representation of $\mathrm{GL}(2n, F)$ is exactly of one of the above types. Moreover, these definitions do not depend on the choice of the group $G'$ (that is, $G'$ can be either $\mathrm{SO}(4n, F)$ or $\mathrm{Sp}(4n, F)$).

Furthermore, the dual group of $\mathrm{GL}(2n)$ is $\mathrm{GL}(2n, \mathbb{C})$. Let $\rho_{2n}$ be the standard representation of $\mathrm{GL}(2n, \mathbb{C})$. Let $\wedge^2 \rho_{2n}$ and $\mathrm{Sym}^2 \rho_{2n}$ be the exterior square and symmetric square representation of $\mathrm{GL}(2n, \mathbb{C})$, respectively. Shahidi has defined local $L$-functions $L(s, \rho', \wedge^2 \rho_{2n})$ and $L(s, \rho', \mathrm{Sym}^2 \rho_{2n})$ [Sh1], [Sh2], and has proved that a self-contragredient representation $\rho'$ has symplectic (resp. orthogonal) type if and only if $L(s, \rho', \wedge^2 \rho_{2n})$ (resp. $L(s, \rho', \mathrm{Sym}^2 \rho_{2n})$) has a pole at $s = 0$.

***Example 3.1*** Let $\rho'$ be a self-contragredient supercuspidal representation of $\mathrm{GL}(2, F)$, and let $\omega'$ be its central character. If $\omega' = 1$ then $\rho'$ is of symplectic type, and if $\omega' \neq 1$ then $\rho'$ is of orthogonal type.

Our first result is

***Theorem 3.1*** *Assume that $\rho$ is a self-contragredient unitarizable supercuspidal representation of $\mathrm{GL}(n, D)$, being the lift of a supercuspidal representation $\rho'$ of $\mathrm{GL}(2n, F)$. Then we have*

*(i) If $\rho'$ has symplectic type, then $I(\rho, \pm 1/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 1/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 1/2$.*
*(ii) If $\rho'$ has orthogonal type, then $I(\rho, 0)$ is reducible, and $I(\rho, s)$ is an irreducible non-unitarizable representation for $s \neq 0$.*

**Proof** As explained before, to find the point of reducibility $s_0$, we need to study the poles and zeroes of $\mu(s, \rho)$. Proposition 2.1 implies $\mu(s, \rho) = \mu(s, \rho')$, and the theorem follows. ∎

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In other words, Theorem 3.1 says that $I(\rho, s)$ reduces if and only if $I(\rho', s)$ reduces. On the other hand, reducibility of $I(\rho', 1/2)$ can be checked as follows. Let $w$ be a non-singular skew-symmetric matrix in $\mathrm{GL}(2n, F)$. Put

$$\mathrm{Sp}(2n, F) = \{g \in \mathrm{GL}(2n, F); g^t wg = w\}.$$

We have the following result of Shahidi [Sh1, Theorem 5.3].

***Proposition 3.2*** *Assume that $\omega'$ is the central character of $\rho'$. For each function $f \in C_c^\infty(\mathrm{GL}(2n, F))$, such that*

$$f_{\rho'}(g) = \int_Z f(zg)\omega^{-1}(z)\, dz$$

*defines a non-trivial matrix coefficient of $\rho'$, we put*

$$I(f) = \int_{\mathrm{Sp}(2n,F)\backslash \mathrm{GL}(2n,F)} f(g^t \cdot wgw^{-1})\, dg.$$

*Then, $I(\rho', 1/2)$ is reducible if and only if there exists $f$ as above, such that $I(f) \neq 0$.*

Finally, we note that Murnaghan and Repka [MR] have computed this integral for a large family of tamely ramified supercuspidal representations.

Now, we will assume that the lift $\rho'$ is not supercuspidal. Hence, by (1.1), $\rho' \cong \delta(\rho_0, 2)$, where $\rho_0$ is an irreducible supercuspidal representation of $\mathrm{GL}(n, F)$ and $n$ is odd. Since $\rho$ is self-contragredient, $\rho_0$ must also be self-contragredient. Now, we have

***Theorem 3.2*** *Assume that $G' = \mathrm{SO}(4n, F)$ ($n$ is odd). Let $\rho$ be a self-contragredient unitarizable supercuspidal representation of $\mathrm{GL}(n, D)$. Assume that $\rho$ corresponds to a discrete series representation $\rho' = \delta(\rho_0, 2)$ of $\mathrm{GL}(2n, F)$. Let $I(\rho, s)$ be the induced representation of $G = G_n(D, -1)$. Then $I(\rho, \pm 1/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 1/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 1/2$.*

**Proof** As explained before, to find the point of reducibility $s_0$, we need to find the poles and zeroes of $\mu(s, \rho) = \mu(s, \rho')$. Let $q$ be the order of the residue field of $F$. Combining (3.16) and (7.4) of [Sh2], the Plancherel measure $\mu(s, \rho')$ is, up to a monomial in $q^s$, equal to

$$\frac{L(1 + 2s, \rho', \wedge^2\rho_{2n})L(1 - 2s, \rho', \wedge^2\rho_{2n})}{L(-2s, \rho', \wedge^2\rho_{2n})L(2s, \rho', \wedge^2\rho_{2n})}. \tag{3.3}$$

In fact, since both $\mu(s, \rho')$ and the function given by the formula (3.3) are even, they are equal up to a non-zero constant. Since $\rho' = \delta(\rho_0, 2)$, by [Sh1, Proposition 8.1],

$$L(s, \rho', \wedge^2\rho_{2n}) = L(s + 1, \rho_0, \wedge^2\rho_n)L(s, \rho_0, \mathrm{Sym}^2\rho_n). \tag{3.4}$$

Since $n$ is odd, we have that $L(s, \rho_0, \mathrm{Sym}^2\rho_n)$ has a pole at $s = 0$, while $L(s, \rho_0, \wedge^2\rho_n)$ is holomorphic there [Sh1, Proposition 3.5]. Since the $L$-functions of supercuspidal representations have all poles on the imaginary axis $\mathrm{Re}(s) = 0$ [Sh2, Proposition 7.3], we see

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that the only real pole of the $L$-function on the left hand-side of (3.4) is $s = 0$. Now, since local $L$-functions never vanish, (3.3) implies that $s_0 = 1/2$. The theorem is proved. ■

***Theorem 3.3*** *Assume that $G' = \mathrm{Sp}(4n, F)$ ($n$ is odd). Let $\rho$ be a self-contragredient unitarizable supercuspidal representation of $\mathrm{GL}(n, D)$. Assume that $\rho$ corresponds to a discrete series representation $\rho' = \delta(\rho_0, 2)$ of $\mathrm{GL}(2n, F)$. Let $I(\rho, s)$ be the induced representation of $G = G_n(D, +1)$. Then:*

*(i) If $n = 1$ and $\rho_0 = 1_{F^\times}$, then $I(\rho, \pm 3/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 3/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 3/2$ (note that the unique irreducible subrepresentation of $I(3/2, \rho)$ is the Steinberg representation of $G$).*
*(ii) If $n = 1$ and $\rho_0^2 = 1_{F^\times}$, $\rho_0 \neq 1_{F^\times}$, then $I(\rho, 0)$ is reducible, and $I(\rho, s)$ is an irreducible non-unitarizable representation for $s \neq 0$.*
*(iii) If $n > 1$ then $I(\rho, \pm 1/2)$ is reducible, and $I(\rho, s)$ is irreducible for $|s| \neq 1/2$. Moreover, $I(\rho, s)$ is in the complementary series if and only if $|s| < 1/2$.*

**Proof** The proof is similar to the proof of Theorem 3.2. This time note that, up to a non-zero constant, the Plancherel measure $\mu(s, \rho')$ is equal to

$$\frac{L(1 + 2s, \rho', \wedge^2 \rho_{2n}) L(1 - 2s, \rho', \wedge^2 \rho_{2n})}{L(-2s, \rho', \wedge^2 \rho_{2n}) L(2s, \rho', \wedge^2 \rho_{2n})} \cdot \frac{L(1 + s, \rho') L(1 - s, \rho')}{L(-s, \rho') L(s, \rho')}, \tag{3.5}$$

where $L(s, \rho')$ is the principal $L$-function [J]. Since $\rho' = \delta(\rho_0, 2)$, by [J, Proposition 3.1.3]

$$L(s, \rho') = L(s + 1/2, \rho_0).$$

Note that $L(s, \rho_0)$ has a real pole if and only if $n = 1$ and $\rho_0 = 1_{F^\times}$. Moreover, $s = 0$ is the only real pole of $L(s, 1_{F^\times})$. The theorem follows from (3.5) and a case by case discussion. ■

***Remark 3.1*** Note that Theorem 3.1 (combined with Example 3.1) and Theorem 3.3 give a classification of the non-cuspidal part of the unitary dual of the rank one, non-split form of $\mathrm{Sp}(4, F)$.

***Remark 3.2*** We could also calculate the reducibility of $I(\rho, 0)$, where $\rho$ is a unitarizable discrete series representation of $\mathrm{GL}(n, D)$ using the theory of $R$-groups and the results of Shahidi. Note that Shahidi has determined all of the reducibilities $I(\rho', 0)$, where $\rho'$ is a discrete series representation of $M' = \mathrm{GL}(2n, F)$ (see Section 9 and Theorem 9.1 in [Sh1]).

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## References

[Be] J. Bernstein, *Draft of: Representations of p-adic groups.* (Lectures at Harvard University), written by Karl E. Rumelhart, 1992.

https://doi.org/10.4153/CMB-2000-014-5 Published online by Cambridge University Press

[DKV] P. Deligne, D. Kazhdan and M. F. Vignéras, *Représentations des algèbres centrales simples p-adiques*. Représentations des Groupes Réductifs sur un Corps Local, Herman, Paris, 1984, 33–117.

[F] Y. Z. Flicker, *Rigidity for automorphic forms*. J. Analyse Math. **49**(1987), 135–202.

[FK] Y. Z. Flicker and D. A. Kazhdan, *A simple trace formula*. J. Analyse Math. **50**(1988), 189–200.

[Go] D. Goldberg, *Some results on reducibility for unitary groups and local Asai L-functions*. J. Reine Angew. Math. **448**(1994), 65–95.

[J] H. Jacquet, *Principal L-functions of the linear group*. Proc. Sympos. Pure Math. **33**(1979), 63–86.

[MR] F. Murnaghan and J. Repka, *Reducibility of some induced representations of split classical p-adic groups*. Comp. Math., to appear.

[MW] C. Moeglin and J. L. Waldspurger, *Spectral Decomposition and Eisenstein series, une paraphrase de l'Ecriture*. Cambridge University Press **133**, 1995.

[Si] A. J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*. Math. Notes **23**, Princeton University Press, Princeton, NJ, 1979.

[Si1] ———, *Special representations of reductive p-adic groups are not integrable*. Ann. of Math. **111**(1980), 571–587.

[Sh1] F. Shahidi, *Twisted endoscopy and reducibility of induced representations for p-adic groups*. Duke Math. J. **66**(1992), 1–41.

[Sh2] ———, *A proof of Langland's conjecture on Plancherel measures; Complementary series for p-adic groups*. Ann. of Math. **132**(1990), 273–330.

[Sh3] ———, *Langland's conjecture on Plancherel measures for p-adic groups*. Progr. Math. **101**(1991), 277–295.

[T] M. Tadić, *Induced representations of* $\mathrm{GL}(n, A)$ *for p-adic division algebras A*. J. Reine Angew. Math. **405**(1990), 48–77.

[We] A. Weil, *Basic Number Theory*. Springer Verlag, 1973.

[Ze] A. V. Zelevinsky, *Induced representations of reductive p-adic groups, II: On irreducible representations of* $\mathrm{GL}(n)$. Ann. Sci. École Norm. Sup. **13**(1980), 165–210.



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