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On the sum and the difference of finite sets of integers

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Let $A = \{a_i\}$ be a finite set of integers and let p and m denote the cardinalities of $A + A = \{a_i+a_j\}$ and $A - A = \{a_i-a_j\}$, respectively. In the paper relations are established between p and m; in particular, if $\max_i \left(a_i-a_{i-1}\right) = 2$ those sets are characterized for which $p = m$ holds.

Let $A = \{a_i \mid 0 \le i \le n\}$ be a finite set of integers where without loss of generality we may assume $a_0 < a_1 < \ldots < a_n$, and let p and m denote the cardinalities of $A + A = \{a_i+a_j \mid a_i, a_j \in A\}$ and $A - A = \{a_i-a_j \mid a_i, a_j \in A\}$, respectively. Spohn [2] remarked that the values of p and m depend only on the n differences $d_i = a_i - a_{i-1}$ $(1 \le i \le n)$ and are unchanged if the d_i are multiplied by a constant or are reversed. Thus we may set $a_0 = 0$ and $\left(a_1, a_2, \ldots, a_n\right) = 1$. Further we use the abbreviation $\max\limits_{1\le i\le n} d_i = D$.

Macdonald and Street [1] proved the following.

THEOREM. *If* $D \le 2$, *then* $p \le m$.

The proof is based on the following results.

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LEMMA A. *If* $d_1 = 1$ *, then*

$$A - A = \{k \in Z \mid -a_n \le k \le a_n\} .$$

LEMMA B. *If*

$$d_1 = \dots = d_\alpha = d_{n-\beta+1} = \dots = d_n = 2 , \quad d_{\alpha+1} = d_{n-\beta} = 1 ,$$

then

$$m = 2a_n + 1 - 2 \min(\alpha, \beta)$$

and

$$p \le 2a_n + 1 - (\alpha+\beta) .$$

REMARK. In Lemma B the value of p can be improved to $p = 2a_n + 1 - (\alpha+\beta)$, since if

$$A = \{0, 2, 4, \dots, 2\alpha, 2\alpha+1, \dots, a_n-2\beta-1, a_n-2\beta, \dots, a_n-2, a_n\} ,$$

we have

$$A + A = \{0, 2, 4, \dots, 2\alpha, 2\alpha+1, \dots, 2a_n-2\beta-1, 2a_n-2\beta, \dots, 2a_n-2, 2a_n\} ,$$

where all integers between 2α and $2a_n - 2\beta$ belong to $A + A$, but for numbers below 2α and beyond $2a_n - 2\beta$ this holds only for even integers.

Using these results we shall derive the following facts.

LEMMA 1. *If* $d_1 = \dots = d_{D-1} = 1$ *, then*

$$A - A = \{k \in Z \mid -a_n \le k \le a_n\} .$$

Proof. The proof is similar to that of Lemma A in [1], namely, by induction on k it is shown that each $k \in \{1, \dots, a_n\}$ is the sum of consecutive d_i's , that is $k = d_s + d_{s+1} + \dots + d_t$, where $s \in \{1, \dots, D\}$ and $t \ge s$.

(1) $k = 1$: $k = d_1$.

(2) Suppose the claim is true for $k - 1 = d_s + \dots + d_t$. If $s > 1$, then

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$$k = d_{s-1} + d_s + \ldots + d_t .$$

If $s = 1$ and $d_{t+1} = 1$, then

$$k = d_s + \ldots + d_t + d_{t+1} .$$

If $s = 1$ and $d_{t+1} > 1$, then

$$k = d_{d_{t+1}} + d_{d_{t+1}+1} + \ldots + d_{t+1} ,$$

since, on account of $d_{t+1} \le D$, we have $d_1 = \ldots = d_{d_{t+1}-1} = 1$. Therefore we subtracted $d_{t+1} - 1$ from $k - 1$ and then we added d_{t+1}.

THEOREM 1. *If*

$$d_1 = \ldots = d_{D-1} = d_{n-D+2} = \ldots = d_n = 1 ,$$

then

(a) $A + A = \{k \in \mathbb{Z} \mid 0 \le k \le 2a_n\}$, *and*

(b) $p = m$

hold.

Proof. We show claim *(a)* in three stages.

1. Let $0 \le k \le a_n - 1$. If $k \in A$, then, on account of $0 \in A$, we have $k = k + 0 \in A + A$. If $k \notin A$, then at least one of the integers between $k - D + 1$ and $k - 1$ is an element of A; otherwise D consecutive integers would not belong to A which contradicts the definition of D. Now if $k - j \in A$ $(j \in \{1, 2, \ldots, D-1\})$, we have, since $j \in A$, also $k = (k-j) + j \in A + A$.

2. If $k = a_n$, then $k = k + 0 \in A + A$.

3. Let $a_n + 1 \le k \le 2a_n$. If $k = a_n + a_j$ for one $j \in \{1, \ldots, n\}$, then $k \in A + A$. Now let us assume that $k \ne a_n + a_j$ for all $j \in \{1, 2, \ldots, n\}$. Then for a suitable $j_1 \in \{1, \ldots, n-1\}$ we have

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$$a_n + a_{j_1} < k < a_n + a_{j_1+1} .$$

From this we derive $2 \le d_{j_1+1} = a_{j_1+1} - a_{j_1} \le D$ and with a suitable $b \in \{1, \ldots, d_{j_1+1}-1\}$ we have

$$k = a_n + a_{j_1} + b = a_n + a_{j_1+1} - d_{j_1+1} + b = (a_n - d_{j_1+1} + b) + a_{j_1+1} .$$

On account of

$$a_n - d_{j_1+1} + b \ge a_n - d_{j_1+1} + 1 \ge a_n - D + 1 ,$$

$a_n - d_{j_1+1} + b$ belongs to A, and therefore $k \in A + A$.

Claim *(b)* follows immediately from *(a)* together with Lemma 1. The example $A = \{0, 1, 2, 3, 4, 9, 10, 12, 13\}$ with $A + A = \{0, 1, \ldots, 26\}$ and $A - A = \{0, \pm 1, \ldots, \pm 13\}$ shows that the converse of Theorem 1 does not hold. But we are able to show the following weaker statement.

LEMMA 2. *Let* $p = m$ *and* $d_1 = \ldots = d_{D-1} = 1$. *Then* $d_n = 1$ *and* $d_{n-1} \in \{1, 2\}$ *hold.*

Proof. By Lemma 1 we have $m = 2a_n + 1$. By hypothesis $p = m$, therefore $A + A$ must consist of all integers between 0 and $2a_n$. If $d_n > 1$, then $2a_n - 1$ is not representable as the sum of two elements of A; the same holds for $2a_n - 3$, if $d_{n-1} > 2$.

After these preliminaries we are able to characterize those sets with $D = 2$, for which $p = m$ holds. For this purpose let α, respectively β, denote the number of differences at the ends of the set A which are equal to 2. ($\alpha = 0$ means $d_1 = 1$; $\alpha > 0$ means $d_1 = \ldots = d_\alpha = 2$, $d_{\alpha+1} = 1$.)

THEOREM 2. *Let* A *be a set of integers with* $D = 2$. *Then* $p = m$ *if and only if* $\alpha = \beta$.

Proof. First we prove the theorem for $\alpha = 0$, that is, $d_1 = 1$. If $\beta = 0$, then by Theorem 1 *(b)*, we have $p = m$. If conversely $p = m$, we

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obtain by Lemma 2 that $d_n = 1$, that is, $\alpha = \beta = 0$.

Now let $\alpha > 0$. The necessity follows from Lemma B, since for $\alpha \neq \beta$ the inequality $p < m$ holds. Finally, the sufficiency is a consequence of our remark to Lemma B, which states that $p = m = 2a_n + 1 - 2\alpha$.

Now we shall investigate which values can be assumed by $p = m$.

THEOREM 3. *(a) Let A be a set with $n + 1$ elements $(n \geq 2)$, $D \leq 2$, and $p = m$. Then p is an odd integer from the interval $2n + 1 \leq p \leq 4n - 3$.*

(b) Conversely, to each such p there exists a set A with $n + 1$ elements and $D \leq 2$, for which $|A+A| = |A-A| = p$.

Proof. *(a)* If $\alpha = \beta = 0$, that is $d_1 = d_n = 1$, for the largest element of A the inequalities $n \leq a_n \leq 2(n-2) + 2 = 2n - 2$ hold, therefore using Theorem 1 *(a)* we obtain $2n + 1 \leq p \leq 4n - 3$.

If, on the other hand $\alpha = \beta > 0$, we have $n + 2\alpha \leq a_n \leq 2n - 2$ for even n and $n + 2\alpha \leq a_n \leq 2n - 1$ for odd n . Then

$$2n + 1 < 2n + 2\alpha + 1 \leq p \leq 3n - 1 \leq 4n - 3 \quad \text{for even } n \geq 2$$

from our remark to Lemma B, and also

$$2n + 1 < 2n + 2\alpha + 1 \leq p \leq 3n \leq 4n - 3 \quad \text{for odd } n \geq 3 .$$

We prove *(b)* showing by induction on n that each possible value of p can be generated by a set with $d_1 = d_n = 1$. For $n = 2$ the set $A = \{0, 1, 2\}$ satisfies our theorem. Now let $p \in \{2n-1, 2n+1, \ldots, 4n-7\}$ and be arbitrary. By hypothesis there exists a set A with n elements and $d_1 = d_{n-1} = 1$, for which $|A+A| = p$ holds.

If we form $A_1 = A \cup \{a_{n-1}+1\}$, we have

$$A_1 + A_1 = (A+A) \cup \{2a_{n-1}+1, 2a_{n-1}+2\} ,$$

therefore $|A_1+A_1| = p + 2$. Finally, by Theorem 1 *(a)* the set A_2 with $d_1 = d_n = 1$ and $d_2 = \ldots = d_{n-1} = 2$ has

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$$|A_2+A_2| = 2a_n + 1 = 2(2n-2) + 1 = 4n - 3 .$$

THEOREM 4. *All values for* $p = m$ *which are possible according to Theorem* 3 *can be generated by symmetric sets* (*that is* $d_i = d_{n-i+1}$), *whilst asymmetric sets generate all values for* $p = m$ *except the smallest* $(2n+1)$ *and the largest* $(4n-3)$.

Proof. First we prove the claim for symmetric sets and even n and we consider sets with the following difference schemes:

(a) $d_1 = \ldots = d_n = 1$;

(b) $d_1 = \ldots = d_{(n-2j)/2} = d_{(n+2j+2)/2} = \ldots = d_n = 1$,

$d_{(n-2j+2)/2} = \ldots = d_{(n+2j)/2} = 2$ for $1 \le j \le (n-2)/2$;

(c) $d_1 = d_{(n-2j+2)/2} = \ldots = d_{(n+2j)/2} = d_n = 2$,

$d_2 = \ldots = d_{(n-2j)/2} = d_{(n+2j+2)/2} = \ldots = d_{n-1} = 1$ for $0 \le j \le (n-4)/2$.

By Theorem 1 *(a)* and the remark to Lemma B for the corresponding values of $p = m$ we obtain:

(a) $p = 2n + 1$;

(b) $p = 2a_n + 1 = 2n + 4j + 1$; on account of the possible values for j follows $p \in \{2n+5, 2n+9, \ldots, 4n-7, 4n-3\}$;

(c) $p = 2a_n + 1 - 2 = 2n + 4j + 3$, therefore $p \in \{2n+3, 2n+7, \ldots, 4n-9, 4n-5\}$.

Now let n be odd. We consider the differences:

(a) $d_1 = \ldots = d_n = 1$;

(b) $d_1 = \ldots = d_{(n-2j-1)/2} = d_{(n+2j+3)/2} = \ldots = d_n = 1$,

$d_{(n-2j+1)/2} = \ldots = d_{(n+2j+1)/2} = 2$ for $0 \le j \le (n-3)/2$;

(c) $d_1 = d_{(n-2j+1)/2} = \ldots = d_{(n+2j+1)/2} = d_n = 2$,

$d_2 = \ldots = d_{(n-2j-1)/2} = d_{(n+2j+3)/2} = \ldots = d_{n-1} = 1$ for

$$0 \le j \le (n-5)/2 .$$

In these cases for the values of $p = m$ we obtain:

(a) $p = 2n + 1$;

(b) $p = 2a_n + 1 = 2n + 4j + 3$, therefore

$$p \in \{2n+3, 2n+7, \ldots, 4n-7, 4n-3\} ;$$

(c) $p = 2a_n + 1 - 2 = 2n + 4j + 5$, therefore

$$p \in \{2n+5, 2n+9, \ldots, 4n-9, 4n-5\} .$$

If we form for $n \ge 4$ those asymmetric sets A which are defined by $d_1 = \ldots = d_j = d_n = 1$, $d_{j+1} = \ldots = d_{n-1} = 2$ $(2 \le j \le n-2)$, then by Theorem 1 *(a)* for $|A+A|$ we obtain the values $4n - 2j - 1$; therefore $p \in \{2n+3, 2n+5, \ldots, 4n-7, 4n-5\}$.

Now let A be an asymmetric set with $p = m$. If $d_1 = d_n = 1$, the inequalities $2n + 3 \le |A+A| \le 4n - 5$ hold. If

$$d_1 = \ldots = d_\alpha = d_{n-\alpha+1} = \ldots = d_n = 2 ,$$

$d_{\alpha+1} = d_{n-\alpha} = 1$, then, on account of the asymmetry, we have $\alpha + 1 \ne n - \alpha$; therefore $n + 2\alpha + 1 \le a_n \le 2n - 3$. Since $\alpha \ge 1$, for $p = 2a_n + 1 - 2\alpha$ we obtain the inequalities

$$2n + 5 \le 2n + 2\alpha + 3 \le p \le 4n - 2\alpha - 5 \le 4n - 7 ,$$

which prove Theorem 4.

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