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Congruence Relations for Shimura Varieties Associated with $GU(n-1,1)$

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Abstract. We prove the congruence relation for the mod-p reduction of Shimura varieties associated with a unitary similitude group $GU(n-1,1)$ over $\mathbb{Q}$ when p is inert and n odd. The case when n is even was obtained by T. Wedhorn and O. Bültel, as a special case of a result of B. Moonen, when the μ-ordinary locus of the p-isogeny space is dense. This condition fails in our case. We show that every supersingular irreducible component of the special fiber of p-$\mathscr{I}$sog is annihilated by a degree one polynomial in the Frobenius element F, which implies the congruence relation.

1 Introduction

Let (G,X) be a Shimura datum where G is a reductive group over $\mathbb{Q}$. We fix a prime p. For every compact open subgroup $K \subset G(\mathbb{A}_f)$, let Sh_K be the associated Shimura variety with reflex field E. The complex points of Sh_K are

$$Sh_K(\mathbb{C}) = G(\mathbb{Q})\diagdown\big(X \times G(\mathbb{A}_f)\diagup K\big).$$

When K is sufficiently small, Sh_K is smooth. Assume that $G_{\mathbb{Q}_p}$ is unramified and $K = K_pK^p$ with $K_p \subset G(\mathbb{Q}_p)$ hyperspecial and $K^p \subset G(\mathbb{A}_f^p)$ Then Sh_K is said to have good reduction at p. Let $\mathfrak{p}$ be a prime in E lying over p. In [1], the authors define a polynomial $H_\mathfrak{p}$ with coefficients in the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$, the set of $\mathbb{Q}$-linear combinations of K_p-double cosets of $G(\mathbb{Q}_p)$. It is made into a ring by convolution. This ring acts on the cohomology of the Shimura variety. Denote by $\mathrm{Fr}_\mathfrak{p}$ the conjugacy class of geometric Frobenius in $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$. Blasius and Rogawski conjectured the following.

Conjecture 1.1 *Let ℓ be a prime $\neq p$. Then $H^i_{et}(Sh_K \times_E \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ is unramified at p, and the relation $H_\mathfrak{p}(\mathrm{Fr}_\mathfrak{p}) = 0$ holds inside* $\mathrm{End}_{\mathbb{Q}_\ell}(H^i_{et}(Sh_K \times_E \overline{\mathbb{Q}}, \mathbb{Q}_\ell))$.

This equation makes sense since the action of Galois commutes with that of $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$. In the PEL case, an integral model over $\mathcal{O}_{E_\mathfrak{p}}$ can be defined explicitely, and the cohomology of $\overline{Sh_K} = Sh_K \times \kappa(\mathcal{O}_{E_\mathfrak{p}})$ coincides in many cases with that of $Sh_K \times E$.

In the case of Shimura curves, this conjecture was proved by Eichler, Shimura, and Ihara and was used to determine completely the eigenvalues of $\mathrm{Fr}_\mathfrak{p}$ acting on $H^i_{et}(Sh_K \times_E \overline{\mathbb{Q}}, \mathbb{Q}_l)$. More general situations have been dealt with. T. Wedhorn proved Conjecture 1.1 in the PEL case for groups that are split over $\mathbb{Q}_p$ in [16], O. Bültel for

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certain orthogonal groups in [2], and together they worked out the unitary case of signature $(n-1,1)$ with n even in [3].

In these articles, the authors use a moduli space $p\text{-}\mathscr{I}\mathrm{sog}$ that parametrizes p-isogenies between points of Sh_K. This space was used by Deligne in his work on the Ramanujan conjecture. It comes with two maps s,t to Sh_K, associating with an isogeny its source and target respectively. For any field L with a map $\mathcal{O}_{E_\mathfrak{p}} \to L$, we consider the $\mathbb{Q}$-algebra of cycles in $p\text{-}\mathscr{I}\mathrm{sog} \times L$, where multiplication is defined by composition of isogenies, and we denote by $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times L]$ the subalgebra generated by the irreducible components. This algebra was first introduced in [5].

In [11,15], the authors define the μ-ordinary locus in the good reduction of a PEL Shimura variety. It is at the same time a Newton polygon stratum and an Ekedahl–Oort stratum. Furthermore, it posesses a unique isomorphism class of p-divisible groups. It can also be defined as the unique open stratum in each of these stratifications. We will denote by $\overline{Sh_K}^{\mathrm{ord}}$ the μ-ordinary locus. Define the μ-ordinary locus $p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_\mathfrak{p}})$ of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_\mathfrak{p}})$ by taking inverse image by s (or t). Finally, define $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})]$ in the same fashion as above. We have a commutative diagram of $\mathbb{Q}$-algebra homomorphisms:

$$
\begin{array}{ccc}
\mathcal{H}_0(G(\mathbb{Q}_p)//K_p) & \xrightarrow{\;h\;} & \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times E] \\
 & & \big\downarrow \sigma \\
\Big\downarrow \dot{S} & & \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_\mathfrak{p}})] \\
 & & \big\downarrow \mathrm{ord} \\
\mathcal{H}_0(M(\mathbb{Q}_p)//(K_p \cap M(\mathbb{Q}_p))) & \xrightarrow[\;\bar{h}\;]{} & \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_\mathfrak{p}})].
\end{array}
$$

Here $M \subset G_{\mathbb{Q}_p}$ is the centralizer of the norm of the minuscule coweight μ of G associated with (G,X). The algebras on the left-hand side of the diagram are subalgebras of the Hecke algebras containing functions with integral support. The morphism $\dot{S}$ is a twisted version of the Satake homomorphism. The map σ is a specialization of cycles; the map ord intersects a cycle in $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_\mathfrak{p}})$ with the μ-ordinary locus. The morphism h is defined in Subsection 4.3. and we refer to [11] for the definition of the map $\bar{h}$. There is a natural Frobenius section of s defined on $\overline{Sh_K}$, defined by mapping an abelian variety to its Frobenius isogeny, which produces a closed subscheme F of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$. Similarly, the multiplication-by-p isogeny defines a section of s and a closed subscheme of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$ denoted by $\langle p \rangle$. These subschemes are μ-ordinary, in the sense that the μ-ordinary locus of F and $\langle p \rangle$ is dense in them. In this context, by "congruence relation" we mean the following conjecture.

Conjecture 1.2 *Consider the polynomial $H_\mathfrak{p}$ inside $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism $\sigma \circ h$. The element F lies in the center of this ring and the relation $H_\mathfrak{p}(F) = 0$ holds.*

This is related to Conjecture 1.1 by using functorial properties of cohomology. The geometric relation $H_{\mathfrak{p}}(F) = 0$ implies the same equality on the cohomology. For PEL-type Shimura varieties, the "μ-ordinary part" of the congruence relation is known [11, Corollary 4.2.15]. More precisely, we have the following theorem.

Theorem 1.3 *Consider the polynomial $H_{\mathfrak{p}}$ inside $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism* $\mathrm{ord} \circ \sigma \circ h$. *In this ring, the following relation holds:*

$$H_{\mathfrak{p}}(F) = 0.$$

When the μ-ordinary locus of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$ is dense, this theorem is equivalent to the congruence relation. This condition is satisfied in almost all the examples where Conjecture 1.2 is known. In the unitary similitude case of signature $(n-1,1)$, this density condition is satisfied if and only if n is even.

From now on, we consider only the unitary case $G = GU(n-1,1)$ when n is odd. In this article, we prove Conjecture 1.2 for these Shimura varieties. We first show that the Hecke polynomial factors into a product $H_p(t) = R(t) \cdot (t - p^{n-1} 1_{pK_p})$. The polynomial R annihilates F in $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_{\mathfrak{p}}})]$; this comes from an easy calculation inside $\mathcal{H}_0(M(\mathbb{Q}_p)//(K_p \cap M(\mathbb{Q}_p)))$ carried out in [16]. It follows that $R(F)$ calculated inside $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_{\mathfrak{p}}})]$ lies in the kernel of ord, thus is a linear combination of supersingular components. The final argument is the following result:

Theorem 1.4 *Let $C \subset p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$ be a supersingular irreducible component. Then*

$$C \cdot \left(F - p^{n-1}\langle p \rangle\right) = 0$$

inside $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})]$.

We will now give an overview on how this article is organized. In the second section, we establish the factorization of the Hecke polynomial. In the third one, we give the moduli problem, and the results from [3] on the stratifications of the special fiber. Section 4 is dedicated to the moduli space of p-isogenies. Section 5 studies the supersingular locus of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$. Here we use mainly [3, 13, 14] for some key results. Finally, in Section 6, we prove Conjecture 1.2.

Notations

1. We fix an odd integer $n \geq 3$ and a prime $p > 2$. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.
2. Let E be an imaginary quadratic extension of $\mathbb{Q}$, such that p is inert in E. We write $\sigma \colon x \mapsto \bar{x}$ for the non trivial automorphism of E and E_p for the completion of E at p. Let $\mathcal{O}_{E_p}$ be the ring of integers of E_p and $\kappa(\mathcal{O}_{E_p}) = (\mathcal{O}_{E_p})/(p\mathcal{O}_{E_p})$ the residual field.
3. We fix an embedding $\vartheta \colon E \hookrightarrow \overline{\mathbb{Q}}$. We denote by $\overline{\mathbb{F}}$ the algebraic closure of $\kappa(\mathcal{O}_{E_p})$ provided by the embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$. We choose an element $\alpha \in E^\times \cap \mathcal{O}_{E_p}^\times$ such that the imaginary part of α is > 0 and $\alpha + \overline{\alpha} = 0$. If $z \in \mathbb{C}$ and $z = a + \alpha b$, $a, b \in \mathbb{R}$, then we call b the α-imaginary part of z.

4. (V, ψ) is a hermitian space of dimension n, *i.e.*, V is an n-dimensional E-vector space, and $\psi\colon V \times V \to E$, a non-degenerate hermitian pairing. We assume the signature of (V, ψ) to be $(n-1, 1)$.
5. Let $\varphi\colon V \times V \to \mathbb{Q}$ be the α-imaginary part of ψ. Then φ is a skew-symmetric form such that $\forall e \in E, \forall x, y \in V, \varphi(ex, y) = \varphi(x, \bar{e}y)$.
6. G is the (connected, reductive) algebraic $\mathbb{Q}$-group of unitary similitudes of (V, ψ).
7. Let $\mathcal{B}_W = (e_1, \dots, e_n)$ be a Witt basis of $V \otimes \mathbb{Q}_p$. This means $V \otimes \mathbb{Q}_p = V_0 \oplus \bigoplus_{1 \le i < k} H_i$ is an orthogonal Witt decomposition, where $V_0 = \mathrm{Vect}_{E_p}(e_k)$ is anisotropic, and $H_i = \mathrm{Vect}_{E_p}(e_i, e_{n+1-i})$ is a hyperbolic plane with $\psi(e_i, e_{n+1-i}) = 1$.
8. In the basis $\mathcal{B}_W$, the diagonal matrices of $G_{\mathbb{Q}_p}$ form a torus T and the upper-triangular matrices of G a Borel subgroup B containing T. Denote by A the maximal split subtorus of T. In the basis $\mathcal{B}_W$, an element of $T(\mathbb{Q}_p)$ has matrix $\mathrm{diag}(x_1, \dots, x_n) \in GL_n(E_p)$ with
$$\overline{x_1}x_n = \overline{x_2}x_{n-1} = \cdots = \overline{x_k}x_k.$$
9. Let $\Omega(T)$ be the Weyl group of T over $\mathbb{Q}_p$. It is the group of permutations of $\{1, \dots, n\}$ fixing the equations above. Thus,
$$\Omega(T) = \left\{ \sigma \in \mathfrak{S}_n; \sigma(i) + \sigma(n+1-i) = n \forall i \in \{1, \dots, n\} \right\}.$$
10. Let ρ be the half-sum of positive roots with respect to (B, T).
11. Let Λ be the $\mathcal{O}_{E_p}$-lattice generated by the e_i. We assume that ψ defines a perfect pairing $\Lambda \times \Lambda \to \mathcal{O}_{E_p}$. This amounts to $\psi(e_k, e_k) \in \mathbb{Z}_p^\times$ and implies that
$$\det(\psi) = 1 \in \frac{\mathbb{Q}_p^\times}{N(E_p^\times)}.$$
12. Let $K_p = \mathrm{Stab}_{G(\mathbb{Q}_p)}(\Lambda)$; this is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. We write $L = K_p \cap B(\mathbb{Q}_p)$ and $T_c = K_p \cap T(\mathbb{Q}_p)$.

2 Hecke Polynomial

2.1 Unitary Similitude Group

There is an isomorphism $V \otimes_{\mathbb{Q}} E \simeq \bigoplus_{\tau \in \mathrm{Gal}(E/\mathbb{Q})} V$. The choice of $\mathrm{id} \in \mathrm{Gal}(E/\mathbb{Q})$ gives an isomorphism

$$G_E \simeq GL_E(V) \times \mathbb{G}_m. \tag{2.1}$$

Let $\mathcal{B}$ be an E-basis of V and let J be the matrix of ψ in $\mathcal{B}$. The group $\mathrm{Gal}(E/\mathbb{Q}) = \{1, \sigma\}$ acts on $G(E) \simeq GL_n(E) \times E^\times$ by

$$\sigma.(A, \lambda) = (\overline{\lambda} J ({}^t\bar{A}^{-1}) J, \overline{\lambda}), \quad \text{for all } (A, \lambda) \in GL_n(E) \times E^\times.$$

2.2 Dual Group

For the diagonal torus $T_{n,\mathbb{Q}} \subset GL_{n,\mathbb{Q}}$, we denote by $\chi_1, \dots, \chi_n$ (resp., $\mu_1, \dots, \mu_n$) the usual characters (resp., cocharacters) of $T_{n,\mathbb{Q}}$. Let χ_0 (resp., μ_0) be the character (resp., cocharacter) of $T_{n,\mathbb{Q}} \times \mathbb{G}_{m,\mathbb{Q}}$ defined by $(A, x_0) \mapsto x_0$ (resp., $x \mapsto (I_n, x)$).

The dual group of G is $\widehat{G} = GL_{n,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. A splitting is a triplet $\Sigma = (\widehat{T}, \widehat{B}, \{X_\alpha\})$ where $(\widehat{B}, \widehat{T})$ is a Borel pair of $\widehat{G}$ and $X_\alpha \in \operatorname{Lie}(\widehat{G})$ an eigenvector for every simple root α of $\widehat{G}$. The $\operatorname{Gal}(E/\mathbb{Q})$-action on $\Psi(\widehat{G}) = \Psi(G)^\vee$ lifts uniquely to an automorphism of $\widehat{G}$ fixing Σ (*cf.* [1, section 1.6]). We make the following standard choices:

$$\widehat{T} = \{\text{diagonal matrices}\} \times \mathbb{G}_{m,\mathbb{C}},$$
$$\widehat{B} = \{\text{upper-triangular matrices}\} \times \mathbb{G}_{m,\mathbb{C}},$$
$$\{X_k\} = (\delta_{i,k}\delta_{j,k+1}) \text{ for } k = 1, 2, \dots, n-1.$$

The vector X_k lies in $\operatorname{Lie}(\widehat{G}) = M_n(\mathbb{C}) \oplus \mathbb{C}$ and is an eigenvector for the simple root $\chi_k - \chi_{k+1}$ of $\widehat{T}$. There is a unique nontrivial automorphism of $\widehat{G}$ fixing Σ, giving the action of σ on $\widehat{G}$:

$$\widehat{G} \longrightarrow \widehat{G}$$
$$(A, \lambda) \longmapsto (J'({}^tA^{-1})J', \det(A)\lambda),$$

where $J' = ((-1)^{i-1}\delta_{i,n+1-j})_{i,j}$ (*cf.* [1, 1.8(c)].

The choice of the basis $\mathcal{B}_W$ gives an identification between

$$\big(G_{\overline{\mathbb{Q}_p}}, T_{\overline{\mathbb{Q}_p}}\big) \quad \text{and} \quad \big(GL_{n,\overline{\mathbb{Q}_p}} \times \mathbb{G}_{m,\overline{\mathbb{Q}_p}}, T_{n,\overline{\mathbb{Q}_p}} \times \mathbb{G}_{m,\overline{\mathbb{Q}_p}}\big)$$

through (2.1). We fix the identification $\Psi(\widehat{G}, \widehat{T}) \simeq \Psi(G, T)^\vee$ given by $\chi_i \leftrightarrow \mu_i$ for $i = 0, \dots, n$. We also identify $\widehat{T}$ and $\operatorname{Hom}(X_*(T), \mathbb{C}^\times)$ such that

$$\big(\operatorname{diag}(x_1, \dots, x_n), x_0\big) \in \widehat{T}$$

corresponds to the map $\mu_i \mapsto x_i$.

2.3 Shimura Datum

Choose a basis $\mathcal{B}$ of $V(\mathbb{R})$ in which the hermitian form ψ admits the diagonal matrix $\operatorname{diag}(1, \dots, 1, -1)$. Consider the morphism $h\colon \mathbb{S} \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$ defined in $\mathcal{B}$ on $\mathbb{R}$-points by

$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G(\mathbb{R})$$
$$z \longmapsto \operatorname{diag}(z, \dots, z, \bar{z}).$$

Let X be the $G(\mathbb{R})$-conjugacy class of h. Then (G, X) is a Shimura datum [10, definition 5.5]. Its reflex field is E. Composing $h_{\mathbb{C}}$ on the right-hand side by $\mathbb{G}_{m,\mathbb{C}} \hookrightarrow \prod_{\sigma \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m,\mathbb{C}} \simeq \mathbb{S}_{\mathbb{C}}$ (given by $\sigma = \operatorname{Id}$) gives a cocharacter $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$. Finally, write $\widehat{\mu}$ for the associated character of $\widehat{T}$ that is dominant relative to $\widehat{B}$. We have $\widehat{\mu} = \chi_1 + \dots + \chi_{n-1} + \chi_0$.

2.4 The Representation r

Let r be the irreducible representation of $GL_{n,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ of highest weight $\widehat{\mu}$ relative to $(\widehat{B}, \widehat{T})$. Let ρ denote the identity representation $GL_{n,\mathbb{C}} \to GL_{n,\mathbb{C}}$; its weights are $(\chi_i)_{1 \le 1 \le n}$. The representation $\det \otimes \rho^\vee$ of $GL_{n,\mathbb{C}}$ is irreducible, and its highest weight is $\chi_1 + \cdots + \chi_{n-1} = \det - \chi_n$. Thus, we can define r as follows:

$$r \colon GL_{n,\mathbb{C}} \times \mathbb{G}_m \longrightarrow GL_{n,\mathbb{C}}$$
$$(A, \lambda) \longmapsto \lambda \det(A)\, {}^tA^{-1}.$$

Definition 2.1 The Hecke polynomial associated with (G, X) is

$$H_p(t) = \det\bigl(t - p^{n-1} r(g(\sigma \cdot g)\bigr).$$

The coefficients of H_p are functions on $\widehat{G}$ invariant under twisted conjugation $c_x \colon g \mapsto xg(\sigma \cdot x^{-1})$, for $x \in \widehat{G}$.

2.5 Hecke Algebra

Definition 2.2 For any $\mathbb{Q}$-algebra R, the Hecke algebra $\mathcal{H}_R(G(\mathbb{Q}_p)//K_p)$ is the set of K_p-biinvariant, compactly supported functions $G(\mathbb{Q}_p) \to R$. Multiplication is defined by convolution:

$$(f \star g)(y) = \int_{G(\mathbb{Q}_p)} f(x) g(x^{-1}y) dx$$

where the Haar measure on $G(\mathbb{Q}_p)$ is normalized by $|K_p| = 1$.

We recall some facts about the Satake isomorphism. We identify $\mathbb{Q}[X_*(A)]$ and $\mathcal{H}_{\mathbb{Q}}(T(\mathbb{Q}_p)//T_c)$ by $\lambda \mapsto 1_{\lambda(p)T_c}$ for $\lambda \in X_*(A)$. In [16, (1.7,1.8)], the twisted Satake homomorphism $\dot{S}_T^G$ is defined by the composition

$$\mathcal{H}_{\mathbb{Q}}\bigl(G(\mathbb{Q}_p)//K_p\bigr) \longrightarrow \mathcal{H}_{\mathbb{Q}}\bigl(B(\mathbb{Q}_p)//L\bigr) \longrightarrow \mathcal{H}_{\mathbb{Q}}\bigl(T(\mathbb{Q}_p)//T_c\bigr),$$

where the first arrow is restriction of functions and the second is the quotient by the unipotent radical of B. It induces an isomorphism between $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p)//K_p)$ and a subalgebra of $\mathcal{H}_{\mathbb{Q}}(T(\mathbb{Q}_p)//T_c)^{\Omega(T),\bullet}$ (the Weyl group acts by the "dot action", see [1, 1.8]). Denote by $S_T^G \colon \mathcal{H}_{\mathbb{C}}(G(\mathbb{Q}_p)//K_p) \to \mathcal{H}_{\mathbb{C}}(T(\mathbb{Q}_p)//T_c)$ the usual Satake isomorphism. Then $S_T^G = \alpha \circ \dot{S}_T^G$ where $\alpha \colon \mathbb{C}[X_*(A)] \to \mathbb{C}[X_*(A)]$ is defined by $\nu \mapsto p^{-2\langle \rho, \nu \rangle} \nu$.

2.6 Hecke Polynomial

The coefficients of H_p are polynomial functions on $\widehat{G}$ invariant under twisted conjugation (the twisted conjugation by $g \in \widehat{G}$ is the map $x \mapsto gx(\sigma \cdot g)^{-1}$). Their restrictions to $\widehat{T}$ are polynomial functions invariant under $\Omega(T)$ and twisted conjugation. This is the same as polynomial functions on $\widehat{A}$ invariant under $\Omega(T)$. By the untwisted Satake isomorphism, they give rise to elements in $\mathcal{H}_{\mathbb{Q}}(G(\mathbb{Q}_p)//K_p)$.

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Lemma 2.3 *The function $\widehat{G} \to \mathbb{C}$ given by $(A, x) \mapsto \det(A)x^2$ is invariant under twisted conjugation. It corresponds to the element 1_{pK_p} in $\mathcal{H}_{\mathbb{C}}(G(\mathbb{Q}_p)//K_p)$.*

Proof The element 1_{pK_p} maps to 1_{pT_c} by the Satake isomorphism. This element corresponds to $\lambda \in \mathbb{Q}[X_*(A)]$ where λ is the cocharacter $u \mapsto u.\mathrm{Id}$. Using the identification (2.1), we have $\lambda = \sum_{i>0} \mu_i + 2\mu_0$. The associated character of $\widehat{T}$ is $\sum_{i>0} \chi_i + 2\chi_0$, which is the function $(A, x_0) \mapsto \det(A)x_0^2$. ■

Let $g = (A, x_0) \in \widehat{T}$, with $A = \operatorname{diag}(x_1, \dots, x_n)$. Then

$$
\begin{aligned}
r\big(g(\sigma \cdot g)\big) &= \det(A)x_0^2 \operatorname{diag}\Big(\frac{x_n}{x_1}, \dots, \frac{x_1}{x_n}\Big),\\
H_p(t) &= \det\big(t - p^{n-1} r\big(g(\sigma \cdot g)\big)\big) = \prod_{i=1}^{n} \Big(t - p^{n-1} \det(A)x_0^2 \frac{x_{n+1-i}}{x_i}\Big)\\
&= R(t) \times \big(t - p^{n-1} \det(A)x_0^2\big),
\end{aligned}
$$

where

$$
R(t) = \prod_{i \neq k} \Big(t - p^{n-1} \det(A)x^2 \frac{x_{n+1-i}}{x_i}\Big).
$$

The polynomial R is invariant under $\Omega(T)$ and twisted conjugation. We deduce the following result.

Theorem 2.4 *The Hecke polynomial H_p in $\mathcal{H}(G(\mathbb{Q}_p)//K_p)$ factors into a product*

$$
H_p(t) = R(t) \cdot (t - p^{n-1} 1_{pK_p}),
$$

where $R(t) \in \mathcal{H}(G(\mathbb{Q}_p)//K_p)[t]$.

Let $[\mu]$ be the $G(E_p)$-conjugacy class of μ and $\mu_T \in [\mu]$ factorizing through T_{E_p}. Let $\mu' = \mu_T \overline{\mu_T}$ be the norm of μ_T. Let M be the Levi subgroup stabilizing μ'. It is defined over $\mathbb{Q}_p$. Write $L_M = K_p \cap M(\mathbb{Q}_p)$. The following easy lemma follows from the calculation in [16, (2.10)].

Lemma 2.5 *The polynomial R is the minimal polynomial of the element $1_{\mu'(p)L_M} \in \mathcal{H}_{\mathbb{Q}}(M(\mathbb{Q}_p)//L_M)$ via the Satake morphism S_M^G.*

3 The Shimura Variety

3.1 The Moduli Problem

Let $K^p \subset G(\mathbb{A}_f^p)$ be a compact open subgroup and denote by Sh_K the moduli space associated with the data $(E, \sigma, V, \psi, \mathcal{O}_{E,(p)}, \Lambda, h, \mu)$ according to Kottwitz (see [9]). We assume K^p to be sufficiently small such that this moduli problem is representable by a smooth quasi-projective scheme over $\mathcal{O}_{E_p}$. For any noetherian $\mathcal{O}_{E_p}$-scheme S, it classifies the following data, up to prime-to-p-isogeny:

(a) an abelian scheme A of dimension n over S,
(b) a $\mathbb{Q}$-homogeneous polarization $\overline{\lambda} = \mathbb{Q}\lambda$ for some prime-to-p polarization λ,

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(c) an action $\iota\colon \mathcal{O}_E \otimes \mathbb{Z}_{(p)} \hookrightarrow \operatorname{End}(A) \otimes \mathbb{Z}_{(p)}$ compatible with $\overline{\lambda}$,

(d) a $\pi_1(S,s)$-stable K^p-orbit of compatible isomorphisms $\overline{\eta}\colon V(\mathbb{A}_f^p) \xrightarrow{\sim} H_1(A_s, \mathbb{A}_f^p)$ for one geometric point s in each connected component of S.

Furthermore, $(A, \iota, \overline{\lambda}, \overline{\eta})$ satisfies the determinant condition: the characteristic polynomial of $e \in \mathcal{O}_E \otimes \mathbb{Z}_{(p)}$ acting on $\operatorname{Lie}(A)$ is $(T-e)^{n-1}(T-\overline{e}) \in \mathcal{O}_S[T]$.

We now give an equivalent moduli problem. Write

$$\widehat{\mathbb{Z}}^{(p)} = \prod_{\ell \neq p} \mathbb{Z}_\ell \subset \mathbb{A}_f^p$$

and for any $\mathcal{O}_E[1/p]$-lattice $L \subset V$, write $\widehat{L}^{(p)} = L \otimes \widehat{\mathbb{Z}}^{(p)} \subset V(\mathbb{A}_f^p)$. We can find a $\mathcal{O}_E[1/p]$-lattice $L \subset V$ satisfying the conditions

$$K^p \subset \left\{ g \in G(\mathbb{A}_f^p), g(\widehat{L}^{(p)}) = \widehat{L}^{(p)} \right\} \tag{3.1}$$

$$\varphi(L, L) \subset \mathbb{Z}\left[\frac{1}{p}\right]$$

(see notations for the definition of φ). The determinant of $\varphi\colon L \times L \to \mathbb{Z}[1/p]$ is a square in $\mathbb{Z}[1/p]$ and is well defined up to an invertible element. Let $d \in \mathbb{Z}$ coprime to p such that $\det(\varphi) = d^2$. We consider the moduli problem $\mathfrak{F}$ classifying the following data, up to isomorphism: for any noetherian $\mathcal{O}_{E_p}$-scheme S,

(a) an abelian scheme A of dimension n over S,

(b) a polarization $\lambda\colon A \to A^\vee$ of degree d^2,

(c) an action $\iota\colon \mathcal{O}_E \hookrightarrow \operatorname{End}(A)$ compatible with λ,

(d) a $\pi_1(S,s)$-stable K^p-orbit of compatible isomorphisms $\overline{\eta}\colon \widehat{L}^{(p)} \xrightarrow{\sim} T_f^p(A_s) = \prod_{\ell \neq p} T_\ell(A_s)$ such that the following diagram commutes

$$\begin{array}{ccccc} \widehat{L}^{(p)} & \times & \widehat{L}^{(p)} & \longrightarrow & \widehat{\mathbb{Z}}^{(p)} \\ \downarrow{\scriptstyle \eta^p} & & \downarrow{\scriptstyle \eta^p} & & \downarrow{\scriptstyle \theta} \\ T_f^p(A_s) & \times & T_f^p(A_s) & \longrightarrow & \widehat{\mathbb{Z}}^{(p)}(1), \end{array}$$

where θ is some $\widehat{\mathbb{Z}}^{(p)}$-linear isomorphism. Further, $(A, \iota, \lambda, \overline{\eta})$ satisfies the determinant condition.

Proposition 3.1 is well known; we will skip the proof.

Proposition 3.1 *The natural map $\mathfrak{F} \to Sh_K$ is an isomorphism of functors.*

Remark 3.2 For K^p small enough, every automorphism of a tuple $(A, \iota, \lambda, \overline{\eta})$ is trivial.

3.2 Dieudonné Modules

We write $\overline{Sh_K} = Sh_K \times \kappa(\mathcal{O}_{E_p})$ for the special fibre of Sh_K. It is equidimensional of dimension $n-1$. In order to study $\overline{Sh_K}$, we use (covariant) Dieudonné theory.

3.2.1 Some Definitions

Let k be an algebraically closed field containing $\kappa(\mathcal{O}_{E_p})$ and let $W = W(k)$ be the ring of Witt vectors and $W_{\mathbb{Q}} = W \otimes \mathbb{Q}$. The choice of k induces an embedding $\varrho\colon E_p \hookrightarrow W_{\mathbb{Q}}$. A *Dieudonné module over k* is a free W-module M of finite rank together with a σ-linear endomorphism F and a σ^{-1}-endomorphism V of M such that $FV = VF = p$.

A *Dieudonné space* over k is a finite-dimensional k-vector space together with a Frob_k-linear endomorphism and a Frob_k^{-1}-endomorphism V of M such that $FV = VF = 0$. If M is a Dieudonné module, then $\overline{M} = \frac{M}{pM}$ is a Dieudonné space that satisfies

$$\mathrm{Im}(F) = \mathrm{Ker}(V) \quad \text{and} \quad \mathrm{Im}(V) = \mathrm{Ker}(F). \tag{3.2}$$

An $\mathcal{O}_{E_p}$*-Dieudonné module over k* is a Dieudonné module endowed with a W-linear $\mathcal{O}_{E_p}$-action commuting with F, V. We define similarly the notion of $\mathcal{O}_{E_p}$*-Dieudonné space.* The $\mathcal{O}_{E_p}$-action induces a decomposition $M = M_e \oplus M_{\overline{e}}$ where M_e (resp. $M_{\overline{e}}$) is the submodule where $\mathcal{O}_{E_p}$ acts via ϱ (resp. $\overline{\varrho}$). We define the *signature* of an $\mathcal{O}_{E_p}$-Dieudonné module to be the pair

$$\left(\dim_k\left(\frac{M_e}{VM_{\overline{e}}}\right), \dim_k\left(\frac{M_{\overline{e}}}{VM_e}\right)\right).$$

If A is an abelian variety over k and $M = \mathbb{D}(A)$, then $\mathrm{Lie}(A) = \frac{M}{VM}$. We define in a similar fashion the signature of an $\mathcal{O}_{E_p}$-Dieudonné space. The signatures of M and $\overline{M}$ coincide.

A *quasi-unitary Dieudonné module over k* is an $\mathcal{O}_{E_p}$-Dieudonné module endowed with a non-degenerate alternating pairing $\langle\cdot,\cdot\rangle\colon M \times M \to W_{\mathbb{Q}}$ such that for all $e \in \mathcal{O}_{E_p}$, for all $x, y \in M$, $\langle ex, y\rangle = \langle x, \overline{e}y\rangle$ and $\langle Fx, y\rangle = \sigma\langle x, Vy\rangle$. We call M *unitary* if $\langle\cdot,\cdot\rangle\colon M \times M \to W$ is perfect. We define similarly the notion of *unitary Dieudonné space.*

A *unitary isocrystal over k* is a finite-dimensional $W_{\mathbb{Q}}$-vector space N together with endomorphisms F, V, an $\mathcal{O}_{E_p}$-action, a $W_{\mathbb{Q}}$-bilinear pairing $\langle\cdot,\cdot\rangle\colon N \times N \to W_{\mathbb{Q}}$ subject to the same hypotheses as above. If M is a quasi-unitary Dieudonné module, then $M \otimes W_{\mathbb{Q}}$ is a unitary isocrystal. If $\lambda \in \mathbb{Q}$, we denote by N_λ "the" simple isocrystal of slope λ. We say that an isocrystal is *supersingular* if all its slopes are $\frac{1}{2}$.

3.2.2 Dieudonné Theory

Dieudonné theory gives an equivalence of categories between unitary Dieudonné modules over k and p-divisible groups over k (with polarization and $\mathcal{O}_{E_p}$-action). For a definition of these objects; see [3, section 2]. Similarly, there is an equivalence of categories between unitary Dieudonné spaces over k that satisfy (3.2) and truncated Barsotti–Tate groups of level 1 (or BT_1) over k (with polarization and $\mathcal{O}_{E_p}$-action); see [8, definition 3.2] for these statements.

3.2.3 Examples

1. Let SS be the following Dieudonné module. It has a W-basis (g, h) such that $SS_e = Wg$, $SS_{\bar{e}} = Wh$, the endomorphisms F, V are defined by $F(g) = h = -V(g)$, and the pairing is given by $\langle g, h\rangle = 1$. This is a unitary Dieudonné module of signature $(1, 0)$ and slope $\frac{1}{2}$.
2. Let $d \geq 1$ be an integer. Define a unitary Dieudonné module $\mathbb{B}(d)$ as follows. It has a W-basis (e_i, f_i), $i \in \{1, \dots, d\}$ with $e_i \in \mathbb{B}(d)_e$ and $f_i \in \mathbb{B}(d)_{\bar{e}}$. The endomorphisms F, V are given by

$$\begin{aligned} F(f_1) &= (-1)^d e_n, \\ F(e_i) &= f_{i-1} \quad \text{for } i = 2, \dots, d \\ V(f_d) &= e_1, \\ V(e_i) &= f_{i+1} \quad \text{for } i = 1, \dots, d-1. \end{aligned}$$

   The alternating form is defined by $\langle e_i, f_j\rangle = (-1)^{i-1}\delta_{i,j}$. This is a unitary Dieudonné module of signature $(d-1, 1)$. If d is odd, every slope of $\mathbb{B}(d) \otimes W_{\mathbb{Q}}$ is $\frac{1}{2}$. If d is even, its slopes are $\frac{1}{2} \pm \frac{1}{d}$ (*cf.* [3, Lemma 3.3]).

3.2.4 Classification

We classify isocrystals and Dieudonné spaces that come into play in our situation. We refer to [3, sections 3.1 and 3.6], respectively for the proofs.

Proposition 3.3 *Let M be a unitary Dieudonné module of signature $(n-1, 1)$ and N its isocrystal. Then*

$$N \simeq N(r) \times (N_{\frac{1}{2}})^{n-2r},$$

where r is an integer $0 \leq r \leq \frac{n-1}{2}$ and

$$N(r) = \begin{cases} 0 & \text{if } r = 0, \\ N_{\frac{1}{2}-\frac{1}{2r}} \oplus N_{\frac{1}{2}+\frac{1}{2r}} & \text{if } r > 0 \text{ is even}, \\ N^2_{\frac{1}{2}-\frac{1}{2r}} \oplus N^2_{\frac{1}{2}+\frac{1}{2r}} & \text{if } r \text{ is odd}. \end{cases}$$

Proposition 3.4 *Let $\overline{M}$ be a unitary Dieudonné space of signature $(n-1, 1)$. There is an integer $1 \leq r \leq n$ such that $\overline{M}$ is isomorphic to $\overline{\mathbb{B}(r)} \oplus \overline{SS}^{n-r}$.*

3.3 Stratifications

3.3.1 Ekedahl–Oort Stratification

Applying Proposition 3.4 to M gives us an integer $1 \leq r \leq n$. This defines a stratification

$$\overline{Sh_K} = \bigsqcup_{r=1}^{n} \mathcal{M}_r,$$

where $\mathcal{M}_r$ is the locus where $\overline{M}$ is isomorphic to $\overline{\mathbb{B}(r)} \oplus \overline{SS}^{n-r}$. A point in $\mathcal{M}_r$ and its Dieudonné module are said of type r. The subsets $\mathcal{M}_r$ are locally closed, equidimensional and their dimensions are the following:

$$\dim(\mathcal{M}_{2i}) = n - i, \quad \dim(\mathcal{M}_{2i+1}) = i$$

(*cf.* [3, 5.4]).

3.3.2 Newton Polygon Stratification

The Newton polygon stratification is given by isomorphism classes of unitary isocrystals. It happens to be coarser than the Ekedahl–Oort stratification. It reads

$$\overline{Sh_K} = \mathcal{M}_2 \sqcup \mathcal{M}_4 \sqcup \cdots \sqcup \mathcal{M}_{n-1} \sqcup \bigsqcup_{r \text{ odd}} \mathcal{M}_r.$$

The stratum $\mathcal{M}_{2r}$ is also the locus where the unitary isocrystal is isomorphic to $N(r) \times (N_{\frac{1}{2}})^{n-2r}$. The stratum $\mathcal{M}_2$ is the only open stratum; it is called the μ-ordinary locus and is denoted by $\overline{Sh_K}^{\text{ord}}$. In [11, 15], the authors show that this locus is dense in $\overline{Sh_K}$. The supersingular locus is $\overline{Sh_K}^{ss} = \bigsqcup_{r \text{ odd}} \mathcal{M}_r$ and has dimension $\frac{n-1}{2}$ [3, Proposition 5.5]. Finally, we state a result on the geometric structure of $\overline{Sh_K}^{ss}$. For the proof, see [14, Theorem 5.2].

Theorem 3.5 *For K^p sufficiently small, the supersingular locus $\overline{Sh_K}^{ss}$ is equidimensional of dimension $\frac{n-1}{2}$ and locally of complete intersection. Its smooth locus is the open Ekedahl–Oort stratum $\mathcal{M}_n$.*

4 Moduli Space of p-isogenies

4.1 The Moduli Problem

We define a moduli space classifying p-isogenies. Let S be an $\mathcal{O}_{E_p}$-scheme and $\underline{A}_i = (A_i, \iota_i, \overline{\lambda}_i, \overline{\eta}_i)$, $i \in \{1, 2\}$ two tuples corresponding to S-valued points of Sh_K. A *p-isogeny* $f\colon \underline{A}_1 \to \underline{A}_2$ is an $\mathcal{O}_{E,(p)}$-linear isogeny compatible with the level structures $\overline{\eta}_1$,$\overline{\eta}_2$ such that $p^c\lambda_1 = f^\vee \circ \lambda_2 \circ f$ for some $c \geq 0$, which we call the multiplicator. This implies $\deg(f) = p^{cn}$.

Let $p\text{-}\mathscr{I}\text{sog}$ be the $\mathcal{O}_{E_p}$-scheme classifying p-isogenies. Two p-isogenies $f\colon \underline{A}_1 \to \underline{A}_2$ and $f'\colon \underline{A}'_1 \to \underline{A}'_2$ are identified if there are prime-to-p-isogenies $h_i\colon \underline{A}_i \to \underline{A}'_i$ for $i \in \{1, 2\}$ such that $f' \circ h_1 = h_2 \circ f$. The p-isogenies of multiplicator c form an open and closed subscheme $p\text{-}\mathscr{I}\text{sog}^{(c)} \subset p\text{-}\mathscr{I}\text{sog}$.

Let S be an $\mathcal{O}_{E_p}$-scheme. Write $\mathfrak{I}(S)$ for the moduli problem classifying p-isogenies between points of $\mathfrak{F}(S)$ (see Subsection 3.1 for the definition of $\mathfrak{F}$), up to isomorphisms. The natural map $\mathfrak{I} \to p\text{-}\mathscr{I}\text{sog}$ is an isomorphism of functors.

Let $s, t\colon p\text{-}\mathscr{I}\text{sog} \to Sh_K$ be the maps sending an isogeny to its source and target, respectively. The restrictions $s, t\colon p\text{-}\mathscr{I}\text{sog}^{(c)} \to Sh_K$ are proper for $c \geq 0$ (see [11, 4.2.1]).

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The "multiplication by p" map sends $\underline{A}$ to the isogeny $p\colon \underline{A} \to \langle p \rangle \underline{A}$, where the operator $\langle x \rangle$ multiplies the level structure of $\underline{A}$ by x, for $x \in G(\mathbb{A}_f^p)$. This defines a section of s. As s is separated, its image is a reduced closed subscheme $\langle p \rangle \subset p\text{-}\mathscr{I}\mathrm{sog}^{(2)}$. On the special fibre, there is a Frobenius section of s. It sends a tuple $\underline{A}$ to the Frobenius isogeny $F_{\underline{A}}\colon \underline{A} \to \underline{A}^{(p^2)}$. The level structure on $\overline{\eta}^{(p^2)}$ on $A^{(p^2)}$ is compatible with $\overline{\eta}$ through F_A. The image of s is a reduced closed subscheme $F \subset p\text{-}\mathscr{I}\mathrm{sog}^{(2)} \times \kappa(\mathcal{O}_{E_p})$, which is a union of irreducible components of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$, because s is finite and flat over the μ-ordinary locus [11, 4.2.2] and $\overline{Sh_K}^{\mathrm{ord}}$ is dense in $\overline{Sh_K}$. This follows also from the fact that $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$ is equidimensional of dimension $n-1$, as we will show later. By duality, we also have the Verschiebung map $V_A\colon A^{(p^2)} \to A$. Notice that $V_A \circ F_A = p^2$, so taking into account level structures, the Verschiebung is actually a map $V_{\underline{A}}\colon \langle p^{-2} \rangle \underline{A}^{(p^2)} \to \underline{A}$.

The μ-ordinary locus $p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})$ is defined as the inverse image of $\overline{Sh_K}^{\mathrm{ord}}$ by s (or t). We define similarly the supersingular locus $p\text{-}\mathscr{I}\mathrm{sog}^{ss} \times \kappa(\mathcal{O}_{E_p})$.

4.2 The $\mathbb{Q}$-algebra $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times L]$

Composition of isogenies defines a morphism

$$c\colon p\text{-}\mathscr{I}\mathrm{sog} \times_{t,s} p\text{-}\mathscr{I}\mathrm{sog} \longrightarrow p\text{-}\mathscr{I}\mathrm{sog}$$

which is proper (*cf.* [11, 4.2.1]). Let L be a field and $\mathcal{O}_{E_p} \to L$ a homomorphism. Let $Z_{\mathbb{Q}}(p\text{-}\mathscr{I}\mathrm{sog} \times L)$ denote the group of algebraic cycles of $p\text{-}\mathscr{I}\mathrm{sog} \times L$, with $\mathbb{Q}$-coefficients. For cycles Y_1, Y_2, we define

$$Y_1 \cdot Y_2 = c_*(Y_1 \times_{t,s} Y_2).$$

Extending this product bilinearly, we get a ring structure on $Z_{\mathbb{Q}}(p\text{-}\mathscr{I}\mathrm{sog} \times L)$, with identity $p\text{-}\mathscr{I}\mathrm{sog}^{(0)} \times L$. Let $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times L]$ be the $\mathbb{Q}$-subalgebra generated by the irreducible components.

Define the $\mathbb{Q}$-algebra $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})]$ in a similar fashion as hereabove. We may view F as an element of $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})]$ or of $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})]$.

4.3 A Commutative Diagram

Let $\mathcal{H}_0(G(\mathbb{Q}_p)//K_p) \subset \mathcal{H}(G(\mathbb{Q}_p)//K_p)$ be the subalgebra of $\mathbb{Q}$-valued functions that have support contained in $G(\mathbb{Q}_p) \cap \mathrm{End}(\Lambda)$. There is a $\mathbb{Q}$-algebra homomorphism

$$h\colon \mathcal{H}_0(G(\mathbb{Q}_p)//K_p) \longrightarrow \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times E_p]$$

which we will explain briefly. Let L be a field containing E_p and let $f\colon \underline{A}_1 \to \underline{A}_2$, corresponding to an L-valued point in $p\text{-}\mathscr{I}\mathrm{sog} \times E_p$. Choose isomorphisms $\alpha_i\colon \Lambda \simeq T_p(A_i)$, $i \in \{0,1\}$. Then $\alpha_2^{-1} \circ V_p f \circ \alpha_1\colon \Lambda \otimes \mathbb{Q}_p \to \Lambda \otimes \mathbb{Q}_p$ is an element of $G(\mathbb{Q}_p) \cap \mathrm{End}(\Lambda)$. Its class $\tau(f)$ in $K_p \backslash G(\mathbb{Q}_p)/K_p$ is independent of the choices involved. The function τ is constant on irreducible components of $p\text{-}\mathscr{I}\mathrm{sog} \times E_p$. Then h maps

$1_{K_pgK_p}$ to the sum of irreducible components $C \subset p\text{-}\mathscr{I}\mathrm{sog} \times E_p$ such that $\tau(C) = K_pgK_p$. The specialization map

$$\sigma\colon \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times E_p] \longrightarrow \mathbb{Q}\big[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})\big]$$

is defined as follows. Let C be an irreducible component of $p\text{-}\mathscr{I}\mathrm{sog} \times E_p$ and $\mathcal{C}$ the scheme-theoretic image of C by the open immersion $p\text{-}\mathscr{I}\mathrm{sog} \times E_p \hookrightarrow p\text{-}\mathscr{I}\mathrm{sog}$. Then $\sigma(C) = [\mathcal{C} \times \kappa(\mathcal{O}_{E_p})]$.

We denote again by $\mathcal{H}_0(M(\mathbb{Q}_p)//L_M)$ the functions with support in $\mathrm{End}(\Lambda)$, where M and L_M are defined as in Section 2.6. We have a commutative diagram of $\mathbb{Q}$-algebra homomorphisms

(4.1)
$$\begin{array}{ccc}
\mathcal{H}_0(G(\mathbb{Q}_p)//K_p) & \xrightarrow{\;h\;} & \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times E_p] \\
 & & \downarrow{\scriptstyle \sigma} \\
\downarrow{\scriptstyle \dot{S}} & & \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})] \\
 & & \downarrow{\scriptstyle \mathrm{ord}} \\
\mathcal{H}_0(M(\mathbb{Q}_p)//L_M) & \xrightarrow[\;\bar{h}\;]{} & \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})].
\end{array}$$

The morphism $\dot{S}$ is the twisted Satake homomorphism (see [16, §1]). The map ord is defined by intersection with the μ-ordinary locus. For the definition of the map $\bar{h}$, we refer to [11, 4.2.12]. In this context, the "congruence relation" means the following conjecture.

Conjecture *Consider the polynomial $H_{\mathfrak{p}}$ inside $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism $\sigma \circ h$. The element F lies in the center of this ring and the relation $H_{\mathfrak{p}}(F) = 0$ holds.*

This relation makes sense, since F belongs to the center of $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})]$, as we shall see in Section 6. The polynomial R annihilates the element $1_{\mu'(p)L_M} \in \mathcal{H}_0(M(\mathbb{Q}_p)//L_M)$ (Lemma 2.5). The proof of [11, Theorem 4.2.14] shows that this element is mapped to F by $\bar{h}$.

Theorem 4.1 *Consider the polynomial R inside $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \kappa(\mathcal{O}_{E_p})]$ via the morphism* ord $\circ\, \sigma \circ h$*. In this ring, the relation $R(F) = 0$ holds.*

5 The Source and Target Morphisms

5.1 The Moduli Space $\mathcal{N}'$

Uniformization theory from [12] can be used in order to study the supersingular locus of $\overline{Sh_K}$. In [13, 14], the authors give the geometric structure of $\overline{Sh_K}^{ss}$. We state their main results below.

Let $K^p \subset G(\mathbb{A}_f^p)$ be an open compact subgroup. We fix a tuple $\underline{A}' = (A', \iota', \overline{\lambda}', \overline{\eta}')$ over $\overline{\mathbb{F}}$. Using the same conventions as in [12], $\overline{\eta}'$ is a K^p-orbit of isomorphisms $\overline{\eta}' \colon H_1(A', \mathbb{A}_f^p) \to V(\mathbb{A}_f^p)$. We assume that A' is supersingular. We denote by $\underline{X}'$ its p-divisible group over $\overline{\mathbb{F}}$, and we write $M' = \mathbb{D}(A')$ and $N' = M' \otimes W_{\mathbb{Q}}$.

The formal scheme $\mathcal{N}'$ over $\overline{\mathbb{F}}$ classifies the following pairs $(\underline{X}, \rho_X)$ up to prime-to-p-isogenies. Given a $\overline{\mathbb{F}}$-scheme S, $\underline{X}$ is a p-divisible group with unitary structure of signature $(n-1, 1)$ over S and $\rho_X \colon \underline{X} \to \underline{X}'_S$ is a quasi-isogeny such that $\rho_X^*(\overline{\lambda}') = p^c \overline{\lambda}$ for some $c \in \mathbb{Z}$.

Dieudonné theory gives a bijection between $\mathcal{N}'(\overline{\mathbb{F}})$ and the set of quasi-unitary Dieudonné modules $M \subset N'$ of signature $(n-1, 1)$ such that $p^c M^\vee = M$ for some $c \in \mathbb{Z}$. If $(\underline{X}, \rho_X) \in \mathcal{N}'(S)$, there is a unique tuple $\underline{A} = (A, \iota, \overline{\lambda}, \overline{\eta})$ over S with a quasi-isogeny $f \colon \underline{A} \to \underline{A}'$ lifting ρ_X. We write $\underline{A} = \rho_X^* \underline{A}'$.

If $g \in G(\mathbb{A}_f)$ and $\underline{A} = (A, \iota, \overline{\lambda}, \overline{\eta})$ is a tuple over S, then we define $\langle g \rangle \underline{A} = (A, \iota, \overline{\lambda}, \overline{g \circ \eta})$. The uniformization morphism is given by

$$\Theta \colon \mathcal{N}' \times G(\mathbb{A}_f^p) \longrightarrow \overline{Sh_K}^{ss} \times \overline{\mathbb{F}},$$
$$(X, \rho_X) \times g \longmapsto \langle g \rangle \rho_X^* \underline{A}'.$$

Let I be the algebraic group over $\mathbb{Q}$ of $\mathcal{O}_{E,(p)}$-linear quasi-isogenies in $\operatorname{End}^0(A')$ compatible with $\overline{\lambda}'$. We have a natural homomorphism $\alpha_p \colon I(\mathbb{Q}_p) \hookrightarrow J(\mathbb{Q}_p)$, where J denotes the $\mathbb{Q}_p$-algebraic group of automorphisms of N' respecting the polarization up to factor. An element $\eta' \in \overline{\eta'}$ provides a homomorphism $\alpha^p \colon I(\mathbb{Q}) \to J(\mathbb{A}_f^p)$ (for more details, see [12, 6.15]). We have the following theorem ([12, Theorem 6.30]).

Theorem 5.1 *The uniformization theorem induces an isomorphism of $\overline{\mathbb{F}}$-schemes:*

$$I(\mathbb{Q}) \diagdown \mathcal{N}'_{\mathrm{red}} \times G(\mathbb{A}_f^p) / K^p \longrightarrow \overline{Sh_K}^{ss} \times \overline{\mathbb{F}}.$$

Write $I(\mathbb{Q}) \diagdown G(\mathbb{A}_f^p) / K^p = \{g_1, \dots, g_m\}$ and $\Gamma_j = I(\mathbb{Q}) \cap g_j K^p g_j^{-1}$. There is a decomposition

$$I(\mathbb{Q}) \diagdown \mathcal{N}'_{\mathrm{red}} \times G(\mathbb{A}_f^p) / K^p = \coprod_{j=1}^{m} \Gamma_j \diagdown \mathcal{N}'_{\mathrm{red}}.$$

We now recall some results from [13, 14]. Notice that in these articles the signature of A' is $(1, n-1)$. That is why we modify slightly the definition of $\mathcal{L}_i(n)$ (the integer i is replaced by $i-1$). The scheme $\mathcal{N}'_{\mathrm{red}}$ has a stratification

$$\mathcal{N}'_{\mathrm{red}} = \bigcup_{i \in 2\mathbb{Z}} \mathcal{N}'_{\mathrm{red},i},$$

where $\mathcal{N}'_{\mathrm{red},i}$ is the open and closed subscheme of elements of multiplicator i. Observe that $\mathcal{N}'_{\mathrm{red},i}$ is empty if i is odd ([14, 1.5.1]). For i even, all the $\mathcal{N}'_{\mathrm{red},i}$ are isomorphic to one another [14, Proposition 1.1]). Write $\mathbf{N}'_0 = \{x \in N'_e, \tau x = x\}$, where $\tau = p^{-1}F^2$. This is a $\mathbb{Q}_{p^2}$-hermitian space for the form $\{x, y\} = \alpha \langle x, Fy \rangle$. Define

$$\mathcal{L}_i(n) = \left\{ L \subset \mathbf{N}'_0, \mathbb{Z}_{p^2}\text{-lattice}, L = p^{i-1} L^\wedge \right\}$$

where $L^\wedge$ is the dual lattice with respect to $\{\cdot\,,\cdot\}$. For each $L \in \mathcal{L}_i(n)$, there is an associated closed subscheme $\mathcal{N}'_L \subset \mathcal{N}'_{\mathrm{red},i}$. We have the following decomposition in irreducible components:

$$\mathcal{N}'_{\mathrm{red},i} = \bigcup_{L\in\mathcal{L}_i(n)} \mathcal{N}'_L$$

([14, Theorem 4.2]). The $\mathcal{N}'_L$ are all isomorphic for $L \in \mathcal{L}_i(n)$, smooth, of dimension $\frac{n-1}{2}$. We say that a point $(\underline{X}, \rho_X) \in \mathcal{N}'_{\mathrm{red}}$ has type r if its Dieudonné module has type r (see Subsection 3.3.1). The smooth locus of $\mathcal{N}'_{\mathrm{red}}$ is the set of points of type n.

Further, there is a bijection between quasi-unitary superspecial Dieudonné modules $M \subset N'$ of signature $(n, 0)$ such that $p^{i-1}M^\vee = M$ and lattices in $\mathcal{L}_i(n)$. The bijection is given by $M \mapsto M_\epsilon^\tau$. If $L \in \mathcal{L}_i(n)$, then write L^+ for the associated superspecial Dieudonné module of signature $(n, 0)$. We thus get a bijection between irreducible components of $\mathcal{N}'_{\mathrm{red},i}$ and quasi-unitary superspecial Dieudonné modules of signature $(n, 0)$ that satisfy $p^{i-1}M^\vee = M$. If y a point in $\mathcal{N}'_{\mathrm{red},i}(\overline{\mathbb{F}})$ with Dieudonné module M, then y lies in $\mathcal{N}'_L(\overline{\mathbb{F}})$ if and only if $M \subset L^+$ ([14, Lemma 3.3]). If $y \in \mathcal{N}'_L(\overline{\mathbb{F}})$ has type n, then $L^+ = \Lambda^+(M)$, the smaller superspecial Dieudonné module containing M.

5.2 Dimension of the Fibers of s, t

Let $c \geq 0$ be a fixed even integer and x be an $\overline{\mathbb{F}}$-valued point of $\overline{Sh_K}^{ss}$, corresponding to a tuple $\underline{A}' = (A', \iota', \overline{\lambda}', \overline{\eta}')$ over $\overline{\mathbb{F}}$. Let M' be its Dieudonné module and N' its isocrystal. We write $t_c^{-1}(x)$ for the fibre of t above x in $p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$. We consider the moduli space $\mathcal{N}'_{\mathrm{red}}$ associated with $\underline{A}'$, as above. We assume that K^p satisfies the condition of Remark 3.2. Then there is a well-defined morphism of $\overline{\mathbb{F}}$-schemes

$$\epsilon\colon t_c^{-1}(x) \longrightarrow \mathcal{N}'_{\mathrm{red}}$$

sending an isogeny $f\colon \underline{A} \to \underline{A}'$ to the induced isogeny $f\colon \underline{X} \to \underline{X}'$ on the p-divisible groups (forgetting the level structure). It can be shown that ϵ is proper using the valuative criterion. Further, ϵ is injective on S-points for all $\overline{\mathbb{F}}$-schemes S, since $\underline{A}$ can be reconstructed from $f\colon \underline{X} \to \underline{X}'$. Thus ϵ is a closed immersion [7, 8.11.6].

The $\overline{\mathbb{F}}$-points of $t_c^{-1}(x)$ are in bijection with the quasi-unitary Dieudonné modules M over $\overline{\mathbb{F}}$ of signature $(n-1, 1)$ satisfying $M \subset M'$ and $p^c M^\vee = M$. The map ϵ induces the natural injection of this set into $\mathcal{N}'_{\mathrm{red}}(\overline{\mathbb{F}})$. If $f\colon \underline{A} \to \underline{A}'$ lies in $t_c^{-1}(x)$, then $\rho_X^* \underline{A}' = \underline{A}$. Embed $\mathcal{N}'_{\mathrm{red}}$ into $\mathcal{N}'_{\mathrm{red}} \times G(\mathbb{A}_f)$ by $\alpha\colon z \mapsto (z, 1)$. There is a commutative diagram

$$\begin{array}{ccc} I(\mathbb{Q})\backslash \mathcal{N}'_{\mathrm{red}} \times G(\mathbb{A}_f^p)/K^p & \xrightarrow{\Theta} & \overline{Sh_K}^{ss} \times k \\ \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle s} \\ \mathcal{N}'_{\mathrm{red}} & \xleftarrow{\epsilon} & t_c^{-1}(x). \end{array}$$

Proposition 5.2 *The restriction of s to $t_c^{-1}(x)$ is a finite morphism.*

Proof The restriction of α to any quasi-compact subscheme of $\mathcal{N}'_{\mathrm{red}}$ is quasi-finite (see [14, 5.4]). ■

Corollary 5.3 *The morphism*

$$p\text{-}\mathscr{I}\mathrm{sog}^{(c),ss}\times\kappa(\mathcal{O}_{E_p})\xrightarrow{(s,t)}\overline{Sh_K}^{ss}\times\overline{Sh_K}^{ss}$$

is finite.

Proof It is proper and quasi-finite. ■

This result also follows by observing the proof of the following much stronger theorem.

Theorem 5.4 *Let $c\geq 0$ be an even integer. There exists $K^p\subset G(\mathbb{A}_f^p)$ such that*

$$p\text{-}\mathscr{I}\mathrm{sog}_K^{(c)}\times\kappa(\mathcal{O}_{E_p})\xrightarrow{(s,t)}\overline{Sh_K}\times\overline{Sh_K}$$

is a closed immersion.

Proof We will use the moduli problems $\mathfrak{F},\mathfrak{I}$ described in Sections 3.1 and 4.1. Choose an $\mathcal{O}_E[\frac{1}{p}]$-lattice $L\subset V$ satisfying condition (3.1). Let $x_i=(A_i,\lambda_i,\iota_i,\overline{\eta_i})$, $i\in\{0,1\}$ be two points of $\mathfrak{F}(\overline{\mathbb{F}})$ with multiplicator c. We assume that there is an isogeny $h\colon(A_0,\lambda_0,\iota_0)\to(A_1,\lambda_1,\iota_1)$. We write $R=\mathrm{Hom}(A_0,A_1)$ for the group of homomorphisms (with no compatibility condition). If $f,g\in R$, we write

$$\langle f,g\rangle=\mathrm{Tr}(d^2\lambda_0^{-1}\circ f^\vee\circ\lambda_1\circ g),$$

where $\mathrm{Tr}\colon\mathrm{End}(A_0)\to\mathbb{Z}$ is the trace morphism. Since λ_0 has degree d^2, the quasi-isogeny $d^2\lambda_0^{-1}$ is an isogeny, thus $\langle f,g\rangle\in\mathbb{Z}$. The form $\langle\cdot,\cdot\rangle$ is symmetric and positive definite. Indeed, h identifies this form with one on $\mathrm{End}(A_0)\otimes\mathbb{Q}$ that is positive definite because the Rosati involution is. Define $q(f)=\langle f,f\rangle$ for $f\in R$. For a p-isogeny f of multiplicator c, we have

$$q(f)=d^2p^c. \tag{5.1}$$

Choose an integer N such that $N^2>4d^2p^c$. Define

$$K'^p=\big\{g\in K^p,\ (g-1)(\widehat{L}^{(p)})\subset N\widehat{L}^{(p)}\big\}$$

and write $\mathfrak{I}_{K'^p}$ for the moduli problem for this new level structure. Let f,g be two isogenies in $\mathfrak{I}_{K'^p}(\overline{\mathbb{F}})$ of multiplicator c such that $s(f)=s(g)$ and $t(f)=t(g)$, denoted respectively x_0 and x_1. Then (5.1) shows that $q(f)=q(g)=d^2p^c$. By definition of K'^p, we have $f=g$ on $A_0[N]$. There exists $h\in R$ such that $f-g=Nh$, thus $N^2q(h)=q(Nh)\leq 4d^2p^c$, because q is positive definite. We deduce $h=0$ and $f=g$. This shows that (s,t) is injective on $\overline{\mathbb{F}}$-points.

Write $R=(\overline{\mathbb{F}}[t])/((t^2))$. Let f,g be two points in $\mathfrak{I}_{K'^p}(R)$ such that $s(f)=s(g)$ and $t(f)=t(g)$. We denote by f_0,g_0 the reduced isogenies on $\overline{\mathbb{F}}$. We have $(s,t)(f_0)=(s,t)(g_0)$, thus $f_0=g_0$, and we deduce $f=g$ by [4, Lemma 3.1]. This shows that (s,t) is a closed immersion. ■

Proposition 5.5 *Let $x \in \overline{Sh_K}^{ss}(\overline{\mathbb{F}})$ and let $c \geq 2$ be an even integer. The dimension of the fibre $t_c^{-1}(x)$ is $\frac{n-1}{2}$.*

Proof Clearly $\dim(t_c^{-1}(x)) \leq \frac{n-1}{2}$. Let M be the Dieudonné module associated with x. Let M_0 be any quasi-unitary Dieudonné module of signature $(n,0)$ such that $M_0 \subset M$ and $M_0^\vee = p^{c-1}M_0$. The irreducible component of $\mathcal{N}'_{\mathrm{red}}$ associated with M_0 is then contained in $t_c^{-1}(x)$. ∎

Remark 5.6 When $c = 2$ and x has type n, there is only one such M_0 (namely $p\Lambda^+(M)$). Therefore, the fibre $t_c^{-1}(x)$ has only one irreducible component of dimension $\frac{n-1}{2}$.

5.3 Irreducible Components of $p\text{-}\mathscr{I}\mathrm{sog} \times \overline{\mathbb{F}}$

Proposition 5.7 *$p\text{-}\mathscr{I}\mathrm{sog} \times \overline{\mathbb{F}}$ is equidimensional of dimension $n-1$. If an irreducible component of $p\text{-}\mathscr{I}\mathrm{sog} \times \overline{\mathbb{F}}$ intersects the μ-ordinary locus, then it is contained in the closure of $p\text{-}\mathscr{I}\mathrm{sog}^{\mathrm{ord}} \times \overline{\mathbb{F}}$. Otherwise, all its points are supersingular.*

Proof Let C be an irreducible component of $p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ for $c \geq 0$. Using [3, Proposition 6.15], we have $\dim(C) \geq n-1$. Suppose that C intersects $p\text{-}\mathscr{I}\mathrm{sog}^{(c),\mathrm{ord}} \times \overline{\mathbb{F}}$. Since the μ-ordinary locus is open, C is contained in its closure and $\dim(C) = n-1$. Suppose that C has no μ-ordinary point and that there exists a non-supersingular point $z \in C$. There is an open subset $U \subset \overline{Sh_K}$ containing $t(z)$ such that $U \cap \overline{Sh_K}^{ss} = \varnothing$. Then z is in $t^{-1}(U) \cap C$ so this is a nonempty dense open subset of C. By [3, Corollary 7.3], the map t is finite over $t^{-1}(U)$, thus

$$\dim\bigl(t^{-1}(U) \cap C\bigr) = \dim\Bigl(t\bigl(t^{-1}(U) \cap C\bigr)\Bigr) \leq \dim\bigl(t(C)\bigr) < n-1,$$

because $t(C)$ does not meet $\overline{Sh_K}^{\mathrm{ord}}$. This contradicts $\dim(C) \geq n-1$. We have shown that a component not intersecting $p\text{-}\mathscr{I}\mathrm{sog}^{(c),\mathrm{ord}} \times \overline{\mathbb{F}}$ is supersingular. Finally, let C be a supersingular irreducible component. We have $\dim(\overline{Sh_K}^{ss}) = \frac{n-1}{2} = \dim(t_c^{-1}(x))$ for all $x \in \overline{Sh_K}^{ss}(\overline{\mathbb{F}})$, so $\dim(C) = n-1$. ∎

6 Congruence Relation

6.1 A Few Lemmas

Theorem 6.1 *Let X, Y be irreducible schemes of finite type over a field. Let $f\colon X \to Y$ be a dominant morphism. Then there is an open dense subset $U \subset Y$ such that for all $y \in U$, we have*

$$\dim(f^{-1}(y)) = \dim(X) - \dim(Y).$$

Proof We may assume that X, Y are reduced. Using [7, théorème 6.9.1], there exists an open dense subset $U \subset Y$ such that $f\colon f^{-1}(U) \to U$ is flat. Then use [7, lemme 13.1.1 and corollaire 14.2.4]. ∎

Corollary 6.2 *Let X, Y be schemes of finite type over a field. Let $f\colon X \to Y$ be a dominant morphism and $r \geq 0$. Assume that for all $y \in Y$, the dimension of $f^{-1}(y)$ is r. Then $\dim(X) - \dim(Y) = r$.*

Proof This is a simple exercise. ∎

Lemma 6.3 *Let $C \subset p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ be a supersingular irreducible component. Then $C_s := s(C)$ and $C_t := t(C)$ are irreducible components of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$.*

Proof They are irreducible closed subsets of dimension $\geq \frac{n-1}{2}$, since the fibres have dimension $\leq \frac{n-1}{2}$. But $\overline{Sh_K}^{ss}$ is equidimensional of dimension $\frac{n-1}{2}$ [14, Theorem 5.2], so the result follows. ∎

Proposition 6.4 *Let $C_1, C_2 \subset p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ be supersingular irreducible components. Assume that the map (s,t) is a closed immersion on $p\text{-}\mathscr{I}\mathrm{sog}_K^{(c),ss}$. Assume further that $C_{1,s} = C_{2,s}$ and $C_{1,t} = C_{2,t}$. Then $C_1 = C_2$.*

Proof The map (s,t) induces a closed immersion $C_1 \hookrightarrow C_{1,s} \times C_{1,t}$. Since $C_{1,s}$ and $C_{1,t}$ are irreducible components of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$, they are smooth of dimension $\frac{n-1}{2}$. The product $C_{1,s} \times C_{1,t}$ is thus irreducible of dimension $n-1$, so (s,t) defines an isomorphism $C_1 \xrightarrow{\sim} C_{1,s} \times C_{1,t}$. The same holds for C_2, and we deduce the result. ∎

6.2 The Frobenius Action

Let $\mathcal{F}$ be the Frobenius map on $\overline{Sh_K}$ and $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$. If C is a cycle, write $|C|$ for its support.

Proposition 6.5 *Let $\widetilde{C}$ be an irreducible component of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$. We have*

$$\mathcal{F}(\widetilde{C}) = \langle p \rangle \widetilde{C}.$$

Proof Let $x \in \overline{Sh_K}^{ss}(\overline{\mathbb{F}})$ be a point of type n whose image lies in $\widetilde{C}$. Write $M = \mathbb{D}(x)$. Then $t_2^{-1}(x)$ has a unique irreducible component C of dimension $\frac{n-1}{2}$ (see Remark 5.6). More precisely, points $y \in p\text{-}\mathscr{I}\mathrm{sog}^{(2)}(\overline{\mathbb{F}})$ whose image lie in C correspond to quasi-unitary Dieudonné modules M' of signature $(n-1,1)$ satisfying $p^2 M'^{\vee} = M'$ and $M' \subset p\Lambda^+(M)$. Clearly the isogenies $p\colon \langle p^{-1} \rangle x \to x$ and $V\colon \langle p^{-2} \rangle Fx \to x$ belong to $t_2^{-1}(x)$. They lie in C because $V^2 M \subset p\Lambda^+(M)$ and $pM \subset p\Lambda^+(M)$. Thus $\langle p^{-2} \rangle \mathcal{F}x$ and $\langle p^{-1} \rangle x$ lie in $s(C)$, which is an irreducible component of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$. Observe that $\langle p^{-2} \rangle \mathcal{F}x \in \langle p^{-2} \rangle \mathcal{F}(\widetilde{C})$ and $\langle p^{-1} \rangle x \in \langle p^{-1} \rangle \widetilde{C}$. We deduce $\langle p^{-2} \rangle \mathcal{F}(\widetilde{C}) = s(C) = \langle p^{-1} \rangle \widetilde{C}$, because a point of type n lies in a unique irreducible component of $\overline{Sh_K}^{ss} \times \overline{\mathbb{F}}$. ∎

Proposition 6.6 *Let $C \subset p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ be an irreducible component. Then*

$$F \cdot \mathcal{F}(C) = C \cdot F.$$

Proof If $\underline{A}_1 \xrightarrow{f} \underline{A}_0$ is an $\overline{\mathbb{F}}$-valued point of $p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ whose image lies in C, then $\mathcal{F}(f)$ is the isogeny $\underline{A}_1^{(q)} \xrightarrow{f^{(q)}} \underline{A}_0^{(q)}$ and we have $f^{(q)} \circ F_{A_1} = F_{A_0} \circ f$, where $F_{A_i}\colon A_i \to A_i^{(q)}$ is the Frobenius isogeny. This shows $|F \cdot \mathcal{F}(C)| = |C \cdot F|$. Write X for this support. We define an isomorphism

$$\alpha\colon p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}} \longrightarrow (p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}) \times_{t,s} F$$

by sending $\underline{A}_1 \xrightarrow{f} \underline{A}_0$ to the pair $(\underline{A}_1 \xrightarrow{f} \underline{A}_0\,,\ \underline{A}_0 \xrightarrow{F} \underline{A}_0^{(q)})$. Similarly, define

$$\beta\colon p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}} \longrightarrow F \times_{t,s} (p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}})$$

by sending $\underline{A}_1 \xrightarrow{f} \underline{A}_0$ to the pair $(\underline{A}_1 \xrightarrow{F} \underline{A}_1^{(q)}\,,\ \underline{A}_1^{(q)} \xrightarrow{f^{(q)}} \underline{A}_0^{(q)})$. It has degree 1. Consider the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \alpha\ } & C \times_{t,s} F \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle c_2} \\ F \times_{t,s} \mathcal{F}(C) & \xrightarrow{\ c_1\ } & X, \end{array}$$

where c_1 and c_2 are the restriction of c to $F \times_{t,s} \mathcal{F}(C)$ and $C \times_{t,s} F$ respectively. We have $F \cdot \mathcal{F}(C) = \deg(c_1)X$ and $C \cdot F = \deg(c_2)X$. Since α and β have degree 1, we deduce $F \cdot \mathcal{F}(C) = C \cdot F$. ■

We have proved some results on $p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$. Observe that diagram (4.1) involves $p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})$. For the relation $H_p(F) = 0$ to make sense, F has to commute with the coefficients of H_p. The pullback by

$$p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}} \to p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})$$

defines a $\mathbb{Q}$-algebra homomorphism

$$\mathbb{Q}\big[\,p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})\big] \hookrightarrow \mathbb{Q}[\,p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}]. \tag{6.1}$$

Corollary 6.7 *The element F belongs to the centre of $\mathbb{Q}[\,p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \kappa(\mathcal{O}_{E_p})]$.*

Proof This follows from Proposition 6.6 using (6.1). ■

6.3 Étale Covering

Let K^p and K'^p be two compact open subgroups of $G(\mathbb{A}_f^p)$ such that $K'^p \subset K^p$. Write $K = K_pK^p$ and $K' = K_pK'^p$. Then we have étale coverings

$$\pi\colon Sh_{K'} \longrightarrow Sh_K$$

$$\Pi\colon p\text{-}\mathscr{I}\mathrm{sog}_{K'} \longrightarrow p\text{-}\mathscr{I}\mathrm{sog}_K$$

Lemma 6.8 *The pushforward by Π defines a $\mathbb{Q}$-algebra homomorphism*

$$\Pi_*\colon \mathbb{Q}[\,p\text{-}\mathscr{I}\mathrm{sog}_{K'}^{(c)} \times \overline{\mathbb{F}}] \longrightarrow \mathbb{Q}[\,p\text{-}\mathscr{I}\mathrm{sog}_K^{(c)} \times \overline{\mathbb{F}}].$$

Further, $\Pi_*(F) = \deg(\pi)F$ and $\Pi_*(\langle p\rangle) = \deg(\pi)\langle p\rangle$.

Proof Consider the commutative diagram

$$\begin{array}{ccc} p\text{-}\mathscr{I}\mathrm{sog}_{K'}^{(c)} \times p\text{-}\mathscr{I}\mathrm{sog}_{K'}^{(c)} & \xrightarrow{c} & p\text{-}\mathscr{I}\mathrm{sog}_{K'}^{(c)} \\ \downarrow{\scriptstyle \Pi\times\Pi} & & \downarrow{\scriptstyle \Pi} \\ p\text{-}\mathscr{I}\mathrm{sog}_{K}^{(c)} \times_{s,t} p\text{-}\mathscr{I}\mathrm{sog}_{K}^{(c)} & \xrightarrow{c} & p\text{-}\mathscr{I}\mathrm{sog}_{K}^{(c)} . \end{array}$$

If C_1, C_2 are cycles, then

$$\begin{aligned} \Pi_*(C_1 \cdot C_2) &= \Pi_* c_*(C_1 \times_{t,s} C_2) = c_*(\Pi \times \Pi)_*(C_1 \times_{t,s} C_2) \\ &= c_*(\Pi_*(C_1) \times_{t,s} \Pi_*(C_2)) = \Pi_*(C_1) \cdot \Pi_*(C_2), \end{aligned}$$

thus Π_* is a ring homomorphism. We have another commutative diagram

$$\begin{array}{ccc} p\text{-}\mathscr{I}\mathrm{sog}_{K'}^{(2)} \times \overline{\mathbb{F}} & \xrightarrow{\Pi} & p\text{-}\mathscr{I}\mathrm{sog}_{K}^{(2)} \times \overline{\mathbb{F}} \\ {\scriptstyle F}\uparrow & & {\scriptstyle F}\uparrow \\ \overline{Sh_{K'}} \times \overline{\mathbb{F}} & \xrightarrow{\pi} & \overline{Sh_{K}} \times \overline{\mathbb{F}}, \end{array}$$

thus $\Pi_*(F) = \deg(\pi)F$, and similarly $\Pi_*(\langle p \rangle) = \deg(\pi)\langle p \rangle$. ∎

6.4 Main Theorem

Lemma 6.9 *Let $C \subset p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ be a supersingular irreducible component. Assume that the map (s,t) is a closed immersion on $p\text{-}\mathscr{I}\mathrm{sog}_K^{(c),ss}$. Then*

$$C \cdot \left(F - p^{n-1}\langle p \rangle\right) = 0.$$

holds in the ring $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \overline{\mathbb{F}}]$.

Proof The proof is twofold. First we show that $C \cdot F$ and $C \cdot \langle p \rangle$ have the same support, then we look at multiplicities. The supports $|C \cdot F|$ and $|C \cdot \langle p \rangle|$ are irreducible, of dimension $n-1$. Indeed, they are the direct images by the composition morphism c of $C \times_{t,s} F$ and $C \times_{t,s} \langle p \rangle$ respectively, which are irreducible. Thus, $|C \cdot F|$ and $|C \cdot \langle p \rangle|$ are irreducible components of $p\text{-}\mathscr{I}\mathrm{sog}^{(c+2)} \times \overline{\mathbb{F}}$. We clearly have $s(C \cdot F) = s(C \cdot \langle p \rangle)$. Using Proposition 6.5, we have

$$t(C \cdot F) = \mathcal{F}(C_t) = \langle p \rangle C_t = t\left(C \cdot \langle p \rangle\right).$$

Proposition 6.4 then shows that $|C \cdot F| = |C \cdot \langle p \rangle|$. We denote by X this closed subset.

The projection on C defines isomorphisms

$$a_F \colon C \times_{t,s} F \to C, \quad a_p \colon C \times_{t,s} \langle p \rangle \to C.$$

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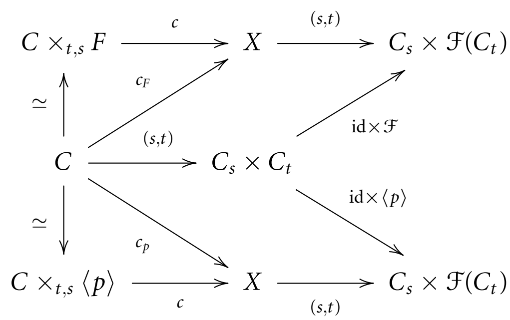
Write $c_F = c \circ a_F^{-1}$ and $c_p = c \circ a_p^{-1}$. There is a commutative diagram:



Recall that $\mathcal{F}(C_t) = \langle p \rangle C_t$. By definition, $C \cdot F = \deg(c_F)X$ and $C \cdot \langle p \rangle = \deg(c_p)X$. The diagram shows that

$$\frac{\deg(c_F)}{\deg(c_p)} = \frac{\deg(\mathrm{id} \times \mathcal{F})}{\deg(\mathrm{id} \times \langle p \rangle)}.$$

The map $\langle p \rangle \colon C_t \to \langle p \rangle C_t$ has degree 1 and $\mathcal{F} \colon C_t \to \mathcal{F}(C_t)$ has degree $p^{2\frac{n-1}{2}} = p^{n-1}$ since C_t has dimension $\frac{n-1}{2}$. Thus, $\deg(c_F) = p^{n-1} \deg(c_p)$, and finally $C \cdot F = p^{n-1} C \cdot \langle p \rangle$. ■

Theorem 6.10 *Let $C \subset p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ be a supersingular irreducible component. In the ring $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}]$, the following relation holds:*

$$C \cdot \left(F - p^{n-1}\langle p \rangle\right) = 0.$$

Proof Let $K'^p \subset K^p$ such that (s,t) is a closed immersion on $p\text{-}\mathscr{I}\mathrm{sog}_{K'}^{(c+2)} \times \overline{\mathbb{F}}$, (by Proposition 5.4), and let C' be a supersingular irreducible component of $p\text{-}\mathscr{I}\mathrm{sog}^{(c)} \times \overline{\mathbb{F}}$ such that $\Pi(C') = C$. We have $C' \cdot (F - p^{n-1}\langle p \rangle) = 0$ (Lemma 6.9), and taking the image by Π_*, we find $C \cdot (F - p^{n-1}\langle p \rangle) = 0$ (Lemma 6.8). ■

Theorem 6.11 *Let H_p be the Hecke polynomial. Consider the coefficients of H_p in $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})]$ through $\sigma \circ h$ (see Diagram (4.1)). We have the relation $H_p(F) = 0$.*

Proof We have $H_p(t) = R(t) \cdot (t - p^{n-1}\langle p \rangle)$ with $R(t) \in \mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})][t]$ (Theorem 2.4). The coefficients of H_p and R are linear combinations of supersingular irreducible components of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$. Indeed, they are specialization of cycles of dimension $n-1$ in $\mathbb{Q}[p\text{-}\mathscr{I}\mathrm{sog} \times E_p]$, and specialization respects dimensions [6, 20.3]. These components are either μ-ordinary or supersingular (Proposition 5.7). If C is an irreducible component of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$, then so is $C \cdot F$. Thus, $R(F)$ is a linear combination of irreducible components of $p\text{-}\mathscr{I}\mathrm{sog} \times \kappa(\mathcal{O}_{E_p})$, which are supersingular by Theorem 4.1. Finally, Theorem 6.10 shows that $H_p(F) = 0$. ■

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References

[1] D. Blasius and J. D. Rogawski, *Zeta functions of Shimura varieties*. In: Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., 55, American Mathematical Society, 1994, pp. 525–571.
[2] O. Bültel, *The congruence relation in the non-PEL case*. J. Reine Angew. Math. **544**(2002), 133–159.
[3] O. Bültel and T. Wedhorn, *Congruence relations for Shimura varieties associated to some unitary groups*. J. Inst. Math. Jussieu **5**(2006), no. 2, 229–261. http://dx.doi.org/10.1017/S1474748005000253
[4] C. Brian, *Gross-Zagier revisited*. In: Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ., 49, Cambridge University Press, Cambridge, 2004, pp. 67–163.
[5] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*. Ergebnisse der Math., 22, Springer-Verlag, Berlin, 1990.
[6] W. Fulton, *Intersection theory*. Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, 3. Folge, Bd. 2, Springer-Verlag GmbH, 1998.
[7] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*. Inst. Hautes Études Sci. Publ. Math. **28**(1966), 1–255.
[8] ———, *Groupes de Barsotti-Tate et cristaux de Dieudonné*. Séminaire de Mathématiques Supérieures, 45, Les Presses de l'Université de Montréal, Montreal, Que., 1974.
[9] R. E.Kottwitz, *Points on some Shimura varieties over finite fields*. J. Amer. Math. Soc. **5**(1992), no. 2, 373–444. http://dx.doi.org/10.1090/S0894-0347-1992-1124982-1
[10] J. S. Milne, *Introduction to Shimura varieties*. In: Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., 4, American Mathematical Society, 2005, pp. 265–378.
[11] B. Moonen, *Serre-Tate theory for moduli spaces of PEL type*. Ann. Sci. École Norm. Sup. (4) **37**(2004), 223–269.
[12] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*. Annals of Mathematics Studies, 141, Princeton University Press, Princeton, 1996.
[13] I. Vollaard, *The supersingular locus of the Shimura variety for* $\mathrm{GU}(1, s)$. Canad. J. Math. **62**(2010), no. 3, 668–720. http://dx.doi.org/10.4153/CJM-2010-031-2
[14] I. Vollaard and T Wedhorn, *The supersingular locus of the Shimura variety of* $\mathrm{GU}(1, n-1)$ *II*. Invent. Math. **184**(2011), no. 3, 591–627. http://dx.doi.org/10.1007/s00222-010-0299-y
[15] T. Wedhorn, *Ordinariness in good reductions of Shimura varieties of PEL-type*. Ann. Sci. École Norm. Sup. (4) **32**(1999), no. 5, 575–618.
[16] ———, *Congruence relations on some Shimura varieties*. J. Reine Angew. Math. **524**(2000), 43–71.

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