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ON THE FACTORIZATION OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction and statement of results. In [**4**] N. Steinmetz used Nevanlinna theory to establish remarkably versatile theorems on the factorization of ordinary differential equations which implied numerous previous results of various authors. (Here factorization is taken in the sense of function composition as introduced by F. Gross in [**2**].) The thrust of Steinmetz' central results on factorization is that if $g(z)$ is entire and $f(z)$ is meromorphic in $\mathbf{C}$ such that the composite $f \circ g$ satisfies an algebraic differential equation, then so do $f(z)$ and, degenerate cases aside, $g(z)$. In addition, the more one knows about the equation for $f \circ g$ (e.g. degree, weight, autonomy), the more one can conclude about the equations for f and g.

In this note we generalize Steinmetz' work to show the following:

a) Steinmetz' two basic results, Satz 1 and Korollar 1 of [**4**] can be seen as one-variable specializations of a single two variable result, and

b) the function $g(z)$ can itself be allowed to be a function of several variables.

The recursive scheme of proof remains as in [**4**], but we apply Nevanlinna theory for functions of several variables, especially the beautiful results of A. Vitter [**5**] establishing the full several variables analogues of the Lemma of the Logarithmic Derivative and the Second Main Theorem, along with the Defect Relation.

Our results on the factorization of differential equations will follow from a theorem which does not explicitly involve differentiation. Nevertheless let us state our results in increasing generality in order to illustrate the use of the central result. First of all, we make specific what we mean by an algebraic differential equation. For a meromorphic function h on an open set in $\mathbf{C}^n$, a *differential polynomial* $\mathcal{D}h$ in h is a polynomial in the variables $z_1, \dots, z_n$ and h and the derivatives (partial if $n > 1$) of h. $\mathcal{D}h$ will be called *autonomous* if none of $z_1, \dots, z_n$ is involved in $\mathcal{D}h$ and *trivial* if, as an ordinary polynomial, $\mathcal{D}h \equiv 0$. A differential polynomial which is a product of a polynomial in $z_1, \dots, z_n$ with a power product of h and its derivatives will be called a *differential monomial* in h. A (non-trivial) *algebraic differential equation* for h is simply an equation of

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the form $\mathscr{D}h = 0$, where $\mathscr{D}h$ is a (non-trivial) differential polynomial in h. The equation will be said to be autonomous or trivial if $\mathscr{D}h$ is autonomous or trivial. The main content of Satz 3 and Satz 5 of [**4**] is given by the case $n = 1$ in the following result, where we write, e.g., $Mf|_g$ for the composite $(Mf) \circ g$.

THEOREM 1. *Let $f:\mathbf{C} \to \mathbf{P}$ be meromorphic and $g:\mathbf{C}^n \to \mathbf{C}$ be entire and non-constant. Let the composite $f \circ g:\mathbf{C} \to \mathbf{P}$ satisfy the algebraic differential equation*

(1) $\quad \mathscr{D}h = 0.$

i) *If we can write*

(2) $$\mathscr{D}(f \circ g) = \sum_{k=1}^{K} M_k f|_g \cdot \delta_k g,$$

with $M_k f$ distinct autonomous differential monomials in f and $\delta_k g$ non-zero differential polynomials in g, $k = 1, \dots, K$, then f satisfies a non-trivial algebraic differential equation of the form

(3) $$\mathscr{D}_1 f = \sum_{k=1}^{K} a_k \cdot M_k f = 0,$$

with the a_k polynomials in the variable of f.

ii) *If we can write*

(4) $$\mathscr{D}(f \circ g) = \sum_{k=1}^{K} \Delta_k f|_g \cdot m_k g,$$

with $m_k g$ distinct differential monomials in g and $\Delta_k f$ non-zero autonomous differential polynomials in f, $k = 1, \dots, K$, then g satisfies a non-trivial algebraic differential equation of the form

(5) $$\mathscr{D}_2 g = \sum_{k=1}^{K} b_k(g) \cdot m_k g = 0,$$

with the b_k polynomials.

The condition of i) is automatically fulfilled by non-trivial differential polynomials $\mathscr{D}$ when $n = 1$, as is pointed out in the proof of Satz 2 of [**4**]. To see that for $n > 1$ some restriction is necessary to obtain non-trivial information on f, it suffices to consider the differential equation

$$\frac{\partial h}{\partial z_1} - \frac{\partial h}{\partial z_2} = 0,$$

which is satisfied by the function $h(z_1, z_2) = f(z_1 + z_2)$ for any meromorphic f. Thus when we view h as a composite of f with $g(z_1, z_2) = z_1 + z_2$, we see that in the representation (2),

$$K = 1, \quad M_1 f = f', \quad \text{and} \quad \delta_1 g = \frac{\partial g}{\partial z_1} - \frac{\partial g}{\partial z_2} = 0.$$

So the non-vanishing of the coefficients $\delta_k g$ of distinct monomials $M_k f$ in (2) is a natural condition from which to conclude anything in particular about f.

The condition of ii) is often satisfied, but it is not automatic even when $n = 1$. (See the Zusatzbedingung for Satz 5 of **[4]**.) The degree of the polynomials a_k and b_k can be bounded above in terms of the degrees of the differential monomials in (2) and (4) (see the Supplement to Theorem 3 below and the proof of Theorem 1). Finding applications is clearly a matter of determining natural conditions under which the hypothesis i) or ii) is satisfied, a question we shall not pursue here. The preceding result is obviously implied by the following more general result, where we let w denote the variable for f, i.e., $f = f(w)$, and $\mathbf{z} = (z_1, \dots, z_n)$ denote the variables for g, i.e., $g = g(\mathbf{z})$.

THEOREM 2. *Let f, g be as in Theorem* 1. *For $i = 1, \dots, K$, let $D_i f$ and $\Delta_i g$ denote pairs of non-trivial differential polynomials in f and g, respectively, satisfying*

$$(6) \qquad D_1 f|_g \Delta_1 g + \dots + D_K f|_g \Delta_K g = 0.$$

Then there exist polynomials $A_{ij}(x, y) \in \mathbf{C}[x, y]$, $1 \leqq i, j \leqq K$, and a non-zero polynomial $A(x)$ such that

$$\text{a)} \qquad \sum_{i,j=1}^{K} A_{ij}(g(\mathbf{z}), w)(D_i f(w))(\Delta_j(g(\mathbf{z}))) = 0,$$

$$\text{b)} \qquad A_{ij}(x, x) = \delta_{ij} A(x),$$

where δ_{ij} denotes the Kronecker delta function, $1 \leqq i, j \leqq K$.

Condition b) shows that when we set $w = g(\mathbf{z})$, then equation a) becomes equation (6) multiplied by a non-zero polynomial in $g(\mathbf{z})$. Thus, since $g(\mathbf{z})$ is not a constant, equation (6) is simply a specialization of a more general partial differential equation satisfied jointly by $f(w)$ and $g(\mathbf{z})$.

To state our general result, we use the Nevalinna characteristic function $T(h, r)$ defined for meromorphic functions h on $\mathbf{C}^n$. For a careful exposition of most of the basic properties, we refer the reader to **[3]**. Here $S(h, r)$ denotes a positive function of $r \geqq 0$ such that $S(h, r) = o(T(h, r))$ outside an exceptional set of r's of finite measure.

THEOREM 3. *Let* $F_1, \ldots, F_K: \mathbf{C} \to \mathbf{P}$ *and* $h_1, \ldots, h_K: \mathbf{C}^n \to \mathbf{P}$ *be meromorphic functions, none of which is identically zero. Let* $g: \mathbf{C}^n \to C$ *be a non-constant entire function. For some* $C > 0$ *suppose that the characteristic functions satisfy*

$$\sum_{k=1}^{K} T(h_k, r) \leqq CT(g, r) + S(g, r).$$

If

$$(F_1 \circ g)h_1 + \ldots + (F_K \circ g)h_K = 0, \tag{7}$$

then there exist polynomials $0 \neq A(x) \in \mathbf{C}[x]$ *and* $A_{ij}(x, y) \in \mathbf{C}[x, y]$, $1 \leqq i, j \leqq K$, *such that*

$$\sum_{i,j} A_{ij}(g(\mathbf{z}), w)F_i(w)h_j(\mathbf{z}) = 0 \tag{8}$$

and

$$A_{ij}(x, x) = \delta_{ij}A(x),$$

$1 \leqq i, j \leqq K$, *where* δ_{ij} *denotes the Kronecker delta function.*

We have then immediately the following corollary:

COROLLARY 1. *Under the above hypotheses,*

i) *there exist non-zero polynomials* $P_1(x), \ldots, P_K(x)$ *such that*

$$P_1(g)h_1 + \ldots + P_K(g)h_K = 0, \tag{9}$$

ii) *there exist non-zero polynomials* $Q_1(x), \ldots, Q_K(x)$ *such that*

$$Q_1F_1 + \ldots + Q_KF_K = 0. \tag{10}$$

The case $n = 1$ of part i) is the central Satz 1 of [**4**], and the case $n = 1$ of part ii) is Korollar 1 of [**4**]. To prove (9), we need only select a in the range of $g(\mathbf{z})$ such that $A(a) \neq 0$ and each $F_i(a) \neq 0$. Then we set

$$P_j(x) = \sum_{i=1}^{K} A_{ij}(x, a)F_i(a),$$

for $j = 1, \ldots, K$. Now $P_j(x) \not\equiv 0, j = 1, \ldots, K$, for when $g(\mathbf{z}') = a$,

$$P_j(g(\mathbf{z}')) = A(a)F_j(a) \neq 0,$$

by our choice of a. Similarly when we choose $\mathbf{b} \in \mathbf{C}^n$ such that $A(g(\mathbf{b})) \neq 0$ and each $h_i(\mathbf{b}) \neq 0$ and set

$$Q_i(x) = \sum_{j=1}^{K} A_{ij}(g(\mathbf{b}), x)h_j(\mathbf{b}),$$

$1 \leqq i \leqq K$, then

$$Q_i(g(\mathbf{b})) = A(g(\mathbf{b}))h_i(\mathbf{b}) \neq 0.$$

2. Proof of Theorem 3.

A. *The Auxiliary Functions* H_κ. Let $t_1, t_2, \ldots$ be a sequence of new parameters. We construct for $\kappa = 1, 2, \ldots$, an auxiliary function

$$H_\kappa(\mathbf{z}, w) = \frac{\sum A_{ij\kappa}(g(\mathbf{z}), w)F_i(w)h_j(\mathbf{z})}{(g(\mathbf{z}) - t_1)\ldots(g(\mathbf{z}) - t_{\kappa-1})(g(\mathbf{z}) - w)},$$

where the sum runs over all $1 \leqq i, j \leqq K$. We carry out this construction inductively in such a fashion that the $A_{ij\kappa}$ have coefficients which are meromorphic functions of $t_1, \ldots, t_{\kappa-1}$ and moreover

(11) $\deg_x A_{ij\kappa}(x, y) \leqq \kappa - 1 - \left[\frac{\kappa - 1}{K}\right]$,

$\left(\text{with strict inequality for } j < \kappa - 1 - \left[\frac{\kappa - 1}{K}\right]\right)$,

(12) $\deg_y A_{ij\kappa} \leqq \kappa - 1$,

and

(13) $A_{ij\kappa}(x, x) = \delta_{ij}A_\kappa(x)$,

$1 \leqq i, j \leqq K$, for a non-zero polynomial $A_\kappa(x)$, whose coefficients are meromorphic functions of $t_1, \ldots, t_{\kappa-1}$. Let

(14) $P_\kappa(\mathbf{z}, w) = \sum_{i,j} A_{ij\kappa}(g(\mathbf{z}), w)F_i(w)h_j(\mathbf{z})$

designate the numerator of $H_\kappa(\mathbf{z}, w)$. Then $H_\kappa(\mathbf{z}, w)$ will satisfy the additional vanishing condition: Whenever the denominator

$$(g(\mathbf{z}) - t_1)\ldots(g(\mathbf{z}) - t_{\kappa-1})(g(\mathbf{z}) - w)$$

vanishes, then so does the numerator $P_\kappa(\mathbf{z}, w)$.

i) Case $\kappa = 1$. We simply set $A_{ij1}(x, y) = \delta_{ij}$, so that

$$H_1(\mathbf{z}, w) = \frac{F_1(w)h_1(\mathbf{z}) + \ldots + F_K(w)h_K(\mathbf{z})}{g(\mathbf{z}) - w}.$$

Conditions (11)-(14) hold trivially. The vanishing condition on $P_1(\mathbf{z}, w)$ is implied directly by equation (7).

ii) Inductive construction of $H_{\kappa+1}(z, w)$. Assume that $H_\kappa(z, w)$ has been constructed as in (11)-(14) satisfying the additional vanishing condition as well. We may assume that $t_1, \ldots, t_{\kappa-1}, w$ are fixed in general position. If

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$H_\kappa(\mathbf{z}, w) \equiv 0$, then set the $A_{ij,x+1} := A_{ijx}$. If $H_\kappa(\mathbf{z}, w) \not\equiv 0$, then some expression of the form

$$(15) \quad \sum_i A_{ij\kappa}(g(\mathbf{z}), w))F_i(w)$$

$j = 1, \ldots, K$, is not identically zero. Let $c_\kappa(w)$ denote the non-zero coefficient of the highest power of $g(\mathbf{z})$ occurring in any expression (15), when it is considered simply as a polynomial in $g(\mathbf{z})$. If the highest power of $g(\mathbf{z})$ occurs in more than one term of the form (15), then choose the expression with j minimal, say $j = j_\kappa$. Then $c_\kappa(w)$ can be expressed as

$$c_\kappa(w) = L_{\kappa 1}(w)F_1(w) + \ldots + L_{\kappa K}(w)F_K(w),$$

where by (12) the $L_{\kappa i}$ are polynomials in w, not all zero, of degree at most $\kappa - 1$ whose coefficients are meromorphic functions of $t_1, \ldots, t_{\kappa-1}$. Then define for $1 \leqq i, j \leqq K$, the numerator of $H_{\kappa+1}(\mathbf{z}, w)$ by setting

$$\begin{aligned} &A_{ij,\kappa+1}(g(\mathbf{z}), w) := \\ &c_\kappa(t_\kappa)(g(\mathbf{z}) - t_\kappa)A_{ij\kappa}(g(\mathbf{z}), w) - L_{\kappa i}(w)(g(\mathbf{z}) - w) \\ &\times \sum_l A_{lj\kappa}(g(\mathbf{z}), t_\kappa)F_l(t_\kappa), \end{aligned}$$

by dropping terms in $A_{ijx}(g(\mathbf{z}), w)$ that cancel in (15), we may assume that no higher power of $g(\mathbf{z})$ occurs in any $A_{ijx}(g(\mathbf{z}), w)$ so that

$$(16) \quad P_{\kappa+1}(\mathbf{z}, w) = c_\kappa(t_\kappa)(g(\mathbf{z}) - t_\kappa)P_\kappa(\mathbf{z}, w) - c_\kappa(w)(g(\mathbf{z}) - w)P_\kappa(\mathbf{z}, t_\kappa)$$

and

$$H_{\kappa+1}(\mathbf{z}, w) = c_\kappa(t_\kappa)H_\kappa(\mathbf{z}, w) - c_\kappa(w)H_\kappa(\mathbf{z}, t_\kappa).$$

Then collecting coefficients of powers of $g(\mathbf{z})$ shows that

$$\deg_x A_{ij,\kappa+1}(x, y) \leqq 1 + \deg_x A_{ij\kappa}(x, y),$$

with strict inequality for $j = j_\kappa$. Thus (11) holds by induction. Inequality (12) follows even more easily. Equation (14) is straightforward from (16). In fact it is easily seen from the definition of $A_{ij,\kappa+1}$ by induction that

$$A_{\kappa+1}(x) = \prod_{j=1}^{\kappa} c_j(t_j)(x - t_j).$$

Therefore $A_{\kappa+1}(g(\mathbf{z}))$ is, up to the factor $(\prod c_j(t_j))/(g(\mathbf{z}) - w)$, the denominator of $H_{\kappa+1}(\mathbf{z}, w)$.

Thus when the denominator of $H_{\kappa+1}(\mathbf{z}, w)$ vanishes because $g(\mathbf{z}) = t_\kappa$, then by induction $P_\kappa(\mathbf{z}, t_x) = 0$ and by (16),

$$P_{\kappa+1}(\mathbf{z}, w) = 0.$$

Using this fact inductively for lower indices as well, we see from (16)

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that when $g(\mathbf{z}) = t_j, j = 1, \ldots, \kappa$, then $P_{\kappa+1}(\mathbf{z}, w) = 0$. On the other hand, when $g(\mathbf{z}) = w$, then by the definition of the $A_{ij,\kappa+1}$, (14) and (7), we see that

$$\begin{aligned} P_{\kappa+1}(\mathbf{z}, w) &= \sum_{i,j} A_{ij,\kappa+1}(g(\mathbf{z}), g(\mathbf{z}))F_i(g(\mathbf{z}))h_j(\mathbf{z}) \\ &= A_{\kappa+1}(g(\mathbf{z})) \sum_i F_i(g(\mathbf{z}))h_i(z) \\ &= 0. \end{aligned}$$

Consequently the vanishing condition holds as well for $H_{ij,\kappa+1}(\mathbf{z}, w)$.

B. *Vanishing of* H_κ, $\kappa \geqq \kappa_0$. We will show that from some index κ_0 on, all $H_\kappa(\mathbf{z}, w) \equiv 0$. Fix κ. If $H_\kappa(\mathbf{z}, w) \equiv 0$, then there is nothing to show. Otherwise we may fix $t_1, \ldots, t_{\kappa-1}, t_\kappa := w$ in general position. Thus in particular the $A_{ij\kappa}(g(\mathbf{z}), t_\kappa)$ are polynomials in $g(\mathbf{z})$, not all zero, and $(t_1, \ldots, t_{\kappa-1})$ lies off the divisor of poles of the coefficients of the $A_{ij\kappa}(x, y)$. For the moment let the integer $q \leqq (\kappa - 1)/K$ be arbitrary and define

$$F(\mathbf{z}) = \prod_{j=1}^{q} (g(\mathbf{z}) - t_j),$$

$$G(\mathbf{z}) = H_\kappa(\mathbf{z}, t_k)F(\mathbf{z}).$$

Then by the First Main Theorem of Nevanlinna theory (Theorem 2.7 of [**3**]), we know that for any $t \in \mathbf{P}$,

$$\begin{aligned} T(h, r) &= N_h(t, r) + m_h(t, r) + O(1) \\ &= T(1/h, r) + O(1). \end{aligned}$$

Thus

$$\begin{aligned} \text{(17)} \quad qT(g, r) &= T(F, r) + O(1) \\ &= T(G/H_\kappa, r) + O(1) \\ &= T(G, r) + T(H_\kappa, r) + O(1). \end{aligned}$$

We remark that the vanishing condition on $P_k(\mathbf{z}, t_k)$ shows that the numerators of G and H_κ vanish whenever $g(\mathbf{z}) = t_j$, $j = 1, \ldots, \kappa$. Thus the multiplicities of points actually on the divisor of poles of G or H_κ which come from the zeros of $\prod (g(\mathbf{z}) - t_j)$ are reduced by at least one from their multiplicities on the latter. Hence for the counting functions of the divisors of poles, we have the inequality

$$\text{(18)} \quad N_G(\infty, r) + N_{H_\kappa}(\infty, r) \leqq 2N(R_g, r) + 2 \sum_{j=1}^{K} N_{h_j}(\infty, r).$$

To estimate the proximity functions, we note that since

$$\deg_x A_{ij\kappa}(x, t_\kappa) \leqq \kappa - 1 - \left[\frac{\kappa - 1}{K}\right] \leqq \kappa - q,$$

the quotients

$$\frac{A_{ij\kappa}(x, t_\kappa)}{\prod_{j=q+1}^{\kappa} (x - t_j)} \quad \text{and} \quad \frac{A_{ij\kappa}(x, t_\kappa)}{\prod_{j=1}^{\kappa} (x - t_j)}$$

are uniformly bounded for $|x - t_j| \geqq 1, j = 1, \ldots, \kappa$. Thus

$$(19) \quad m_G(\infty, r) + m_{H_\kappa}(\infty, r)$$
$$\leqq 2 \sum_{j=1}^{\kappa} m_g(t_j, r) + 2 \sum_{i=1}^{K} m_{h_i}(\infty, r) + O(1).$$

From (17), (18), (19), it follows that

$$qT(g, r)$$
$$\leqq 2 \sum_{j=1}^{\kappa} m_g(t_j, r) + 2N(R_g, r) + 2 \sum_{j=1}^{K} m_{h_j}(\infty, r) + O(1).$$

Then by the hypotheses of Theorem 3,

$$(20) \quad qT(g, r) \leqq 2 \sum_{j=1}^{\kappa} m_g(t_j, r) + 2N(R_g, r) + 2CT(g, r) + S(g, r).$$

The Second Fundamental Theorem (e.g., Theorem 3.1 of [**3**] for the version without an extraneous $O(\log r)$) says that for any distinct $t_1, \ldots, t_s$, even if g were only meromorphic,

$$\sum_{j=1}^{s} N_g(t_j, r) \geqq (s - 2)T(g, r) + N(R_g, r) + S(g, r),$$

where R_g is the ramification divisor of g, which counts each point of $\mathbf{C}^n$ with one less multiplicity that the multiplicity of the value of g there. Thus in the standard way one has the Defect Relation

$$2T(g, r) \geqq \sum_{j=1}^{s} m_g(t_j, r) + N(R_g, r) + S(g, r).$$

When we apply this inequality to (20) with $s = \kappa$, we find that

$$qT(g, r) \leqq 4T(g, r) + 2CT(g, r) + S(g, r).$$

Thus

(21) $q \leqq 2C + 4.$

The only assumptions used in the derivation of this inequality were that $H_\kappa(\mathbf{z}, t_\kappa) \not\equiv 0$ and that $q \leqq (\kappa - 1)/K$. Thus as soon as

(22) $\kappa \geqq [2C]K + 5K + 1,$

we can choose

$$q = \left[\frac{\kappa - 1}{K}\right] \geqq [2C] + 5,$$

to contradict (21). Consequently for κ this large, we must have

$$H_\kappa(\mathbf{z}, w) = 0$$

as a function of $\mathbf{z}$. Since this holds for a generic choice of w, $H_\kappa(\mathbf{z}, w)$ vanishes identically as a function of $\mathbf{z}$ and w, as was to be shown to establish Theorem 3.

In fact the proof provides quantitative information.

SUPPLEMENT. *In Theorem* 3, *we may take*

$$\deg_x A_{ij}(x, y) \leqq [2C + 5](K - 1),$$
$$\deg_y A_{ij}(x, y) \leqq [2C + 5]K.$$

Proof. Choose $\kappa = [2C]K + 5 + 1$ in (22). Then the first inequality follows from (11) and the second from (12).

3. Proof of theorem 2. In order to deduce Theorem 2 from Theorem 3, we apply A. Vitter's generalization **[5]** of the Lemma of the Logarithmic Derivative to functions of n variables in a version without an extra term involving $\log r$ (e.g., Theorem 3.11 of **[3]**). This result shows that for any non-constant meromorphic function $h:\mathbf{C}^n \to \mathbf{P}$ and for any first order partial derivative δh, the proximity function satisfies

$$m_{\delta h/h}(\infty, r) = S(h, r).$$

It follows that

$$m_{\delta h}(\infty, r) \leqq m_h(\infty, r) + S(h, r).$$

Moreover it is clear that the counting function for the divisor of poles satisfies

$$N_{\delta h}(\infty, r) \leqq 2N_h(\infty, r).$$

Consequently

$$T(\delta h, r) \leqq 2T(h, r).$$

This together with the usual virtual subadditivity of m and N for sums and products shows that in the situation of Theorem 2,

$$T(\Delta_j g, r) \leqq c_j T(g, r) + S(g, r),$$

where c_j is the maximal weight of any monomial in $\Delta_j g$ occurring. Thus the hypotheses of Theorem 3 are satisfied with $h_j = \Delta_j g$ for an easily calculated $C > 0$, and Theorem 2 follows.

It would be very interesting to have quantitative versions of, say, Theorem 1. In particular, the first part of that theorem says that if (3) is impossible, then so is (2). A quantitative version would say that if the characteristic function for the left side of (3) is large for every non-trivial choice of a_k's, then the characteristic function of any related left-hand side of (2) must also be large. For example, is it true that for every transcendental g, whenever $\mathscr{D}$ and $\mathscr{D}_1$ are related as in (2) and (3), then we must have

$$T(\mathscr{D}_1 f, r) = o(T(\mathscr{D}(f \circ g), r)), \quad //$$

where // allows a possible exceptional set of r's of finite total length? Similarly is it true that if f is transcendental, then

$$\mathrm{Tr}(\mathscr{D}_2 g, r) = o(T(\mathscr{D}(f \circ g), r)) \quad //$$

whenever $\mathscr{D}$ and $\mathscr{D}_2$ are related as in (4) and (5)? At least is the ratio of the right- to left-hand sides unbounded for transcendental f, g?

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