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Centralizers and Twisted Centralizers: Application to Intertwining Operators

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Abstract. The equality of the centralizer and twisted centralizer is proved based on a case-by-case analysis when the unipotent radical of a maximal parabolic subgroup is abelian. Then this result is used to determine the poles of intertwining operators.

1 Introduction

The purpose of this paper is to prove the equality of the centralizer and twisted centralizer (defined in Section 2.1, originally defined by Shahidi [8]), when the unipotent radical of a maximal parabolic subgroup is abelian. In that case it is known that the adjoint action of the Levi subgroup on the Lie algebra of the unipotent radical has a finite number of orbits, the union of which is an open dense subset [4, 11]. Then it allows the treatment in [8] of determining the poles of intertwining operators.

To be more precise, let F be a non-archimedean local field of characteristic zero and $\bar{F}$ its algebraic closure. Suppose G is a split connected reductive algebraic group over F, T a maximal split torus of G. Let Δ be a set of simple roots, $\theta = \Delta \setminus \{\alpha\}$, where α is a simple root. Let $P = MN = M_\theta N$ be a maximal parabolic subgroup of G. Denote by $\{n_i\}$ a set of representatives for the corresponding open orbits of M in N under the adjoint action of M on $\mathfrak{N} = \mathrm{Lie}(N)$. Let N^- be the opposite of N and suppose one can write $w_0^{-1} n_i = m_i n_i' n_i^-$ where $m_i \in M$, $n_i' \in N$, $n_i^- \in N^-$ and w_0 is a representative for $\widetilde{w_0}$, the longest element in the Weyl group of A_0 (the maximal split torus of T in G) modulo that of A_0 in M.

Define

$$M_{n_i} = \{m \in M \mid \mathrm{Int}(m) \circ n_i = n_i\},$$
$$M^t_{m_i} = \{m \in M \mid w_0(m) m_i m^{-1} = m_i\}.$$

Observe that $M_{n_i} \subset M^t_{m_i}$ (*cf.* [8]).

It is clear that each n_i determines m_i uniquely (as well as n_i' and n_i^-). But the converse with respect to m_i is not true: several n_i could have the same m_i. The primary result of this paper proves this converse if N is abelian. This is the case where the number of open orbits $\{n_i\}$ is finite [11]. The main result of Section 3 is:

Theorem 1.1 *If N is abelian, then $M_{n_i} = M^t_{m_i}$.*

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Our proof of the main theorem is based on a case-by-case analysis; all the cases where N can be abelian have been listed and proved. For the exceptional groups G_2, F_4 and E_8, there is no maximal parabolic subgroup P such that its unipotent subgroup N is abelian. So these groups are not listed nor considered.

The method we adopt to prove this theorem is an extension of Gaussian elimination. Namely, for each orbit, we find a representative for it under $\mathrm{Ad}(M)$, which is a single element from a one dimensional subgroup corresponding to a positive root in N or a product of two elements from two unipotent subgroups, attached to the longest and shortest roots in N, respectively. Explicitly computing the Bruhat decomposition and using the uniqueness of this decomposition, we can show that $M_{n_i} = M^t_{m_i}$.

This result is crucial in determining the poles of intertwining operators in [8]. To be more precise, let $X(\mathbf{M})_F$ be the group of F-rational characters of $\mathbf{M}$. Denote by $\mathbf{A}$ the split component of the center of $\mathbf{M}$. Then $\mathbf{A} \subset \mathbf{A}_0$. Let

$$\mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R}) = \mathrm{Hom}(X(\mathbf{A})_F, \mathbb{R})$$

be the real Lie algebra of $\mathbf{A}$. Set $\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ to denote its real and complex dual.

For $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and σ an irreducible admissible representation of M, let

$$I(\nu, \sigma) = \mathrm{Ind}_{MN\uparrow G}\ \sigma \otimes q^{\langle \nu, H_P(\cdot)\rangle} \otimes 1,$$

where H_P is the extension of the homomorphism $H_M \colon M \to \mathfrak{a} = \mathrm{Hom}(X(\mathbf{M})_F, \mathbb{R})$ to P, extended trivially along N, defined by $q^{\langle \chi, H_P(m)\rangle} = |\chi(m)|_F$ for all $\chi \in X(\mathbf{M})_F$. Let $V(\nu, \sigma)$ be the space of $I(\nu, \sigma)$, for $h \in V(\nu, \sigma)$, let

$$A(\nu, \sigma, w)h(g) = \int_{N_{\tilde{w}}} h(w^{-1}ng)\, dn,$$

where $N_{\tilde{w}} = U \cap wN^-w^{-1}$, be the standard intertwining operator from $I(\nu, \sigma)$ into $I(w(\nu), w(\sigma))$.

Determining the reducibility of $I(\nu, \sigma)$ at $\nu = 0$ is equivalent to determining the pole of $\int_N h(w_0^{-1}n)\, dn$ at $\nu = 0$ for any h in $V(\nu, \sigma)$ which is supported in PN^-, *cf.* [6–8]. For the purpose of computing the residue we may assume that there exists a Schwartz function ϕ on $\mathfrak{N}^-$, the Lie algebra of N^-, such that $h(\exp(\mathfrak{n}^-) = \phi(\mathfrak{n}^-)h(e)$, where $\mathfrak{n}^- \in \mathfrak{N}^-$. Let $n_i^- = exp(\mathfrak{n}_i^-)$ with $\mathfrak{n}_i^- \in \mathfrak{N}^-$. Given a representation σ, let $\psi(m)$ be among the matrix coefficients of σ, *i.e,* choose an arbitrary element $\tilde{v}$ in the contragredient space of σ, let $\psi(m) = \langle \sigma(m)h(e), \tilde{v}\rangle$.

With these notations and by Theorem 2.2, $M^t_{m_i}/M_{n_i} = 1$, (not merely finite as suggested in [8]). Proposition 2.4 [8] can be refined as:

Proposition 1.2 *Let σ be an irreducible admissible representation of M. Then the poles of $A(\nu, \sigma, w_0)$ are the same as those of*

$$\sum_{\mathfrak{n}_i \in O_i} \int_{M/M_{n_i}} q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \phi(\mathrm{Ad}(m^{-1})\mathfrak{n}_i^-)\psi(w_0(m)m_i m^{-1})\, d\dot{m},$$

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where O_i runs through a finite number of open orbits of $\mathfrak{N}$ under $\mathrm{Ad}(M)$, $\mathfrak{n}_i$ is a representative of O_i under the correspondence that $w_0^{-1}n_i = m_i n_i' n_i^-$ with $n_i = \exp(\mathfrak{n}_i)$, $n_i^- = \exp(\mathfrak{n}_i^-)$. Furthermore $d\dot{m}$ is the measure on M/M_{n_i} induced from d^*n_i.

Let $\tilde{A}$ be the center of M. Then there exists a function $f \in C_c^\infty(M)$ such that $\psi(m) = \int_{\tilde{A}} f(am)\omega^{-1}(a)\,da$, where ω is the central character of σ.

Define

$$\theta\colon M \to M,\ \theta(m) = w_0^{-1} m w_0,\ \forall m \in M.$$

Given $f \in C_c^\infty(M)$ and $m_0 \in M$, define the θ-twisted orbit integral for f at m_0 by:

$$\phi_\theta(m_0, f) = \int_{M/M_{\theta,m_0}} f(\theta(m)m_0m^{-1})\,d\dot{m},$$

where

$$M_{\theta,m_0} = M_{\theta,m_0}(F) = \{m \in M(F) \mid \theta(m)m_0m^{-1} = m_0\}$$

is the θ-twisted centralizer of m_0 in $M(F)$, $d\dot{m}$ is the measure on $M/M_{\theta,m_0}$ induced from dm.

Applying Theorem 2.2, we can restate Theorem 2.5 of [8] as:

Proposition 1.3 *Assume σ is supercuspidal and $w_0(\sigma) \cong \sigma$. The intertwining operator $A(\nu, \sigma, w_0)$ has a pole at $\nu = 0$ if and only if*

$$\sum_i \int_{Z(G)/Z(G)\cap w_0(\tilde{A})\tilde{A}^{-1}} \phi_\theta(zm_i, f)\omega^{-1}(z)\,dz \neq 0,$$

for f as above. Here $Z(G)$ is the center of G and

$$\phi_\theta(zm_i, f) = \int_{M/M_{n_i}} f(z\theta(m)m_im^{-1})\,d\dot{m},$$

is the θ-twisted orbital integral for f at zm_i, where m_i corresponds to the representatives $\{n_i\}$ for the open orbits in N under $\mathrm{Int}(M)$ with $w_0^{-1}n_i = m_in_i'n_i^-$ as n_i runs through the finite number of open orbits in N.

2 Preliminaries

Let F be a non-Archimedean local field of characteristic zero. Denote by $\mathcal{O}$ its ring of integers and let $\mathcal{P}$ be the unique maximal ideal of $\mathcal{O}$. Let q be the number of elements in $\mathcal{O}/\mathcal{P}$ and fix a uniformizing element ϖ for which $|\varpi| = q^{-1}$, where $|\cdot| = |\cdot|_F$ denotes an absolute value for F normalized in this way. Let $\bar{F}$ be the algebraic closure of F.

Let $\mathbf{G}$ be a split connected reductive algebraic group over F. Fix an F-Borel subgroup $\mathbf{B}$ and write $\mathbf{B} = \mathbf{TU}$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ is a maximal torus there. Let $\mathbf{A}_0$ be the maximal split torus of $\mathbf{T}$ and let Δ be the set of simple roots of $\mathbf{A}_0$ in the Lie algebra of $\mathbf{U}$.

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Denote by $\mathbf{P} = \mathbf{MN}$ a maximal parabolic subgroup of $\mathbf{G}$ in the sense that $\mathbf{N} \subset \mathbf{U}$. Assume $\mathbf{T} \subset \mathbf{M}$ and let $\theta = \Delta \setminus \{\alpha\}$ such that $\mathbf{M} = \mathbf{M}_\theta$. As usual, we use $W = W(\mathbf{A}_0)$ to denote the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. Given $\tilde{w} \in W$, we use w to denote a representative for $\tilde{w}$. Particularly, let $\tilde{w}_0$ be the longest element in W modulo the Weyl group of $\mathbf{A}_0$ in $\mathbf{M}$.

We use G, P, M, N, B, T, U, A_0 to denote the subgroups of F-rational points of the groups $\mathbf{G}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \mathbf{B}, \mathbf{T}, \mathbf{U}, \mathbf{A_0}$, respectively. We also use $\tilde{G}, \tilde{P}, \tilde{M}, \tilde{N}, \tilde{B}, \tilde{T}, \tilde{U}, \tilde{A_0}$ to denote the $\bar{F}$ points of $\mathbf{G}, \mathbf{P}, \mathbf{M}, \mathbf{N}, \mathbf{B}, \mathbf{T}, \mathbf{U}, \mathbf{A_0}$, respectively.

For any $g \in \mathbf{G}$, we will use $\mathrm{Int}(g)$ to denote the inner morphism of $\mathbf{G}$ induced by g, *i.e.*, for any $u \in \mathbf{G}$, $\mathrm{Int}(g) \circ u = gug^{-1}$. Let $\mathfrak{g} = \mathrm{Lie}(G)$, the Lie algebra of G. We will use $\mathrm{Ad}(g)$ to denote the adjoint action on $\mathfrak{g}$ induced from $\mathrm{Int}(g)$.

Suppose R is the root system of G. For each root $\beta \in R$ we choose a root vector $\mathfrak{g}_\beta$ in $\mathfrak{g}$. For $\beta \in R$, let U_β be the one dimensional root subgroup of β and for $x \in F$, let $U_\beta(x) = \exp(x\mathfrak{g}_\beta)$.

Let $\mathfrak{N} = \mathrm{Lie}(N)$, the Lie algebra of N. Then $\mathfrak{N} = \bigoplus \mathfrak{N}_i$, where $\mathfrak{N}_i$ is graded according to α. M acts on $\mathfrak{N}$ by adjoint action. In particular, each $\mathfrak{N}_i$ is invariant under $\mathrm{Ad}(M)$.

For each root $\beta \in R$, there is a one dimensional subtorus $H_\beta(F)$, dual to β, such that the subgroup generated by H_β, U_β and $U_{-\beta}$ is a simply connected group of rank one which is split over F. So it is isomorphic to $SL_2(F)$. Let Φ_β be the isomorphism from $SL_2(F)$ to the subgroup generated by H_β, U_β and $U_{-\beta}$. Then for any $\gamma \in R$ and $t \in F^*$,

$$\gamma\left(\Phi_\beta\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\right) = t^{\langle\gamma,\beta\rangle}.$$

Lemma 2.1 ([10, Proposition 8.2.3]) *Let $\beta, \gamma \in R$, with $\beta \neq \gamma$. Then there exist constants $C_{\beta,\gamma;i,j} \in \bar{F}$, such that*

$$(U_\beta(x), U_\gamma(y)) = \prod_{\substack{i\beta+j\gamma\in R\\ i,j>0}} U_{i\beta+j\gamma}(C_{\beta,\gamma;i,j}x^i y^j),$$

where the order of the factors in the right side are prescribed by a fixed ordering of R. Actually, the constants $C_{\beta,\gamma;i,j}$ can be normalized so that $C_{\beta,\gamma;i,j} \in \mathbb{Z}$. Moreover, if γ is the longer element in the two dimensional root space spanned by β and γ. Then $C_{\beta,\gamma;i,j}$ can be normalized such that $C_{\beta,\gamma;1,1} = 1$ if $\beta + \gamma \in R$. (Then $C_{\gamma,\beta,1,1} = -1$).

2.1 Centralizer and Twisted Centralizer

Let $n_1 \in N$, suppose $w_0^{-1}n_1 \in PN^-$, and write $w_0^{-1}n_1 = p_1 n_1^- = m_1 n_1' n_1^-$ with $m_1 \in M$, $n_1' \in N$ and $n_1^- \in N^-$. Let $\mathrm{Cent}_M(n_1) = M_{n_1}$ be the centralizer of n_1 in M, *i.e.*,

$$M_{n_1} = \{m \in M \mid \mathrm{Int}(m) \circ n_1 = n_1\},$$

and let $M_{n_1'} = \mathrm{Cent}_M(n_1')$ and $M_{n_1^-} = \mathrm{Cent}_M(n_1^-)$, respectively. Let $M^t_{m_1} = \mathrm{Cent}^t_{m_1} = \{m \in M \mid w_0(m)m_1m^{-1} = m_1\}$ be the twisted (by means of w_0) centralizer of m_1 in M. Then by the uniqueness of PN^- decomposition of $w_0^{-1}n_1$, it is not hard to see

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that the groups M_{n_1}, $M_{n_1^-}$ and M'_{n_1} are all equal and are all contained in $M^t_{m_1}$, *cf.* [8]. Let $n_i = \exp(\mathfrak{n}_i), \mathfrak{n}_i \in \mathfrak{N}$, and assume the set $\{\mathfrak{n}_i\}$ generates a dense subset of $\mathfrak{N}$ under the action of M.

The main result in this paper is the following:

Theorem 2.2 *Let $n_1 = \exp(\mathfrak{n}_1)$, where $\mathfrak{n}_1 \in \{\mathfrak{n}_i\}$ is one of the generators of a dense subset of $\mathfrak{N}$ under the action of M. Then $M_{n_1} = M^t_{m_1}$.*

From the above notations, we have:

$$w_0^{-1} n_1 = m_1 n_1' n_1^- . \tag{2.1}$$

If $m \in M^t_{m_1}$, then

$$\begin{aligned} w_0^{-1} m n_1 m^{-1} &= (w_0(m) m_1 m^{-1})(m n_1' m^{-1})(m n_1^- m^{-1}) \\ &= m_1 (m n_1' m^{-1})(m n_1^- m^{-1}). \end{aligned} \tag{2.2}$$

For convenience of notation, Let

$$n_2 = \operatorname{Int}(m) \circ n_1, \quad n_2' = \operatorname{Int}(m) \circ n_1', \quad n_2^- = \operatorname{Int}(m) \circ n_1^- .$$

Then equation (2.2) will be changed to:

$$w_0^{-1} n_2 = m_1 n_2' n_2^- . \tag{2.3}$$

Multiplying the inverse of equation (2.3) by equation (2.1), we have:

$$n_2^{-1} n_1 = (n_2^-)^{-1} (n_2')^{-1} n_1' n_1^- . \tag{2.4}$$

Let

$$s_1 = n_2^{-1} n_1 \in N, \quad s_1^- = (n_1^-)^{-1} \in N^-;$$
$$s_2^- = (n_2^-)^{-1} \in N^-, \quad s_2 = (n_2')^{-1} n_1' \in N.$$

Then equation (2.4) becomes

$$s_1 s_1^- = s_2^- s_2. \tag{2.5}$$

Let

$$\begin{aligned} n_1 &= \exp(\mathfrak{n}_1), & n_2 &= \exp(\mathfrak{n}_2); \\ s_1 &= \exp(r_1), & s_2 &= \exp(r_2); \\ s_1^- &= \exp(r_1^-), & s_2^- &= \exp(r_2^-). \end{aligned}$$

Then $\mathfrak{n}_2 = \operatorname{Ad}(m) \circ \mathfrak{n}_1$ is one of the generators of a dense orbit of $\mathfrak{N}$ under $\operatorname{Ad}(M)$ since $\mathfrak{n}_1$ is. Similarly it is not hard to see that both r_1^- and r_2^- are generators of a dense orbit of $\mathfrak{N}^-$.

Our goal is to prove:

Claim *Under the assumption in Theorem 2.2, we must have: $s_1^- = s_2^-$.*

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Once this has been proved, it implies $n_1^- = n_2^-$, which will lead to $n_1 = n_2$ by the uniqueness of PN^- decomposition. Since $m \in M^t_{m_1}$ and $n_2 = \mathrm{Int}(m) \circ n_1$, we get $m \in M_{n_1}$ if $m \in M^t_{m_1}$. So $M^t_{m_1} \subset M_{n_1}$. But we already have $M_{n_1} \subset M^t_{m_1}$, *cf.* [8]. So $M_{n_1} = M^t_{m_1}$ as desired.

Remark We can always assume that $s_2^- \neq 1$, since otherwise there is nothing that needs to be done. We are going to prove the claim according to the type of Dynkin diagram of G since the Gaussian elimination essentially depends on the structure of the root system.

Strategy of Proof Except for some simple cases (like A_l, C_l), our proof relies on Gaussian elimination for $\mathfrak{N}$. Namely, $\mathfrak{N}$ can be generated by $\mathfrak{g}_\beta$ with β a positive root in N, or by $\mathfrak{g}_\beta, \mathfrak{g}_\gamma$ under $\mathrm{Ad}(M)$, where $\mathfrak{g}_\beta$, $\mathfrak{g}_\gamma$ are root vectors attached to the shortest and longest roots in N. Thus by acting with a suitable $m \in M$ on both sides of equation (2.5), we can always assume that $s_2 = U_\beta(a_1)U_\gamma(a_2)$ or $U_\beta(a_1)$.

We will multiply both sides of equation (2.5) by $U_\beta(x)U_\gamma(y)$ from the right, where x, y are variables. Then the M-parts of $s_1 s_1^- U_\beta(x)U_\gamma(y)$ and $s_2^- s_2 U_\beta(x)U_\gamma(y)$ can be calculated and compared explicitly since they are in the simplest form. We can then conclude that their M-parts will never be equal unless $s_1^- = s_2^-$.

3 Proof of the Main Theorem

Now suppose N is abelian, then $\mathrm{Ad}(M)$ acts on $\mathfrak{N}$ having finite number of orbits, *cf.* [4, 11].

3.1 Roots in Unipotent Radical

Lemma 3.1 *Suppose N is abelian. If*

$$\beta = c\alpha + \sum_{\alpha_i \neq \alpha} c_i \alpha_i$$

is a positive root of N where α_i's are simple roots from θ, then $c = 1$.

Proof Using [3, Corollary of Lemma A §10.2], β can be written in the form $\beta_1 + \beta_2 + \cdots + \beta_k$ with $\beta_i \in \Delta$ (β_i not necessary distinct) such that each partial sum $\beta_1 + \beta_2 + \cdots + \beta_j$ is a root ($1 \leq j \leq k$). Suppose $c \geq 2$, then there is j such that $\beta_j = \alpha$ and in the remaining partial sum $\beta_1 + \beta_2 + \cdots + \beta_{j-1}$, there is still one α. Let $\gamma = \beta_1 + \beta_2 + \cdots + \beta_{j-1}$, then $\mathfrak{g}_\gamma, \mathfrak{g}_{\beta_j} \in \mathfrak{N}$, and $[\mathfrak{g}_\gamma, \mathfrak{g}_{\beta_j}] = \mathfrak{g}_{\beta_1+\beta_2+\cdots+\beta_j} \neq 0$. This is a contradiction to $\mathfrak{N}$ being abelian. ∎

If

$$P = \sum_{\substack{\alpha_i \in \Delta \\ i=1}}^{k} c_i \alpha_i$$

is a root, choose k points in a plane representing each α_i and draw a line connecting α_i, α_j, if $\langle \alpha_i, \hat{\alpha}_j \rangle \neq 0$. Then the graph obtained is obviously a subgraph of the Dynkin diagram and is composed of several connected pieces. For each connected piece C_i of this graph, we set

$$P_i = \sum_{\alpha_i \in C_i} c_i \alpha_i.$$

Then

$$P = \sum_{i}^{m} P_i,$$

where m is the number of connected pieces. All the C_i's are disjoint. We call P_i a connected piece of P. Call P_i positive if each c_i is positive, and negative if each c_i is negative. In particular, we call P a connected root if P is composed of only one connected piece.

Lemma 3.2 *Every positive root is connected.*

Proof Let

$$r = \sum_{i=1}^{k} P_i$$

be a positive root with all P_i's being positive connected and disjoint with each other. Then by [3, Corollary of Lemma A §10.2], r can be written as

$$r = \sum_{i=1}^{n} \alpha_i,$$

such that every partial sum

$$r_s = \sum_{i=1}^{s} \alpha_i, \quad 1 \leq s \leq n,$$

is a root. If $k > 1$, then there must be one s, $s > 1$, and one i, $1 \leq i \leq k$, such that in the sum for r_s, there is only one element, say $\alpha_j, 1 \leq j \leq s$, which comes from P_i. Then for all $\alpha_i, 1 \leq i \leq s, i \neq j, \langle \alpha_i, \hat{\alpha}_j \rangle = 0$ since α_i, α_j are not in the same connected piece. So

$$S_{\alpha_j}(r_s) = r_s - \langle r_s, \widehat{\alpha_j} \rangle \alpha_j = \sum_{i=1}^{s} \alpha_i - 2\alpha_j = \sum_{\substack{i=1 \\ i \neq j}}^{s} \alpha_i - \alpha_j,$$

where S_{α_j} is the reflection about α_j in the Weyl group of G. Since none of the α_i's in the sum

$$\sum_{i=1, i \neq j}^{s} \alpha_i$$

can be α_j, and all α_i are simple roots, $S_{\alpha_j}(r_s)$ is not a root. This is a contradiction to $S_{\alpha_j}(r_s)$ being a root since r_s is a root. ∎

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3.2 Type A_l

Equation (2.5) implies $\exp(r_1)\exp(r_1^-) = \exp(r_2^-)\exp(r_2)$.

Since $r_1^2 = r_2^2 = (r_1^-)^2 = (r_2^-)^2 = 0$, we have:

$$r_1 + r_1^- + r_1 r_1^- = r_2^- + r_2 + r_2^- r_2. \tag{3.1}$$

Choose $t \in T$ and let $\mathrm{Ad}(t)$ act on both sides of equation (3.1). We get

$$\alpha(t)r_1 + \alpha^{-1}(t)r_1^- + r_1 r_1^- = \alpha^{-1}(t)r_2^- + \alpha(t)r_2 + r_2^- r_2.$$

Since this is true for all $t \in T$, we must have $r_1 = r_2, r_1^- = r_2^-$. Consequently, $s_1^- = s_2^-$.

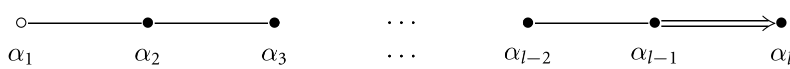
3.3 Type B_l

In this case, we may assume that T can be chosen to be the set of matrices of the form:

$$\mathrm{diag}(x_1, x_2, \dots, x_l, x_1^{-1}, x_2^{-1}, \dots, x_l^{-1}, 1),$$

since the unipotent subgroups remain unchanged in every adjoint action.

The Dynkin diagram of G is:



Let $e_i \in \mathrm{Hom}(T, F^*)$, $1 \le i \le l$ such that $e_i(T) = x_i$. Then $\alpha_i = e_i - e_{i+1}$, $1 \le i \le l-1$; $\alpha_l = e_l$. The only case when N can be abelian is $\alpha = \alpha_1$. Then the positive roots in N are: $\{e_1 \pm e_i \mid 2 \le i \le l\} \cup \{e_1\}$.

We choose a root vector for each positive root in G as follows:

$$\begin{aligned} \mathfrak{g}_{e_i - e_j} &= E_{i,j} - E_{l+j,l+i}, && 1 \le i < j \le l, \\ \mathfrak{g}_{e_i + e_j} &= E_{i,l+j} - E_{j,l+i}, && 1 \le i < j \le l, \\ \mathfrak{g}_{e_i} &= E_{i,2l+1} - E_{2l+1,l+i}, && 1 \le i \le l. \end{aligned}$$

We also choose a root vector for each negative root in G as follows:

$$\begin{aligned} \mathfrak{g}_{-e_i + e_j} &= E_{j,i} - E_{l+i,l+j}, && 1 \le i < j \le l, \\ \mathfrak{g}_{-e_i - e_j} &= E_{l+j,i} - E_{l+i,j}, && 1 \le i < j \le l, \\ \mathfrak{g}_{-e_i} &= E_{l+i,2l+1} - E_{2l+1,i}, && 1 \le i \le l, \end{aligned}$$

where the $E_{i,j}$'s are elementary matrices in $M_{(2l+1)\times(2l+1)}$ such that its (i, j) entry is 1, all other entries are 0.

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Lemma 3.3 *Given any nonzero element*

$$r = \sum_{i=2}^{l} a_i \mathfrak{g}_{e_1-e_i} + \sum_{i=2}^{l} b_i \mathfrak{g}_{e_1+e_i} + c\mathfrak{g}_{e_1} \in \mathfrak{N},$$

there is an $m \in M$, *such that* $\mathrm{Ad}(m) \circ r = c_0 \mathfrak{g}_{e_1-e_2} + c_1 \mathfrak{g}_{e_1+e_2}$ *with* $c_0 \neq 0$.

Proof This is [9, Lemma 4.2]. ■

Lemma 3.4 *For an element* $r = c_0\mathfrak{g}_{e_1-e_2} + c_1\mathfrak{g}_{e_1+e_2} \in \mathfrak{N}$ *from Lemma* 3.3 *with* $c_1 \neq 0$, *there is* $m \in \tilde{M}$ *such that* $\mathrm{Ad}(m) \circ r = a\mathfrak{g}_{e_1}$ *with* $a \neq 0$.

Proof Choose $x \in \bar{F}$ such that $\frac{1}{2}c_0x^2 = c_1$. Let $m = U_{-e_2}\left(\frac{1}{x}\right)U_{e_2}(x)$. Then $\mathrm{Ad}(m) \circ r = -c_0x\mathfrak{g}_{e_1}$. Setting $a = -c_0x$ finishes the proof. ■

We start with equation (2.5). If $s_2 = 1$, then it immediately follows $s_1^- = s_2^-$, and there is nothing to do. So suppose $s_2 \neq 1$. By the above two lemmas, applying a suitable $\mathrm{Int}(m)$, $m \in \tilde{M}$ on both sides if necessary, we can assume $s_2 = U_{e_1}(a)$ or $U_{e_1-e_2}(a)$ with $a \neq 0$. By taking a suitable finite extension of F, we can always assume that $m \in M$ and consequently $a \in F$. Without loss of generality, we assume $s_2 = U_{e_1}(a)$.

Suppose

$$s_1^- = \prod_{k=2}^{l} U_{-e_1-e_k}(a_k) \prod_{k=2}^{l} U_{-e_1+e_k}(b_k) U_{-e_1}(x_0),$$

$$s_2^- = \prod_{k=2}^{l} U_{-e_1-e_k}(c_k) \prod_{k=2}^{l} U_{-e_1+e_k}(d_k) U_{-e_1}(y_0).$$

Multiply both sides of (2.5) by $u = U_{e_1}(x) \in N$ on the right, where $x \in F$. Decompose both $s_1s_1^-u$ and $s_2^-s_2u$ into PN^- form, and compare their M part. Their M part will never be equal unless $s_1^- = s_2^-$. The reason for multiplying u is to exclude the possibility of occurrence of some Weyl group elements (when $ay_0 = -1$).

First we have

$$U_{-e_1}(y_0)U_{e_1}(a+x) = U_{e_1}\left(\frac{a+x}{1+y_0(a+x)}\right) h_{2,x} U_{-e_1}\left(\frac{y_0}{1+y_0(a+x)}\right),$$

where

$$h_{2,x} = \Phi_{e_1}\begin{pmatrix} \frac{1}{1+y_0(a+x)} & 0 \\ 0 & 1+y_0(a+x) \end{pmatrix} \in T.$$

Set

$$a_x = \frac{a+x}{1+y_0(a+x)}.$$

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For any $k, 2 \le k \le l$, by Lemma 2.1,

$$U_{-e_1+e_k}(d_k)U_{e_1}(a_x) = U_{e_1}(a_x)U_{e_k}(d_k a_x)U_{-e_1+e_k}(d_k),$$
$$U_{-e_1-e_k}(c_k)U_{e_1}(a_x) = U_{e_1}(a_x)U_{-e_k}(c_k a_x)U_{-e_1-e_k}(c_k).$$

Then by recursively applying Lemma 2.1 and using the fact that N and N^- are normal in P and P^- respectively, it can be calculated that the M part of $s_2^- s_2 u$ is:

$$m_2 = \prod_{k=2}^{l} U_{-e_k}(c_k a_x) \prod_{k=2}^{l} U_{e_k}(d_k a_x) h_{2,x}.$$

Similarly, if we set

$$b_x = \frac{x}{1 + x_0 x} \quad \text{and} \quad h_{1,x} = \Phi_{e_1}\begin{pmatrix} \frac{1}{1+x_0x} & 0 \\ 0 & 1 + x_0 x \end{pmatrix} \in T,$$

then the M part of $s_1 s_1^- u$ is:

$$m_1 = \prod_{k=2}^{l} U_{-e_k}(a_k b_x) \prod_{k=2}^{l} U_{e_k}(b_k b_x) h_{1,x}.$$

From equation (2.5), $s_1 s_1^- u = s_2^- s_2 u$. By the uniqueness of MNN^- decomposition, we must have $m_1 = m_2$. Since m_1 and m_2 are products of unipotent groups attached to roots in M in the same order, we must have $c_k a_x = a_k b_x$ and $d_k a_x = b_k b_x$ for almost all $x \in F$ and all $k, 2 \le k \le l$. These equations lead to:

$$(3.2) \qquad (c_k x_0 - a_k y_0)x^2 + (c_k x_0 + a c_k x_0 - a_k - a a_k y_0)x + a c_k = 0,$$

$$(3.3) \qquad (d_k x_0 - b_k y_0)x^2 + (d_k x_0 + a d_k x_0 - b_k - a b_k y_0)x + a d_k = 0.$$

For equations (3.2) and (3.3) to have infinitely many solutions, one must have $a_k = b_k = c_k = d_k \equiv 0, \forall k, 2 \le k \le l$, since $a \ne 0$ by assumption. Moreover, we have $h_{1,x} = h_{2,x}$ for almost all x, which means the equation

$$(y_0 - x_0)x + a y_0 = 0$$

has infinitely many solutions, thus $y_0 = 0$, so $s_2^- = 1$, which is a contradiction. So in order that equation (2.5) holds, we must have $s_2 = 1$, which leads to $s_1^- = s_2^-$. When $s_2 = U_{e_1-e_2}(a)$, we can also prove that $s_1^- = s_2^-$ in a similar way. That finishes the proof of the main theorem in case G is of type B_l.

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3.4 Type C_l

In this case, we may assume T is the set of matrices of the form:

$$\operatorname{diag}(x_1, x_2, \ldots, x_l, x_1^{-1}, x_2^{-1}, \ldots, x_l^{-1}),$$

since the unipotent subgroups remain unchanged in every adjoint action.

Let $e_i \in \operatorname{Hom}(T, F^*)$ such that $e_i(H) = x_i$. Then $R = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_k\}$. N is abelian only in case $\alpha = 2e_l$. In this case, $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq l-1\} \cup \{2e_l\}$. The positive roots in N are: $R^+ \setminus \theta^+ = \{e_i + e_j \mid i \neq j\} \cup \{2e_i \mid 1 \leq i \leq l\}$. And $\mathfrak{N}$ is all the $2l \times 2l$ matrices of the form:

$$\begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}$$

where $Y \in M_l(F)$ and $Y^t = Y$. So for each $\mathfrak{n} \in \mathfrak{N}, \mathfrak{n}^2 = 0$, and for each $\mathfrak{n}^- \in \mathfrak{N}^-, \mathfrak{n}^{-2} = 0$. It can be seen that the proof for the A_l case also applies in this case which implies $s_1^- = s_2^-$.

3.5 Type D_l

In this case, again T may be considered to be the set of matrices of the form:

$$\operatorname{diag}(x_1, x_2, \ldots, x_l, x_1^{-1}, x_2^{-1}, \ldots, x_l^{-1}),$$

because the unipotent subgroups remain unchanged in every adjoint action.

Let $e_i \in \operatorname{Hom}(T, F^*)$ such that $e_i(H) = x_i$. Then $R = \{\pm e_i \pm e_j \mid i \neq j\}$, $\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq l-1\} \cup \{e_{l-1} + e_l\}$. Let $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq l-1$, and let $\alpha_l = e_{l-1} + e_l$. For N to be abelian, α must be α_1, α_{l-1} or α_l. If $\alpha = \alpha_{l-1}$, then every element $\mathfrak{n} \in \mathfrak{N}$ has the form:

$$\begin{pmatrix} A & Y \\ 0 & -A^t \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{l-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

and $B \in M_{l-1}(F), B = -B^t$. Then it is easily checked that $\mathfrak{n}^2 = 0$ and consequently for each $\mathfrak{n}^- \in \mathfrak{N}^-$, $\mathfrak{n}^{-2} = 0$. Again we can use the same method as in A_l or C_l to prove that $s_1^- = s_2^-$.

The symmetry between α_l and α_{l-1} takes care of the case $\alpha = \alpha_l$.

If $\alpha = \alpha_1$, then $\mathfrak{N}$ does not have the property that for each $\mathfrak{n} \in \mathfrak{N}$, $\mathfrak{n}^2 = 0$. In this case, the positive roots in N are: $\{e_1 - e_i \mid 1 < i \leq l\} \cup \{e_1 + e_j \mid 1 < j \leq l\}$.

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We choose a root vector for each positive root in G as follows:

$$\mathfrak{g}_{e_i-e_j} = E_{i,j} - E_{l+j,l+i}, \quad 1 \leq i < j \leq l,$$
$$\mathfrak{g}_{e_i+e_j} = E_{i,l+j} - E_{j,l+i}, \quad 1 \leq i < j \leq l.$$

We also choose a root vector for each negative root in G as follows:

$$\mathfrak{g}_{-e_i+e_j} = E_{j,i} - E_{l+i,l+j}, \quad 1 \leq i < j \leq l,$$
$$\mathfrak{g}_{-e_i-e_j} = E_{l+j,i} - E_{l+i,j}, \quad 1 \leq i < j \leq l,$$

where the $E_{j,k}$'s are elementary matrix in $M_{2l\times 2l}$. Then $\{\mathfrak{g}_{e_1\pm e_i} | 1 < i \leq l\}$ is a basis for $\mathfrak{N}$.

Theorem 3.5 (Gaussian Elimination) *For any nonzero $r \in \mathfrak{N}$, there exist $m \in M$ and $k_0, k_1 \in F$, with $k_0 \neq 0$, such that* $\operatorname{Ad}(m) \circ r = k_0\mathfrak{g}_{e_1-e_2} + k_1\mathfrak{g}_{e_1+e_2}$.

Proof Suppose

$$r = \sum_{i=1}^{l-1} a_i \mathfrak{g}_{e_1-e_{i+1}} + \sum_{i=1}^{l-1} a_i' \mathfrak{g}_{e_1+e_{i+1}}.$$

We first prove that by applying a suitable $m' \in M$ on r if necessary, we can always assume that $a_1 \neq 0$.

Assume $a_1 = 0$. Let

$$m' = \begin{cases} U_{-e_2+e_{i+1}}(1) & \exists\, i, 2 \leq i \leq l-1, \text{ such that } a_i \neq 0, \\ U_{-e_2-e_{i+1}}(1) & \exists\, i, 2 \leq i \leq l-1, \text{ such that } a_i' \neq 0, \\ s_{e_2} & \text{otherwise,} \end{cases}$$

where s_{e_2} is a representative of the Weyl group element S_{e_2}, which is the reflection about e_2.

By applying the formula $\operatorname{Ad}(\exp(x\mathfrak{g}_\beta)) = e^{\operatorname{ad}(x\mathfrak{g}_\beta)}$ for each root $\beta \in R$, it is easily checked that the coefficient of $\mathfrak{g}_{e_1-e_2}$ in $\operatorname{Ad}(m') \circ r$ is nonzero.

Let $k_0 = a_1$, and

$$m = \left[\prod_{i=3}^{l} \exp\Big(\frac{a_{i-1}'}{k_0}\mathfrak{g}_{e_2+e_i}\Big)\right] \cdot \left[\prod_{i=3}^{l} \exp\Big(\frac{a_{i-1}}{k_0}\mathfrak{g}_{e_2-e_i}\Big)\right].$$

Then a direct calculation shows that

$$\operatorname{Ad}(m) \circ r = k_0\mathfrak{g}_{e_1-e_2} + \Big(a_{l-1}' + \sum_{i=3}^{l} \frac{a_{i-1}\cdot a_{i-1}'}{k_0}\Big)\,\mathfrak{g}_{e_1+e_2}.$$

Let k_1 denote the coefficient of $\mathfrak{g}_{e_1+e_2}$ from the right-hand side of the above equation, then $\operatorname{Ad}(m) \circ r = k_0\mathfrak{g}_{e_1-e_2} + k_1\mathfrak{g}_{e_1+e_2}$ as desired. ■

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Considering equation (2.5), if $s_2 \neq 0$, by Theorem 3.5, applying an $m \in M$ on both sides, we can assume $s_2 = U_{e_1-e_2}(k_0) \cdot U_{e_1+e_2}(k_1)$ with $k_0 \neq 0$.

Suppose

$$s_1^- = \prod_{i=2}^{l} U_{-e_1-e_i}(a_i) \prod_{i=2}^{l} U_{-e_1+e_i}(b_i),$$

$$s_2^- = \prod_{i=2}^{l} U_{-e_1-e_i}(c_i) \prod_{i=2}^{l} U_{-e_1+e_i}(d_i).$$

We will adopt the strategy we have used in the case of B_l: multiply both sides of (2.5) by $u = U_{e_1-e_2}(x)U_{e_1+e_2}(y) \in N$ on the right, where x, y are variables in F. Decompose both $s_1 s_1^- u$ and $s_2^- s_2 u$ into PN^- form and compare their M parts.

Now let us consider the PN^- decomposition of $s_1 s_1^- u$ and $s_2^- s_2 u$. For $s_1^- U_{e_1-e_2}(x)$, first we have:

$$U_{-e_1+e_2}(b_2)U_{e_1-e_2}(x) = U_{e_1-e_2}(x')h_{0,x}U_{-e_1+e_2}\left(\frac{b_2}{1+b_2x}\right), \tag{3.4}$$

where

$$x' = \frac{x}{1+b_2x} \quad \text{and} \quad h_{0,x} = \Phi_{e_1-e_2}\begin{pmatrix} \frac{1}{1+b_2x} & 0 \\ 0 & 1+b_2x \end{pmatrix} \in T.$$

For each $i, 3 \leq i \leq l$, by applying Lemma 2.1, we get:

$$U_{-e_1+e_i}(b_i)U_{e_1-e_2}(x') = U_{e_1-e_2}(x')U_{-e_2+e_i}(b_ix')U_{-e_1+e_i}(b_i), \tag{3.5}$$

$$U_{-e_1-e_i}(a_i)U_{e_1-e_2}(x') = U_{e_1-e_2}(x')U_{-e_2-e_i}(a_ix')U_{-e_1-e_i}(a_i). \tag{3.6}$$

And $U_{e_1-e_2}$ commutes with $U_{-e_1-e_2}$.

From equations (3.5), (3.6) and using the fact that both N and N^- are normal in P and P^-, respectively, we reach the following:

$$s_1^- U_{e_1-e_2}(x) = U_{e_1-e_2}(x') \prod_{i=3}^{l} U_{-e_2-e_i}(a_ix') \prod_{i=3}^{l} U_{-e_2+e_i}(b_ix')h_{0,x}s_1^{-\prime}. \tag{3.7}$$

for a suitable $s_1^{-\prime} \in N^-$. When $s_2^- s_2 u = s_2^- U_{e_1-e_2}(k_0+x)U_{e_1+e_2}(k_1+y)$, a similar calculation shows that

$$s_2^- U_{e_1-e_2}(k_0+x) = U_{e_1-e_2}(k_{0,x}) \prod_{i=3}^{l} U_{-e_2-e_i}(c_ik_{0,x}) \prod_{i=3}^{l} U_{-e_2+e_i}(d_ik_{0,x})h'_{0,x}s_2^{-\prime} \tag{3.8}$$

for a suitable $s_2^{-\prime} \in N^-$, where

$$k_{0,x} = \frac{k_0+x}{1+d_2(k_0+x)} \quad \text{and} \quad h'_{0,x} = \Phi_{e_1-e_2}\begin{pmatrix} \frac{1}{1+d_2(k_0+x)} & 0 \\ 0 & 1+d_2(k_0+x) \end{pmatrix} \in T.$$

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Suppose

$$s_1^{-\prime} = \prod_{i=2}^{l} U_{-e_1-e_i}(a_i{}') \prod_{i=2}^{l} U_{-e_1+e_i}(b_i{}'),$$

$$s_2^{-\prime} = \prod_{i=2}^{l} U_{-e_1-e_i}(c_i{}') \prod_{i=2}^{l} U_{-e_1+e_i}(d_i{}').$$

Then with a similar calculation as above, by applying Lemma 2.1 recursively, we get:

$$(s_1^-)'U_{e_1+e_2}(y) = U_{e_1+e_2}(y') \prod_{i=3}^{l} U_{e_2-e_i}(-a_i'y') \prod_{i=3}^{l} U_{e_2+e_i}(-b_i'y')h_{1,y}s_1^{-\prime\prime} \tag{3.9}$$

with a suitable $s_1^{-\prime\prime} \in N^-$, where

$$y' = \frac{y}{1+a_2'y} \quad \text{and} \quad h_{1,y} = \Phi_{e_1+e_2}\begin{pmatrix} \frac{1}{1+a_2'y} & 0 \\ 0 & 1+a_2'y \end{pmatrix} \in T.$$

While

(3.10)

$$(s_2^-)'U_{e_1+e_2}(k_1+y) = U_{e_1+e_2}(k_{1,y}) \prod_{i=3}^{l} U_{e_2-e_i}(-c_i'k_{1,y}) \prod_{i=3}^{l} U_{e_2+e_i}(-d_i'k_{1,y})h'_{1,y}s_2^{-\prime\prime}$$

with a suitable $s_2^{-\prime\prime} \in N^-$, where

$$k_{1,y} = \frac{k_1+y}{1+c_2'(k_1+y)} \quad \text{and} \quad h'_{1,y} = \Phi_{e_1+e_2}\begin{pmatrix} \frac{1}{1+c_2'(k_1+y)} & 0 \\ 0 & 1+c_2'(k_1+y) \end{pmatrix} \in T.$$

Thus from (3.7), (3.9), the M-part of $s_1 s_1^- u$ is:

$$m_1 = \prod_{i=3}^{l} U_{-e_2-e_i}(a_i x') \prod_{i=3}^{l} U_{-e_2+e_i}(b_i x')h_{0,x} \prod_{i=3}^{l} U_{e_2-e_i}(-a_i'y') \times \prod_{i=3}^{l} U_{e_2+e_i}(-b_i'y')h_{1,y}.$$

While from (3.8) (3.10), the M-part of $s_2^- s_2 u$ is:

$$m_2 = \prod_{i=3}^{l} U_{-e_2-e_i}(c_i k_{0,x}) \prod_{i=3}^{l} U_{-e_2+e_i}(d_i k_{0,x})h'_{0,x} \prod_{i=3}^{l} U_{e_2-e_i}(-c_i'k_{1,y}) \times \prod_{i=3}^{l} U_{e_2+e_i}(-d_i'k_{1,y})h'_{1,y}.$$

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Because both m_1 and m_2 are products of one dimensional unipotent subgroups of different root vectors in the same order, for $m_1 = m_2$ to be true, then $a_i x' = c_i k_{0,x}$ and $b_i x' = d_i k_{0,x}$ must hold for all $i, 3 \le i \le l$, and for almost all $x \in F$. These equations lead to:

$$(c_i b_2 - a_i d_2)x^2 + (c_i k_0 + c_i b_2 - a_i d_2 k_0 - a_i)x + c_i k_0 = 0, \tag{3.11}$$

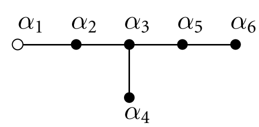
$$(d_i b_2 - b_i d_2)x^2 + (d_i k_0 + d_i b_2 - b_i d_2 k_0 - b_i)x + d_i k_0 = 0. \tag{3.12}$$

For equations (3.11) and (3.12) to have infinitely many solutions, since $k_0 \neq 0$, we must have $a_i = b_i = c_i = d_i \equiv 0$ for all $i, 3 \le i \le l$. Moreover, we must have $h_{0,x} h_{1,y} = h'_{0,x} h'_{1,y}$, which implies $h_{0,x} = h'_{0,x}$ for almost all $x \in F$, since $h_{0,x}(h'_{0,x}), h_{1,y}(h'_{1,y})$ are dual to $e_1 - e_2, e_1 + e_2$, respectively. So $(d_2 - b_2)x + d_2 k_0 = 0$ has infinitely many solutions in F, and consequently $d_2 = b_2 = 0$.

So $s_1^- = U_{-e_1-e_2}(a_2)$, $s_2^- = U_{-e_1-e_2}(c_2)$. And it can be easily calculated that $m_1 = h_{1,y}$, $m_2 = h'_{1,y}$ with $a'_2 = a_2$ in $h_{1,y}$ and $c'_2 = c_2$ in $h'_{1,y}$. Thus for $m_1 = m_2$ to be true for almost all $y \in F$, we must have $(c_2 - a_2)y + c_2 k_1 = 0$. Since $s_2^- \neq 0$, $c_2 \neq 0$, so we must have $k_1 = 0$ and $a_2 = c_2$. So $s_2^- s_2 = U_{-e_1-e_2}(c_2) U_{e_1-e_2}(k_0) = U_{e_1-e_2}(k_0) U_{-e_1-e_2}(c_2) = s_2 s_2^- = s_1 s_1^-$.

By the uniqueness of Bruhat decomposition, $s_1^- = s_2^-$. That finishes the proof of the main theorem for the case G is of type D_l.

3.6 Type E_6



In this case, N is abelian only when $\alpha = \alpha_1$ or α_6 by Lemma 2.1. Since α_1 is symmetric to α_6 on the Dynkin diagram, we need only prove the claim when $\alpha = \alpha_1$.

Let

$$\begin{aligned}\theta_1 = \{&\alpha_1;\ \alpha_1+\alpha_2;\ \alpha_1+\alpha_2+\alpha_3;\ \alpha_1+\alpha_2+\alpha_3+\alpha_5;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4;\\ &\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5;\\ &\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6;\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5;\\ &\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6;\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\}\end{aligned}$$

$$\begin{aligned}\theta_2 = \{&\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5;\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6;\\ &\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\ \alpha_1+2\alpha_2+3\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\\ &\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+\alpha_6.\}\end{aligned}$$

Then the positive roots in N are $R^+ \setminus \theta^+ = \theta_1 \cup \theta_2$. Notice that for each root $\beta \in \theta_1, \beta - \alpha_1$ is still a root, while for $\beta \in \theta_2, \beta - \alpha_1$ is not a root. Also notice that the coefficient of α_2 of roots in θ_1 is 1, while the coefficient of α_2 of roots in θ_2 is 2.

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Let $\beta_1, \beta_2, \dots, \beta_{11}$ be the roots in θ_1 according to the order listed in θ_1, and let $\gamma_1, \gamma_2, \dots, \gamma_5$ be the roots in θ_2 accordingly. Let $\tau_i = \beta_i - \alpha_1, i = 1, \dots, 11$; $\nu_i = \gamma_i - \alpha_1, i = 1, \dots, 5$, ($\nu_i$ is not a root).

The roots of N are divided into these two sets because each element in U_β with $\beta \in \theta_1, \beta \neq \alpha_1$ can be eliminated by an element in U_{α_1} and each element in U_{γ_i} with $i \neq 5$ can be eliminated by an element in U_{γ_5}. Elements in U_{γ_i} cannot be eliminated directly by elements in U_{α_1} since $\gamma_i - \alpha_1$ is not a root.

We will define an order on R: suppose $\beta, \gamma \in R$ and

$$\beta - \gamma = \sum_{i=1}^{6} c_i \alpha_i.$$

If

$$\sum_{i=1}^{6} c_i > 0,$$

then $\beta \succ \gamma$. If

$$\sum_{i=1}^{6} c_i = 0,$$

and if the first nonzero coefficient is > 0, then $\beta \succ \gamma$, otherwise $\beta \prec \gamma$. In particular, if $\beta \in R$ is a positive root, then $\beta \succ 0$. It is easily verified that this order is well defined and we have $\beta_i \prec \beta_j$ if $1 \leq i < j \leq 11$ and $\gamma_i \prec \gamma_j$ if $1 \leq i < j \leq 5$.

Let

$$N_1 = \Big\{ \prod_{i=1}^{11} U_{\beta_i} \Big\} \in N, \quad N_2 = \Big\{ \prod_{i=1}^{5} U_{\gamma_i} \Big\} \in N.$$

be the subgroups (because N is abelian) of N consisting of the unipotent subgroups of roots in θ_1, θ_2, respectively. We will prove that N_1 can be generated by $U_{\beta_1} = U_{\alpha_1}$ and N_2 can be generated by U_{γ_5} under the adjoint action of M.

For each pair of roots $\beta, \gamma \in R$, by Lemma 2.1 we know that

$$U_\gamma(x) U_\beta(y) U_\gamma(-x) = \prod_{i,j>0} U_{i\gamma + j\beta \in R}(C_{\gamma,\beta,i,j} x^i y^j) U_\beta(y). \tag{3.13}$$

Suppose the structure constants are normalized as in Lemma 2.1.

Lemma 3.6 *For each $u \in N$, if $u = u_1 u_2$, $u_i \in N_i, i = 1, 2$, with $u_1 \neq 1$. Then there exists $m \in M$ such that*

$$\operatorname{Int}(m) \circ u = \prod_{i=1}^{11} U_{\beta_i}(x_i') \prod_{i=1}^{5} U_{\gamma_i}(y_i')$$

with $x_1' \neq 0$.

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Proof Suppose

$$u = u_1 u_2 = \prod_{i=1}^{11} U_{\beta_i}(x_i) \prod_{i=1}^{5} U_{\gamma_i}(y_i).$$

If $x_1 \neq 0$, then there is nothing we need to do. Otherwise, let k be the smallest i such that $x_i \neq 0$. Notice such i exists since $u_1 \neq 1$, and by the assumption,

$$u_1 = \prod_{i=k}^{11} U_{\beta_i}(x_i).$$

Let $m = U_{-\tau_k}(1)$. For any pair $\{i, j\}$ of positive integers, $i\beta_k + j(-\tau_k)$ is a root only when $i = j = 1$, and $\beta_k + (-\tau_k) = \beta_1$. So we apply equation (3.13):

$$\mathrm{Int}(m) \circ U_{\beta_k}(x_k) = U_{\beta_1}(x_k) U_{\beta_k}(x_k),$$

since $C_{-\tau_k,\beta_k,1,1}$ is normalized to be 1.

For any $n > k, n \leq 11$, there is no pair $\{i, j\}$ of positive integers such that $i\beta_n + j(-\tau_k)$ is a root. To verify this, we need only to check the coefficients of α_1 and α_2 in $i\beta_n + j(-\tau_k)$. Namely, since N is abelian, the coefficient of α_1 in any root in N must be 1, so $i = 1$. Meanwhile the coefficient of α_2 of $i\beta_n + j(-\tau_k)$ is $1 - j \leq 0$, so j must be 1, too, and if this is the case, the coefficient of α_2 in $\beta_n - \tau_k$ is 0. Then $\beta_n - \tau_k = \beta_1$, since β_1 is the only root in N that has coefficient of α_2 equal to 0. But $\beta_n \succ \beta_k = \beta_1 + \tau_k$, this is a contradiction. So by Lemma 2.1 $\mathrm{Int}(m)$ fixes U_{β_n}.

Also for each n with $1 \leq n \leq 5$, $i\gamma_n + j(-\tau_k)$ can possibly be a root only when $i = j = 1$. (Since N is abelian, i must be 1 and we can exclude the possibility $j = 2$ since $\gamma_n + 2(-\tau_k)$ would not be connected by just applying Lemma 3.2.) If $\gamma_n - \tau_k$ is a root, then $\gamma_n - \tau_k \succ \alpha_1 = \beta_1$. So by Lemma 2.1

$$\mathrm{Int}(m) \circ U_{\gamma_n} \subset \prod_{\beta \succ \beta_1} U_\beta.$$

With these facts,

$$\mathrm{Int}(m) \circ u \in U_{\beta_1}(x_k) \prod_{\beta \succ \beta_1} U_\beta. \qquad \blacksquare$$

Lemma 3.7 *For each $u_2 \neq 1 \in N_2$, there exists an $m \in M$ such that* $\mathrm{Int}(m)$ *fixes U_{β_1} and*

$$\mathrm{Int}(m) \circ u_2 = \prod_{i=1}^{5} U_{\gamma_i}(y_i), \quad \text{with } y_5 \neq 0.$$

Proof Suppose

$$u_2 = \prod_{i=1}^{5} U_{\gamma_i}(x_i).$$

If $x_5 \neq 0$, then nothing needs to be done. Otherwise, let k be the smallest i such that $x_i \neq 0$. So $x_i \neq 0$ only when $k \leq i \leq 4$. Let $\gamma = \gamma_5 - \gamma_k$, and $m = U_\gamma(1)$.

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For each pair $\{i, j\}$ of positive integers, $i\gamma + j\gamma_k$ can be a root only when $i = j = 1$, since otherwise $i\gamma + j\gamma_k \succ \gamma_5$, and γ_5 is the longest element in R such that its α_1 part is nonzero. Moreover, in this case $\gamma + \gamma_k = \gamma_5$. So by applying Lemma 2.1, we have: $\mathrm{Int}(m) \circ U_{\gamma_k}(x_k) = U_{\gamma_k}(x_k)U_{\gamma_5}(C_{\gamma,\gamma_k,1,1}x_k)$, where $C_{\gamma,\gamma_k,1,1}$ is a structure constant, so is nonzero.

For all other q with $k < q \leq 4$, $i\gamma + j\gamma_q$ could not be a root since $i\gamma + j\gamma_q \succ \gamma_5$ for any positive integers i, j. So $\mathrm{Int}(m)$ fixes all these U_{γ_q}.

With these two facts, it is easily calculated that $\mathrm{Int}(m) \circ u = uU_{\gamma_5}(C_{\gamma,\gamma_k,1,1}x_k)$. Now, set $y_5 = C_{\gamma,\gamma_k,1,1}x_k$. Then $y_5 \neq 0$ as we have shown. Because $\gamma \subset \mathrm{span}\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, for each pair $\{i, j\}$ of positive integers, $i\gamma + j\beta_1$ cannot be a root by Lemma 3.2. So $\mathrm{Int}(m)$ fixes U_{β_1} by Lemma 2.1. ∎

Theorem 3.8 (Gaussian Elimination) *For each $u \neq 1 \in N$, there exists $m \in M$ such that* $\mathrm{Int}(m) \circ u = U_{\beta_1}(k_0)U_{\gamma_5}(k_1)$.

Proof We can write u as

$$u = \prod_{\beta_i \in \theta_1} U_{\beta_i}(x_i) \prod_{\gamma_i \in \theta_2} U_{\gamma_i}(x_i') = u_1 u_2, \quad u_1 \in N_1, u_2 \in N_2.$$

If $u_1 = 1$, then just set $m_1 = 1$. If $u_1 \neq 1$, by applying Lemma3.6 and a suitable $\mathrm{Int}(m')$, if necessary, we can assume $x_1 \neq 0$.

Let

$$m_1 = \prod_{i=2}^{11} U_{\tau_i}\left(\frac{x_i}{x_1}\right).$$

Then

$$\mathrm{Int}(m_1) \circ u = \Big[\prod_{\beta_i \in \theta_1} \mathrm{Int}(m_1) \circ U_{\beta_i}(x_i)\Big] \cdot \Big[\prod_{\gamma_i \in \theta_2} \mathrm{Int}(m_1) \circ U_{\gamma_i}(x_i')\Big].$$

For each fixed k, with $2 \leq k \leq 11$, $i\beta_1 + j\tau_k$ is a root for $i, j > 0$ only when $i = j = 1$, and in this case $\beta_1 + \tau_k = \beta_k$. So by applying Lemma 2.1,

$$\mathrm{Int}\left(U_{\tau_k}\left(\frac{x_k}{x_1}\right)\right) \circ U_{\beta_1}(x_1) = U_{\beta_1}(x_1) \cdot U_{\beta_k}(-x_k).$$

For each $q, 2 \leq q \leq 11, q \neq k$, and each pair of positive integers $\{i, j\}, i\beta_q + j\tau_k$ can possibly be a root only when $i = j = 1$. And in this case $\beta_q + \tau_k \in \theta_2$ if it is a root, since the coefficient of α_2 in $\beta_q + \tau_k$ is 2. So

$$\mathrm{Int}\left(U_{\tau_k}\left(\frac{x_k}{x_1}\right)\right) \circ U_{\beta_q}(x_q) = U_{\beta_q}(x_q) \cdot n_q \quad \text{with } n_q \in N_2.$$

For each pair of positive integers $\{i, j\}$, none of $i\beta_k + j\tau_k$ can be a root. So also by Lemma 2.1,

$$\mathrm{Int}\left(U_{\tau_k}\left(\frac{x_k}{x_1}\right)\right) \quad \text{fixes } U_{\beta_k}.$$

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With these facts, one can conclude from

$$\mathrm{Int}(m_1) = \prod_{i=2}^{11} \mathrm{Int}\Big(U_{\tau_i}\Big(\frac{x_i}{x_1}\Big)\Big)$$

that

$$\mathrm{Int}(m_1) \circ U_{\beta_1}(x_1) = U_{\beta_1}(x_1) \prod_{i=2}^{11} U_{\beta_i}(-x_i) \cdot n_1 \quad \text{with } n_1 \in N_2,$$

$$\mathrm{Int}(m_1) \circ U_{\beta_i}(x_i) = U_{\beta_i}(x_i) \cdot n_i' \quad \text{with } n_i' \in N_2,\ \forall i, 2 \le i \le 11.$$

By the last two equations, one can get

$$\mathrm{Int}(m_1) \circ (u_1) = U_{\beta_1}(x_1) \cdot n' \quad \text{where } n' = n_1 \cdot \prod_{i=2}^{11} n_i' \in N_2.$$

For each $\gamma \in \theta_2$, none of $i\tau_k + j\gamma$ is a root for any pair of positive integers $\{i, j\}$, since in the decomposition of $i\tau_k + j\gamma_i$ as a summation of simple roots, the coefficient of α_2 will be $i + 2j \ge 3$, which is not possible. So $\mathrm{Int}(m_1) \circ u_2 = u_2$.

Now we have $\mathrm{Int}(m_1) \circ u = \mathrm{Int}(m_1) \circ (u_1 u_2) = U_{\beta_1}(x_1) n' u_2$. Suppose

$$n' u_2 = \prod_{i=1}^{5} U_{\gamma_i}(y_i').$$

If $n' u_2 = 1$, *i.e.*, $y_i = 0$ for $1 \le i \le 5$, then we are done. Otherwise, let m_2 be the element in m that comes from Lemma 3.7. Then

$$\mathrm{Int}(m_2 m_1) \circ u = U_{\beta_1} \cdot \prod_{i=1}^{5} U_{\gamma_i}(y_i), \quad \text{with } y_5 \ne 0.$$

Now let

$$m_3 = \prod_{i=1}^{4} U_{\gamma_i - \gamma_5}\Big(-\frac{y_i}{y_5}\Big).$$

Then by Lemma 2.1, for any fixed i,

$$\mathrm{Int}\Big(U_{\gamma_i - \gamma_5}\Big(-\frac{y_i}{y_5}\Big)\Big) \circ U_{\gamma_5}(y_5) = U_{\gamma_5}(y_5) \cdot U_{\gamma_i}(-y_i).$$

It can be easily shown, by checking the coefficients of α and α_4, that for any pair $\{j, k\}$ of positive integers, and any q, with $1 \le q \le 4$, none of $j(\gamma_i - \gamma_5) + k\gamma_q$ can be a root. (Namely, for $j(\gamma_i - \gamma_5) + k\gamma_q$ to be a root, k must be 1 since the coefficient of α in $j(\gamma_i - \gamma_5) + k\gamma_q$ is k. Then the coefficient of α_2 in $j(\gamma_i - \gamma_5) + k\gamma_q$ is 2, so $j(\gamma_i - \gamma_5) + k\gamma_q \in \theta_2$ if it is a root. Then the coefficient of α_4 in $j(\gamma_i - \gamma_5) + k\gamma_q$ is

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$1 - j \leq 0$ which is not possible since every root in θ_2 has its coefficient of α_4 equal to 1.) So again by Lemma 2.1, $\mathrm{Int}(U_{\gamma_i - \gamma_5})$ fixes all other $U_{\gamma_q}(y_q)$. Thus

$$\mathrm{Int}(m_3) \circ U_{\gamma_5}(y_5) = U_{\gamma_5}(y_5) \prod_{i=1}^{4} U_{\gamma_i}(-y_i),$$

$$\mathrm{Int}(m_3) \circ \Big(\prod_{i=1}^{4} U_{\gamma_i}(y_i) \Big) = \prod_{i=1}^{4} U_{\gamma_i}(y_i).$$

So

$$\mathrm{Int}(m_3) \circ \Big(\prod_{i=1}^{5} U_{\gamma_i}(y_i) \Big) = U_{\gamma_5}(y_5).$$

Moreover, for each $i, \mathrm{Int}(U_{\gamma_i - \gamma_5})$ fixes U_{β_1} since, from the proof of Lemma 3.7, $\gamma_i - \gamma_5 \subset \mathrm{span}\{\alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Consequently, $\mathrm{Int}(m_3)$ fixes $U_{\beta_1}(x_1)$. Now let $m = m_3 m_2 m_1$. Then $\mathrm{Int}(m) \circ u = U_{\beta_1}(x_1) U_{\gamma_5}(y_5)$. Setting $k_0 = x_1, k_1 = y_5$ proves the theorem. ■

Returning to equation (2.5), by the above lemma and applying $\mathrm{Int}(m)$ on both sides, we can assume $s_2 = U_{\beta_1}(k_0) U_{\gamma_5}(k_1)$. Since without loss of generality we can always assume $s_2 \neq 1$ (otherwise nothing needs to be proved), we assume $k_0 \neq 0$.

Now suppose

$$s_1^- = \prod_{i=1}^{11} U_{-\beta_i}(a_i) \cdot \prod_{i=1}^{5} U_{-\gamma_i}(b_i),$$

$$s_2^- = \prod_{i=1}^{11} U_{-\beta_i}(c_i) \cdot \prod_{i=1}^{5} U_{-\gamma_i}(d_i).$$

Multiply both sides of (2.5) by $u = U_{\beta_1}(x) U_{\gamma_5}(y)$ on the right, where x, y are variables in F. We will decompose both $s_1 s_1^- u$ and $s_2^- s_2 u$ into PN^- form, and compare their M parts.

First for $s_1 s_1^- U_{\beta_1}(x)$, we have:

$$U_{-\beta_1}(a_1) U_{\beta_1}(x) = U_{\beta_1}(x') h_x U_{-\beta_1}\Big(\frac{a_1}{1 + a_1 x} \Big), \tag{3.14}$$

where

$$x' = \frac{x}{1 + a_1 x}, \quad \text{and} \quad h_x = \Phi_{\beta_1} \begin{pmatrix} \frac{1}{1+a_1 x} & 0 \\ 0 & 1 + a_1 x \end{pmatrix} \in T.$$

For each k with $2 \leq k \leq 11$, by Lemma 2.1 we have

$$U_{-\beta_k}(a_k) U_{\beta_1}(x') = U_{\beta_1}(x') U_{-\tau_k}(-a_k x') U_{-\beta_k}(a_k). \tag{3.15}$$

For any k with $1 \leq k \leq 5$, and any pair $\{i, j\}$ of positive integers, none of $i\beta_1 + j(-\gamma_k)$ can be a root. So by Lemma 2.1, U_{β_1} commutes with $U_{-\gamma_k}$ for all k.

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Since N^- is normal in $P^- = MN^-$, from equations (3.14), (3.15) and the above fact, the PN^- decomposition of $s_1^- U_{\beta_1}(x)$ is as follows:

$$s_1^- U_{\beta_1}(x) = U_{\beta_1}(x')\Big[\prod_{i=2}^{11} U_{-\tau_i}(-a_i x')\Big] h_x \cdot (s_1^-)', \tag{3.16}$$

with a suitable $(s_1^-)' \in N^-$. Then for $s_2^- s_2 u = s_2^- U_{\beta_1}(k_0 + x) U_{\gamma_5}(k_1 + y)$. Similarly, the PN^- decomposition of $s_2^- U_{\beta_1}(k_0 + x)$ is:

$$s_2^- U_{\beta_1}(k_0 + x) = U_{\beta_1}(k_x)\Big[\prod_{i=2}^{11} U_{-\tau_i}(-k_x c_i)\Big] h'_x (s_2^-)', \tag{3.17}$$

with a suitable $(s_2^-)' \in N^-$, where

$$k_x = \frac{k_0 + x}{1 + c_1(k_0 + x)}, \quad \text{and} \quad h'_x = \Phi_{\beta_1}\begin{pmatrix} \frac{1}{1+c_1(k_0+x)} & 0 \\ 0 & 1 + c_1(k_0 + x)\end{pmatrix} \in T.$$

For convenience of notation, we will set $U_{\nu_i} \equiv 1$ if ν_i is not a root. Suppose

$$(s_1^-)' = \prod_{i=1}^{11} U_{-\beta_i}(a'_i) \cdot \prod_{i=1}^{5} U_{-\gamma_i}(b'_i), \quad (s_2^-)' = \prod_{i=1}^{11} U_{-\beta_i}(c'_i) \cdot \prod_{i=1}^{5} U_{-\gamma_i}(d'_i).$$

Then with a similar discussion on roots and applying Lemma 2.1, following a similar process of calculation, we get:

$$(s_1^-)' U_{\gamma_5}(y) = U_{\gamma_5}(y')\Big[\prod_{i=1}^{11} U_{\gamma_5 - \beta_i}(a'_i y')\Big]\Big[\prod_{i=1}^{4} U_{\gamma_5 - \gamma_i}(b'_i y')\Big] h_y (s_1^-)'', \tag{3.18}$$

with a suitable $(s_1^-)'' \in N^-$, where

$$y' = \frac{y}{1 + b'_5 y} \quad \text{and} \quad h_y = \Phi_{\gamma_5}\begin{pmatrix} \frac{1}{1+b'_5 y} & 0 \\ 0 & 1 + b'_5 y\end{pmatrix} \in T.$$

Meanwhile,

$$(s_2^-)' U_{\gamma_5}(k_1 + y) = U_{\gamma_5}(k_y)\Big[\prod_{i=1}^{11} U_{\gamma_5 - \beta_i}(k_y c'_i)\Big]\Big[\prod_{i=1}^{4} U_{\gamma_5 - \gamma_i}(k_y d'_i)\Big] h'_y (s_2^-)'', \tag{3.19}$$

with a suitable $(s_2^-)'' \in N^-$, where

$$k_y = \frac{k_1 + y}{1 + d'_5(k_1 + y)} \quad \text{and} \quad h'_y = \Phi_{\gamma_5}\begin{pmatrix} \frac{1}{1+d'_5(k_1+y)} & 0 \\ 0 & 1 + d'_5(k_1 + y)\end{pmatrix} \in T.$$

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Thus, from equations (3.16) and (3.18), the M-part of $s_1 s_1^- u$ is:

$$M_1(x,y) = \Big[\prod_{i=2}^{11} U_{-\tau_i}(-a_i x')\Big] h_x \Big[\prod_{i=1}^{11} U_{\gamma_5-\beta_i}(a_i' y')\Big] \Big[\prod_{i=1}^{4} U_{\gamma_5-\gamma_i}(b_i' y')\Big] h_y.$$

While from equation (3.17) and (3.19), the M-part of $s_2^- s_2 u$ is:

$$M_2(x,y) = \Big[\prod_{i=2}^{11} U_{-\tau_i}(-k_x c_i)\Big] h_x' \Big[\prod_{i=1}^{11} U_{\gamma_5-\beta_i}(k_y c_i')\Big] \Big[\prod_{i=1}^{4} U_{\gamma_5-\gamma_i}(k_y d_i')\Big] h_y'.$$

Notice that all $-\tau_i$ are distinct negative roots while all $\gamma_5 - \beta_i$ and $\gamma_5 - \gamma_i$ are distinct positive roots (if they are roots). For $M_1(x,y) = M_2(x,y)$, the unipotent groups of the corresponding root vector must be equal, and their T parts must be equal as well, as in the previous cases.

So we have $a_i x' = k_x c_i$, for all $i, 2 \le i \le 11$, and almost all $x \in F$. Moreover, $h_x = h_x'$ since $h_x(h_x'), h_y(h_y')$ are dual to β_1, γ_5, respectively. As an analog of the proof in the $B_l(D_l)$ case, we get $a_i = c_i \equiv 0$, $\forall 1 \le i \le 11$. Thus from equation (3.16) and (3.17),

$$s_1^- = s_1^{-\,\prime} = \prod_{i=1}^{5} U_{-\gamma_i}(b_i), \quad s_2^- = s_2^{-\,\prime} = \prod_{i=1}^{5} U_{-\gamma_i}(d_i)$$

and from equations (3.18), (3.19),

$$M_1(x,y) = \Big[\prod_{i=1}^{4} U_{\gamma_5-\gamma_i}(b_i y')\Big] h_y, \quad M_2(x,y) = \Big[\prod_{i=1}^{4} U_{\gamma_5-\gamma_i}(d_i k_y)\Big] h_y'.$$

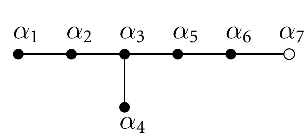
Since $s_2^- \neq 1$, there is one $i, 1 \le i \le 5$, such that $d_i \neq 0$. Together with the fact that $M_1(x,y) = M_2(x,y)$ for almost all $x, y \in F$, following the previous proofs, we can get $k_1 = 0$. So

$$s_2^- s_2 = \Big[\prod_{i=1}^{5} U_{-\gamma_i}(d_i)\Big] U_{\beta_1}(k_0) = U_{\beta_1}(k_0)\Big[\prod_{i=1}^{5} U_{-\gamma_i}(d_i)\Big] = s_2 s_2^- = s_1 s_1^-.$$

By the uniqueness of Bruhat decomposition, it must have $s_1^- = s_2^-$.

If at the beginning of this proof, we assume $k_1 \neq 0$ instead of assuming $k_0 \neq 0$, the proof will be similar.

3.7 Type E_7



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The longest root in this case is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, by Lemma 3.1; N is abelian only when $\alpha = \alpha_7$.

Let

$$\begin{aligned}\theta_1 = \{&\alpha;\ \ \alpha+\alpha_6;\ \ \alpha+\alpha_5+\alpha_6;\ \ \alpha+\alpha_3+\alpha_5+\alpha_6;\ \ \alpha+\alpha_3+\alpha_4+\alpha_5+\alpha_6;\\ &\alpha+\alpha_2+\alpha_3+\alpha_5+\alpha_6;\ \ \alpha+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6;\\ &\alpha+\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6;\quad \alpha+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6;\\ &\alpha+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6;\quad \alpha+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\\ &\alpha+\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6;\quad \alpha+\alpha_1+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\\ &\alpha+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6;\quad \alpha+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\\ &\alpha+\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+2\alpha_5+\alpha_6;\ \ \alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+\alpha_6;\}\end{aligned}$$

$$\begin{aligned}\theta_2 = \{&\alpha+2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6;\\ &\alpha+\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6;\quad \alpha+\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6;\\ &\alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+3\alpha_5+2\alpha_6;\quad \alpha+\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+3\alpha_5+2\alpha_6;\\ &\alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6;\quad \alpha+\alpha_1+2\alpha_2+3\alpha_3+\alpha_4+2\alpha_5+2\alpha_6;\\ &\alpha+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+2\alpha_6;\quad \alpha+\alpha_1+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+2\alpha_6;\\ &\alpha+\alpha_2+2\alpha_3+\alpha_4+2\alpha_5+2\alpha_6\}.\end{aligned}$$

Then the positive roots in N are $R^+ \setminus \theta^+ = \theta_1 \cup \theta_2$.

Let $\beta_1, \beta_2, \dots, \beta_{17}$ denote the roots in θ_1 as the order listed, $\gamma_1, \gamma_2, \dots, \gamma_{10}$ denote the roots in θ_2 similarly. For any $\beta \in \theta_1$, $\beta-\alpha$ is a root (as is E_6); for $i = 2, \dots, 9$, $\gamma_1 - \gamma_i$ is a root while $\gamma_1 - \gamma_{10}$ is not; for each i, $1 \le i \le 10$, $\gamma_i - \beta_1$ is not a root. Notice for each root in θ_1, the coefficient of α_6 is 1, and for each root in θ_2, the coefficient of α_6 is 2.

We will define an order on R: suppose $\beta, \gamma \in R$ and

$$\beta - \gamma = \sum_{i=1}^{7} c_i \alpha_i.$$

If

$$\sum_{i=1}^{7} c_i > 0,$$

then $\beta \succ \gamma$; if

$$\sum_{i=1}^{7} c_i = 0,$$

and if the first nonzero coefficient is > 0, then $\beta \succ \gamma$, otherwise $\beta \prec \gamma$. In particular, if $\beta \in R$ is a positive root, then $\beta \succ 0$. It is easily verified that this order is well defined and we have $\beta_i \prec \beta_j$ if $1 \le i < j \le 17$ and $\gamma_i \succ \gamma_j$ if $1 \le i < j \le 10$.

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Suppose the root vectors are so chosen that the structure constants are normalized as in Lemma 2.1. Let

$$N_1 = \Big\{ \prod_{i=1}^{17} U_{\beta_i} \Big\} \subset N, \quad N_2 = \Big\{ \prod_{i=1}^{10} U_{\gamma_i} \Big\} \subset N.$$

Every element of $u \in N$ can be written as $u = u_1 u_2$ with $u_i \in N_i, i = 1, 2$. And we can similarly define N_1^-, N_2^- as subgroups of N^-. The roots of N are divided into these two sets because, as we will prove, U_{β_1} generates N_1 and U_{γ_1} generates N_2 under the adjoint action of M. Each element in U_{γ_i}, with $1 \le i \le 10$, cannot be eliminated directly by an element in U_{β_1} since $\gamma_i - \beta_1$ is not a root.

Lemma 3.9 *For each $u \in N$, if $u = u_1 u_2$, $u_i \in N_i, i = 1, 2$, with $u_1 \neq 1$. Then there exists $m \in M$, such that*

$$\mathrm{Int}(m) \circ u = \Big\{ \prod_{i=1}^{17} U_{\beta_i}(x_i') \Big\} \Big\{ \prod_{i=1}^{10} U_{\gamma_i}(y_i') \Big\} \quad \text{with } x_1' \neq 0.$$

Proof This is analogous to Lemma 3.6, since for each $i, 2 \le i \le 17$, $\beta_i - \beta_1$ is a root, the proof is almost the same as of the proof for Lemma 3.6. The indices are the only changes. ■

Lemma 3.10 *If*

$$u_2 = \Big\{ \prod_{i=1}^{10} U_{\gamma_i}(x_i) \Big\} \subset N_2$$

and $u_2 \neq 1$, we can find an $m \in M$ such that $\mathrm{Int}(m) \circ u_2 = U_{\gamma_1}(a_2)$ with $a_2 \neq 0$ and $\mathrm{Int}(m)$ fixes every element in U_{β_1}.

Proof First we prove the following claim:

Claim *There is $m_1 \in M$, such that*

$$\mathrm{Int}(m_1) \circ u_2 = \prod_{i=1}^{10} U_{\gamma_i}(x_i') \quad \text{with } x_1' \neq 0, x_2' \neq 0.$$

(This claim is needed because $\gamma_1 - \gamma_{10}$ is not a root, and $U_{\gamma_{10}}$ cannot be eliminated directly through U_{γ_1}. So we use U_{γ_2} to eliminate it.)

Let k be the smallest positive integer such that $x_k \neq 0$. If $k = 1$, *i.e.*, $x_1 \neq 0$. And if $x_2 \neq 0$, then the claim is trivial.

Case** $k = 1$**,** $x_2 = 0$**: Let $m_1 = U_{\gamma_2 - \gamma_1}(1)$. For any $i, 3 \le i \le 10$, by Lemma 2.1,

$$\mathrm{Int}(U_{\gamma_2-\gamma_1}(1)) \circ U_{\gamma_i}(x_i) = \Big(\prod_{\substack{k,n>0 \\ k(\gamma_2-\gamma_1)+n\gamma_i \in R}} U_{k(\gamma_2-\gamma_1)+n\gamma_i}(C_{\gamma_2-\gamma_1,\gamma_i,k,n} x_i^n) \Big) \cdot U_{\gamma_i}(x_i),$$

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where the $C_{\gamma_2-\gamma_1,\gamma_i,k,n}$'s are structure constants.

Since $\gamma_2 - \gamma_1 \in \operatorname{span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, for any pair of positive integers $\{k, n\}$, the coefficient of α_6 in $k(\gamma_2-\gamma_1)+n\gamma_i$ is $2n$. For it to be a root, n must be 1. Moreover, if this is the case, then $k(\gamma_2 - \gamma_1) + \gamma_i \in \theta_2$.

Since $\gamma_2 - \gamma_1 \prec 0$, $k(\gamma_2 - \gamma_1) + \gamma_i \prec \gamma_i$. So

$$\operatorname{Int}(m_1) \circ U_{\gamma_i}(x_i) \subset \prod_{j\geq i} U_{\gamma_j},$$

consequently,

$$\operatorname{Int}(m_1) \circ \Big(\prod_{i=3}^{10} U_{\gamma_i}(x_i)\Big) \subset \prod_{i=3}^{10} U_{\gamma_i}.$$

And by Lemma 2.1, $\operatorname{Int}(m_1) \circ U_{\gamma_1}(x_1) = U_{\gamma_1}(x_1)U_{\gamma_2}(x_1)$. Therefore, $\operatorname{Int}(m_1) \circ u = U_{\gamma_1}(x_1)U_{\gamma_2}(x_1) \cdot u'$ with

$$u' \in \prod_{i=3}^{10} U_{\gamma_i}.$$

Set $x_1' = x_2' = x_1$, and the claim is proved.

Case $k = 2$***:*** Let $m_1 = U_{\gamma_1-\gamma_2}(1) = U_{\alpha_1}(1)$. For each $i, 3 \leq i \leq 10$, and each pair $\{k, n\}$ of positive integers, the coefficient of α_1 in $k\alpha_1 + n\gamma_i$ is $k + n$. So for $k\alpha_1 + n\gamma_i$ to be a root, we must have $k = n = 1$. But it is easily checked that $\alpha_1 + \gamma_i$ is not a root when $i \geq 3$. So by Lemma 2.1, $\operatorname{Int}(U_{\alpha_1}(1)) \circ U_{\gamma_i}(x_i) = U_{\gamma_i}(x_i)$. Also for any pair $\{k, n\}$ of positive integers, $k(\gamma_1 - \gamma_2) + n\gamma_2$ can be a root only when $k = n = 1$. So by applying Lemma 2.1, $\operatorname{Int}(U_{\alpha_1}(1)) \circ U_{\gamma_2}(x_2) = U_{\gamma_1}(x_2)U_{\gamma_2}(x_2)$, with $x_2 \neq 0$. Then

$$\operatorname{Int}(U_{\alpha_1}(1)) \circ u = U_{\gamma_1}(x_2)U_{\gamma_2}(x_2)\Big[\prod_{i=3}^{10} U_{\gamma_i}(x_i)\Big].$$

Setting $x_1' = x_2$ will prove our claim.

Case $3 \leq k < 10$***:*** Let $m_1 = U_{\gamma_1-\gamma_k}(1)U_{\gamma_2-\gamma_k}(1)$, with a similar discussion as the second case, but this time take the coefficients of α_1 and α_2 into account. We can figure out that the $U_{\gamma_1}U_{\gamma_2}$ part of $\operatorname{Int}(m_1) \circ u$ is $U_{\gamma_1}(x_k)U_{\gamma_2}(x_k)$.

Case $k = 10$***:*** This case is handled separately because $\gamma_1 - \gamma_{10}$ is not a root. Let $m_1 = U_{\gamma_2-\gamma_{10}}(1)$, then $\operatorname{Int}(m_1) \circ u = \operatorname{Int}(m_1) \circ U_{\gamma_{10}}(x_{10}) = U_{\gamma_2}(x_{10})U_{\gamma_{10}}(x_{10})$ by Lemma 2.1, since for any positive integers k and n, $k(\gamma_2 - \gamma_{10}) + n\gamma_{10}$ is a root only when $k = n = 1$. Now it will fall into the second case which has already been proved.

Now

$$\operatorname{Int}(m_1) \circ u = \prod_{i=1}^{10} U_{\gamma_i}(x_i') \quad \text{with } x_1' \neq 0, x_2' \neq 0.$$

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Let

$$m_2 = U_{\gamma_{10}-\gamma_2}\Big(-\frac{x'_{10}}{x'_2}\Big).$$

It can be checked for any $i \geq 3$, and any pair of positive integers $\{k, n\}$, that $k(\gamma_{10} - \gamma_2) + n\gamma_i$ is not a root. So $\mathrm{Int}(m_2)$ fixes all U_{γ_i}.

For any pair of positive integers $\{k, n\}$, $k\gamma_1 + n(\gamma_{10} - \gamma_2)$ or $k\gamma_2 + n(\gamma_{10} - \gamma_2)$ can be a root only when $k = n = 1$. And $\gamma_1 + (\gamma_{10} - \gamma_2) = \gamma_9$; $\gamma_2 + (\gamma_{10} - \gamma_2) = \gamma_{10}$.

By Lemma 2.1,

$$\mathrm{Int}(m_2) \circ U_{\gamma_2}(x'_2) = U_{\gamma_{10}}(-x'_{10})U_{\gamma_2}(x'_2),$$

$$\mathrm{Int}(m_2) \circ U_{\gamma_1}(x'_1) = U_{\gamma_9}\Big(\frac{x'_1x'_{10}}{x'_2}\Big)U_{\gamma_1}(x'_1).$$

Consequently,

$$\mathrm{Int}(m_2) \circ \Big(\prod_{i=1}^{10} U_{\gamma_i}(x'_i)\Big) = \Big[\prod_{i=1}^{8} U_{\gamma_i}(x'_i)\Big] U_{\gamma_9}\Big(x'_9 - \frac{x'_1x'_{10}}{x'_2}\Big).$$

For convenience of notation, let the right side of the above equation be

$$\prod_{i=1}^{9} U_{\gamma_i}(y_i).$$

Let

$$m_3 = \prod_{i=2}^{9} U_{\gamma_i-\gamma_1}\Big(-\frac{y_i}{y_1}\Big).$$

By Lemma 2.1, we have:

$$\mathrm{Int}\Big(U_{\gamma_9-\gamma_1}\Big(-\frac{y_9}{y_1}\Big)\Big) \circ U_{\gamma_1}(y_1) = U_{\gamma_1}(y_1)U_{\gamma_9}(-y_9). \tag{3.20}$$

Remark For all i with $i \neq 1$, and any pair $\{k, n\}$ of positive integers, the coefficient of α in $k(\gamma_9 - \gamma_1) + n\gamma_i$ is n. So for it to be a root, n must be 1. Then the coefficient of α_1 in $k(\gamma_9 - \gamma_1) + n\gamma_i$ is $n - k = 1 - k$. For $k(\gamma_9 - \gamma_1) + n\gamma_i$ to be a root, $1 - k = 1$ or 2 which is impossible. So $\mathrm{Int}(U_{\gamma_9-\gamma_1})$ fixes all U_{γ_i} with $i \neq 1$.

So by equation (3.20) and the above remark,

$$\mathrm{Int}\Big(U_{\gamma_9-\gamma_1}\Big(-\frac{y_9}{y_1}\Big)\Big) \circ \Big(\prod_{i=1}^{9} U_{\gamma_i}(y_i)\Big) = \prod_{i=1}^{8} U_{\gamma_i}(y_i).$$

By induction, and with the same discussion on the cases of roots as in the remark, we can prove:

$$\mathrm{Int}\Big(\prod_{i=j}^{9} U_{\gamma_i-\gamma_1}\Big(-\frac{y_i}{y_1}\Big)\Big) \circ \Big(\prod_{i=1}^{9} U_{\gamma_i}(y_i)\Big) = \prod_{i=1}^{j-1} U_{\gamma_i}(y_i).$$

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And in particular when $j = 1$, then

$$\operatorname{Int}(m_3) \circ \Big(\prod_{i=1}^{9} U_{\gamma_i}(y_i) \Big) = U_{\gamma_1}(y_1).$$

Set $m = m_3 m_2 m_1$. We can then see from the above process that $\operatorname{Int}(m) \circ u = U_{\gamma_1}(y_1)$.

For any $1 \leq i, j \leq 10, \gamma_i - \gamma_j \in \operatorname{span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. But for any $\gamma \in \operatorname{span}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and any pair $\{k, n\}$ of positive integers, $k\beta_1 + n\gamma$ cannot be a root by Lemma 3.2. So each $\operatorname{Int}(U_{\gamma_i - \gamma_j})$ fixes U_{β_1} and consequently, all $\operatorname{Int}(m_1)$, $\operatorname{Int}(m_2)$, $\operatorname{Int}(m_3)$ fix U_{β_1} and therefore $\operatorname{Int}(m)$ fixes U_{β_1}. ■

Theorem 3.11 (Gaussian Elimination) *For any $u \in N$, there exists $m \in M$, such that* $\operatorname{Int}(m) \circ u = U_{\beta_1}(a_1) U_{\gamma_1}(a_2)$, *with $a_1, a_2 \in F$.*

Proof Write $u = u_1 u_2$, where

$$u_1 = \prod_{i=1}^{17} U_{\beta_i}(x_i) \in N_1, \quad u_2 = \prod_{i=1}^{10} U_{\gamma_i}(y_i) \in N_2.$$

If $u_1 = 1$, then it is the case of Lemma 3.10.

If $u_1 \neq 1$, by applying a suitable $\operatorname{Int}(m)$ on u from Lemma 3.9, we can assume $x_1 \neq 1$. Let

$$m_1 = \prod_{i=2}^{17} U_{\beta_i - \beta_1}\Big(\frac{x_i}{x_1}\Big).$$

then $\beta_i - \beta_1$ is a positive root and the coefficient of α_6 in $\beta_i - \beta_1$ is 1.

For any fixed j, with $2 \leq j \leq 17$, and for each pair of positive integers $\{k, n\}$, the coefficient of α_6 in $k(\beta_i - \beta_1) + n\beta_j$ is $k + n \geq 2$, so $k(\beta_i - \beta_1) + n\beta_j \in \theta_2$ if it is a root. Moreover, for any $\gamma \in \theta_2$, the coefficient of α_6 in $k(\beta_i - \beta_1) + n\gamma$ is $k + 2n \geq 3$, so $k(\beta_i - \beta_1) + n\gamma$ cannot be a root, hence $\operatorname{Int}(U_{\beta_i - \beta_1})$ fixes every element in N_2.

So by Lemma 2.1, we have:

$$\operatorname{Int}\Big(U_{\beta_i - \beta_1}\Big(\frac{x_i}{x_1}\Big)\Big) \circ U_{\beta_j}(x_j) = U_{\beta_j}(x_j) \cdot n_{i,j}, \quad \text{with } n_{i,j} \in N_2.$$

Consequently, $\operatorname{Int}(m_1) \circ U_{\beta_j}(x_j) = U_{\beta_j}(x_j) n_j$ with

$$n_j = \prod_{i=2}^{17} n_{i,j} \in N_2,$$

and

$$\operatorname{Int}(m_1) \circ U_{\beta_1}(x_1) = U_{\beta_1}(x_1) \cdot \prod_{i=2}^{17} U_{\beta_i}(-x_i) \cdot n_1 \quad \text{with } n_1 \in N_2.$$

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So

$$\operatorname{Int}(m_1) \circ u_1 = \operatorname{Int}(m_1) \circ \Big(\prod_{i=1}^{17} U_{\beta_i}(x_i) \Big) = U_{\beta_1}(x_1) \cdot n \text{ where } n = \prod_{i=1}^{17} n_i \in N_2.$$

Now let $u_2' = n \cdot u_2$ and apply Lemma 3.10 to u_2'. There exists $m_2 \in M$ such that $\operatorname{Int}(m_2) \circ u_2' = U_{\gamma_1}(a_2)$ and $\operatorname{Int}(m_2) \circ U_{\beta_1}(x_1) = U_{\beta_1}(x_1)$. Let $m = m_2 m_1$ and $a_1 = x_1$. Then $\operatorname{Int}(m) \circ u = U_{\beta_1}(a_1) U_{\gamma_1}(a_2)$. ■

Now start from $s_1 s_1^- = s_2^- s_2$ acting on $\operatorname{Int}(m)$ on both sides, we can assume $s_2 = U_{\beta_1}(a_1) U_{\gamma_1}(a_2)$. The proof of the main theorem is almost the same as that of E_6. We need only make a small justification of the fact that $\gamma_1 - \gamma_{10}$ is not a root, but this does not make much difference. Each step in the proof of the E_6 case can be paralleled to finish the proof in the E_7 case.

4 Application to Intertwining Operators

Now by Theorem 2.2, $M_{m_i}^t = M_{n_i}$. This can be used to refine the main results in [8]. To be more precise, let $X(\mathbf{M})_F$ be the group of F-rational characters of $\mathbf{M}$. Denote by $\mathbf{A}$ the split component of the center of $\mathbf{M}$. Then $\mathbf{A} \subset \mathbf{A}_0$. Let

$$\mathfrak{a} = \operatorname{Hom}(X(\mathbf{M})_F, \mathbb{R}) = \operatorname{Hom}(X(\mathbf{A})_F, \mathbb{R})$$

be the real Lie algebra of $\mathbf{A}$. Set $\mathfrak{a}^* = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ to denote its real and complex duals.

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ and σ an irreducible admissible representation of M, let $I(\nu, \sigma) = \operatorname{Ind}_{MN \uparrow G}\ \sigma \otimes q^{\langle \nu, H_P(\cdot) \rangle} \otimes 1$, where H_P is the extension of the homomorphism $H_M \colon M \to \mathfrak{a} = \operatorname{Hom}(X(\mathbf{M})_F, \mathbb{R})$ to P, extended trivially along N, defined by $q^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F$ for all $\chi \in X(\mathbf{M})_F$. Let $V(\nu, \sigma)$ be the space of $I(\nu, \sigma)$, for $h \in V(\nu, \sigma)$, and let

$$A(\nu, \sigma, w) h(g) = \int_{N_{\tilde{w}}} h(w^{-1} n g)\, dn,$$

where $N_{\tilde{w}} = U \cap w N^- w^{-1}$, be the standard intertwining operator from $I(\nu, \sigma)$ into $I(w(\nu), w(\sigma))$.

Let $I(\sigma) = I(0, \sigma)$ and $V(\sigma) = V(0, \sigma)$ be the induced representation and its space at $\nu = 0$, respectively. Since $w_0(M) = M$, $I(\sigma)$ is irreducible if and only if $A(\nu, \sigma, w_0)$ has a pole at $\nu = 0$ (*cf.* [6–8]). By [7, Lemma 4.1], it is enough to determine the pole of $\int_N h(w_0^{-1} n)\, dn$ at $\nu = 0$ for any h in $V(\nu, \sigma)$ which is supported in PN^-.

For $n_i \in N$, suppose n_i is inside an open orbit under $\operatorname{Int}(M)$, with $w_0^{-1} n_i \in PN^-$. Write $w_0^{-1} n_i = m_i n_i' n_i^-$ as before, define $d^* n_i = q^{\langle \rho, H_M(m_i) \rangle} dn$ where ρ is half the summation of the positive roots in N. Then by [8, Lemma 2.3], the measure $d^* n_i$ is an invariant measure on M/M_{n_i} and thus induces a measure on the quotient M/M_{n_i}.

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For the purpose of computing the residue we may assume that there exists a Schwartz function ϕ on $\mathfrak{N}^-$, the Lie algebra of N^-, such that

$$h(\exp(\mathfrak{n}^-)) = \phi(\mathfrak{n}^-)h(e),$$

where $\mathfrak{n}^- \in \mathfrak{N}^-$. Let $n_i^- = \exp(\mathfrak{n}_i^-)$, with $\mathfrak{n}_i^- \in \mathfrak{N}^-$. Given a representation σ, let $\psi(m)$ be among the matrix coefficients of σ, *i.e,* choose an arbitrary element $\tilde{v}$ in the contragredient space of σ. Let $\psi(m) = \langle \sigma(m)h(e), \tilde{v}\rangle$. With these notations and applying Theorem 2.2, [8, Proposition 2.4] can be restated as:

Proposition 4.1 *Let σ be an irreducible admissible representation of M. Then the poles of $A(\nu, \sigma, w_0)$ are the same as those of*

$$\sum_{n_i \in O_i} \int_{M/M_{n_i}} q^{\langle \nu, H_M(w_0(m)m_i m^{-1})\rangle} \phi(\mathrm{Ad}(m^{-1})\mathfrak{n}_i^-)\psi(w_0(m)m_i m^{-1})\, d\dot{m}$$

where O_i runs through a finite number of open orbits of $\mathfrak{N}$ under $\mathrm{Ad}(M)$; $\mathfrak{n}_i$ *is a representative of O_i, under the correspondence that $w_0^{-1}n_i = m_i n_i' n_i^-$, with $n_i =$* $\exp(\mathfrak{n}_i)$, *$n_i^- = exp(\mathfrak{n}_i^-)$ and $d\dot{m}$ is the measure on M/M_{n_i} induced from d^*n_i.*

Let $\tilde{\mathbf{A}}$ be the center of $\mathbf{M}$. Then there exists a function $f \in C_c^\infty(M)$ such that $\psi(m) = \int_{\tilde{A}} f(am)\omega^{-1}(a)\, da$, where ω is the central character of σ.

Define

$$\theta\colon M \to M, \quad \theta(m) = w_0^{-1} m w_0, \forall\, m \in M.$$

Given $f \in C_c^\infty(M)$ and $m_0 \in M$, define the θ-twisted orbit integral for f at m_0 by:

$$\phi_\theta(m_0, f) = \int_{M/M_{\theta,m_0}} f(\theta(m)m_0 m^{-1})\, d\dot{m},$$

where

$$M_{\theta,m_0} = M_{\theta,m_0}(F) = \{m \in M(F) \mid \theta(m)m_0 m^{-1} = m_0\}$$

is the θ-twisted centralizer of m_0 in $M(F)$, $d\dot{m}$ is the measure on $M/M_{\theta,m_0}$ induced from dm.

Applying our Theorem 2.2, the main theorem in [8] (Theorem 2.5) can be modified as:

Proposition 4.2 *Assume σ is supercuspidal and $w_0(\sigma) \cong \sigma$. The intertwining operator $A(\nu, \sigma, w_0)$ has a pole at $\nu = 0$ if and only if*

$$\sum_i \int_{Z(G)/Z(G)\cap w_0(\tilde{A})\tilde{A}^{-1}} \phi_\theta(zm_i, f)\omega^{-1}(z)\, dz \neq 0$$

for f as above. Here $Z(G)$ is the center of G and

$$\phi_\theta(zm_i, f) = \int_{M/M_{n_i}} f(z\theta(m)m_i m^{-1})\, d\dot{m},$$

is the θ-twisted orbital integral for f at zm_i, where m_i corresponds to the representatives $\{n_i\}$ for the open orbits in N under $\mathrm{Int}(M)$, *with $w_0^{-1}n_i = m_i n_i' n_i^-$, as n_i runs through the finite number of open orbits in N.*

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