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UNIQUE CONTINUATION AT INFINITY OF SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH COMPLEX-VALUED POTENTIALS

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Dedicated to the memory of Professor Olgierd A. Biberstein (1921–1997)

We obtain optimal L^2-lower bounds for nonzero solutions to $-\Delta\Psi + V\Psi = E\Psi$ in $\mathbf{R}^n$, $n \geq 2$, $E \in \mathbf{R}$, where V is a measurable complex-valued potential with $V(x) = O(|x|^{-\epsilon})$ as $|x| \to \infty$, for some $\epsilon \in \mathbf{R}$. We show that if $3\delta = \max\{0, 1 - 2\epsilon\}$ and $\exp(\tau|x|^{1+\delta})\Psi \in L^2(\mathbf{R}^n)$ for all $\tau > 0$, then Ψ has compact support. This result is new for $0 < \epsilon < 1/2$ and generalizes similar results obtained by Meshkov for $\epsilon = 0$, and by Froese, Herbst, M. Hoffmann-Ostenhof, and T. Hoffmann-Ostenhof for both $\epsilon \leq 0$ and $\epsilon \geq 1/2$. These L^2-lower bounds are well known to be optimal for $\epsilon \geq 1/2$ while for $\epsilon < 1/2$ this last is only known for $\epsilon = 0$ in view of an example of Meshkov. We generalize Meshkov's example for $\epsilon < 1/2$ and thus show that for complex-valued potentials our result is optimal for all $\epsilon \in \mathbf{R}$.

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1. Introduction

Let $\epsilon \in \mathbf{R}$ be given and suppose V is a measurable complex-valued function on $\mathbf{R}^n$ that satisfies

$$V(x) = O(|x|^{-\epsilon}), \qquad \text{as} \quad |x| \to \infty. \tag{1.1}$$

In this paper we investigate the fastest possible rate of decay of the solutions to

$$-\Delta\Psi + V\Psi = E\Psi, \tag{1.2}$$

on $\mathbf{R}^n$, where Δ is the Laplacian on $\mathbf{R}^n$, $n \geq 2$, and $E \in \mathbf{R}$. Without further assumptions on V we prove:

Theorem 1. *Let V, E, and ϵ be as in* (1.1) *and* (1.2) *and let* $3\delta = \max\{0, 1 - 2\epsilon\}$. *Let* $\Psi \in H^2_{\mathrm{loc}}(\mathbf{R}^n)$ *be a nonzero solution of* (1.2) *that satisfies*

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$$\exp(\tau|x|^{1+\delta})\Psi \in L^2(\mathbf{R}^n) \tag{1.3}$$

for all $\tau > 0$. *Then* Ψ *has compact support.*

We view this theorem as a *unique continuation at infinity* result and prove it through a Carleman-like estimate that generalizes work of Meshkov [**10**] for $\epsilon = 0$. A similar result for both $\epsilon \leq 0$ and $\epsilon \geq 1/2$ has been obtained using different methods by Froese, Herbst, T. Hoffmann-Ostenhof, and M. Hoffmann-Ostenhof [**8**]. It is well known that Theorem 1 is optimal for $\epsilon \geq 1/2$ while for $\epsilon < 1/2$ this last is only known for $\epsilon = 0$ in view of an example due to Meshkov [**10**]. We generalize Meshkov's example for $\epsilon < 1/2$ and prove:

Theorem 2. *Let* $\epsilon < 1/2$ *and* $\delta > 0$ *satisfy* $2\epsilon + 3\delta = 1$. *Then there exist a continuous complex-valued function* V *on* $\mathbf{R}^2$ *satisfying* (1.1), *and a* C^2*-function* Ψ *which does not have compact support and satisfies* $\Delta\Psi = V\Psi$ *on* $\mathbf{R}^2$ *and*

$$\Psi(x) = O(\exp(-\beta|x|^{1+\delta})), \qquad \text{as} \quad |x| \to \infty, \tag{1.4}$$

for some $\beta > 0$.

Thus, for complex-valued potentials Theorem 1 is optimal for all $\epsilon \in \mathbf{R}$.

The above results are closely related to the following question posed by B. Simon [**12**]. Let V be a real-valued potential and suppose that $\Psi \not\equiv 0$ satisfies $(-\Delta + V - E)\Psi = 0$ in $\mathbf{R}^n$. Is it true that if $-\Delta + V$ does not have compact resolvent, then $\exp(\tau|x|)\Psi \notin L^2(\mathbf{R}^n)$ for $\tau > 0$ sufficiently large? The answer to this question is not known but a positive response is suggested by the sharp exponential upper and lower bounds of different kinds already established for several classes of potentials [**1**, **2**, **3**, **4**, **5**, **6**, **7**, **8**]. The results for $0 < \epsilon < 1/2$ presented in this paper show that a proof of an affirmative answer to Simon's question has to use in an essential way the fact that V is real-valued, even if $V(x)$ goes to zero as $|x|$ goes to infinity.

The general strategy to prove Theorems 1 and 2 is that of Meshkov [**10**]; however, the author also benefited from [**11**]. In Section 2 we obtain Carleman-like estimates near infinity and use these to prove Theorem 1. In Section 3 we generalize Meshkov's example.

2. Carleman-like estimates in exterior domains

Theorem 1 will be derived from the following Carleman-like estimate at infinity.

Lemma 2.1. *Let* $\rho > 0$, $E \in \mathbf{R}$, *and* $\delta \geq 0$ *be given. Set* $\Omega_\rho = \{x \in \mathbf{R}^n : |x| > \rho\}$ *and fix* κ *such that* $-3\delta < \kappa \leq 2 - \delta$. *Then there exist a constant* K *independent of* τ *and* $\tau_0 > 0$ *such that for any* $v \in C_0^\infty(\Omega_\rho)$ *and* $\tau > \tau_0$ *we have*

$$\int_{\Omega_\rho} |x|^{3\delta-n+\kappa}|v(x)|^2 \exp(2\tau|x|^{1+\delta})dx \leq \frac{K}{\tau^3}\int_{\Omega_\rho} |x|^{\kappa+1-n}|(\Delta+E)v(x)|^2 \exp(2\tau|x|^{1+\delta})dx. \quad (2.1)$$

Remark. The constant E in (2.1) is important only for $0 \leq \delta < 1/3$.

Proof. Since $E \in \mathbf{R}$ we may assume without loss of generality that v is real-valued. Set $r = |x|, \omega = x/|x|$, and $\alpha = 1+\delta$. For $\tau > 0$ set $u = \exp(\tau r^\alpha)v$ and $\Delta_\tau = \exp(\tau r^\alpha)\Delta\exp(-\tau r^\alpha)$, where $\exp(\tau r^\alpha)$ and $\exp(-\tau r^\alpha)$ are multiplication operators. Using this notation we find that (2.1) is equivalent to

$$\int r^{3\delta+\kappa-1}|u|^2 dr\,d\omega \leq \frac{K}{\tau^3}\int r^\kappa |(\Delta_\tau + E)u|^2 dr\,d\omega, \quad (2.2)$$

where $d\omega$ denotes the Lebesgue measure on S^{n-1}. Defining $L = 2\tau\alpha r^{\alpha-1} + \partial_r$ we have

$$L + \Delta_\tau = \partial_{rr} + \tau^2\alpha^2 r^{2\alpha-2} + \frac{n-1}{r}\partial_r - \tau\alpha(n+\alpha-2)r^{\alpha-2} + \frac{1}{r^2}\Lambda,$$

where Λ is the Laplace-Beltrami operator on the unit sphere S^{n-1}. Setting $u_r = \partial_r u$ and integrating by parts with respect to r we obtain

$$\begin{aligned}
\int r^\kappa|(\Delta_\tau+E)u|^2 dr\,d\omega &\geq \int r^\kappa(|Lu|^2 - 2Lu(L+\Delta_\tau+E)u)dr\,d\omega \\
&= \int (4\tau^2\alpha^2 r^{2\alpha+\kappa-2} + 2\tau\alpha(\alpha+\kappa-2n+1)r^{\alpha+\kappa-2})u_r^2 dr\,d\omega \\
&\quad + 2\tau\alpha\int \big[(\tau^2\alpha^2(3\alpha+\kappa-3)r^{3\alpha+\kappa-4} \\
&\quad - \tau\alpha(2\alpha-3+\kappa)(n+\alpha-2)r^{2\alpha+\kappa-4})u^2 \\
&\quad + (\alpha+\kappa-3)r^{\alpha+\kappa-4}u\Lambda u - \alpha E(\alpha+\kappa-1)r^{\alpha+\kappa-2}u^2\big]dr\,d\omega \\
&= \int 2\tau^3\alpha^3(3\delta+\kappa)r^{3\delta+\kappa-1} \\
&\quad \left[1 - \frac{1}{\tau\alpha(3\delta+\kappa)}\left(\frac{(2\alpha+\kappa-3)(n+\alpha-2)}{r^\alpha} + \frac{E(\delta+\kappa)}{\tau\alpha r^{2\delta}}\right)\right]u^2 dr\,d\omega \\
&\quad + \int 2\tau\alpha\left[2\tau\alpha r^{\kappa+2\delta}\left(1 + \frac{\alpha+\kappa-2n+1}{2\tau\alpha r^\alpha}\right)u_r^2 + (\delta+\kappa-2)r^{\alpha+\kappa-4}u\Lambda u\right]dr\,d\omega.
\end{aligned}$$

Since $-3\delta < \kappa \leq 2-\delta$ and Λ is a negative operator on $L^2(S^{n-1}, d\omega)$ we have

$$\int r^\kappa|(\Delta_\tau+E)u|^2 dr\,d\omega \geq \int 2\tau^3\alpha^3(3\delta+\kappa)r^{3\delta+\kappa-1}u^2\left(1 + O\left(\frac{1}{\tau}\right)\right)dr\,d\omega,$$

from which (2.2) follows.

Proof of Theorem 1. Theorem 1 follows from (2.1) using standard arguments that we sketch here for the sake of completeness. Let V, E, ϵ, δ and Ψ be as in Theorem 1. Assuming that

$$\int_{\mathbf{R}^n} |\Psi(x)|^2 \exp(2\tau|x|^{1+\delta})dx < \infty \tag{2.3}$$

for all $\tau > 0$, we will prove that Ψ has compact support. Using L^2-interior estimates [**9**], it follows from (2.3) that

$$\int_{\mathbf{R}^n} |D^\beta\Psi(x)|^2 \exp(2\tau|x|^{1+\delta})dx < \infty \tag{2.4}$$

for all $\tau > 0$ and all multi-indices β such that $|\beta| \leq 2$. Let κ be as in Lemma 2.1 and fix $\rho \geq 0$ so that $|V(x|) \leq C|x|^{-\epsilon}$ for $x \in \Omega_\rho$. A simple estimate, using (2.1) and $3\delta = \max\{0, 1-2\epsilon\}$, shows that there exist a constant K independent of τ and $\tau_0 > 0$ such that for any $v \in C_0^\infty(\Omega_\rho)$ and $\tau > \tau_0$ we have

$$\int_{\Omega_\rho} |x|^{3\delta-n+\kappa}|v(x)|^2 \exp(2\tau|x|^{1+\delta})dx \leq \frac{K}{\tau^3}\int_{\Omega_\rho} |x|^{\kappa+1-n}|(\Delta - V + E)v(x)|^2 \exp(2\tau|x|^{1+\delta})dx. \tag{2.5}$$

Let h be a C^∞-function on $\mathbf{R}^n$ which takes values between 0 and 1, vanishes on $|x| \leq \rho + 1/2$, and equals 1 on $|x| \geq \rho + 1$. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ be a function which equals 1 on $|x| \leq 1$ and set $\phi_\lambda(x) = \phi(x/\lambda)$ for $\lambda > 0$. An approximation argument shows that (2.5) holds for every $v \in H^2(\mathbf{R}^n)$ with compact support contained in Ω_ρ. Hence (2.5) holds on Ω_ρ for $v_\lambda = \phi_\lambda\Psi h$ and therefore, using (2.4), for $v = \Psi h$. Since Ψ satisfies (1.2) and $h(x) = 1$ on $\Omega_{\rho+1}$ we obtain

$$\int_{\Omega_{\rho+1}} |x|^{3\delta-n+\kappa}|\Psi(x)|^2dx \leq \frac{K}{\tau^3}\int_{\rho\leq|x|\leq\rho+1} |x|^{\kappa+1-n}|(\Delta - V + E)v(x)|^2dx.$$

Letting τ go to infinity in this last estimate we find that $\Psi \equiv 0$ on $\Omega_{\rho+1}$.

3. Examples

Although the essential idea of the construction given below is that of Meshkov [**10**], we present the details for the reader's convenience. For $\rho > 0$ we will denote by $A(\alpha, \beta)$ the annulus in $\mathbf{R}^2$ defined by $\rho + \alpha\rho^{(1-\delta)/2} \leq r \leq \rho + \beta\rho^{(1-\delta)/2}$.

Lemma 3.1. *Let $\delta > 0$ and $\epsilon \in \mathbf{R}$ satisfy $2\epsilon + 3\delta = 1$. For a fixed and large $\rho > 0$ let n and k be positive integers such that $|n - \rho^{1+\delta}| \leq 1$ and $|k - 6(1+\delta)\rho^{(1+\delta)/2}| \leq 1 + 20\delta(1+\delta)$. Then there exist complex-valued functions u and V on $A(0, 6)$ possessing the following properties:*

(a) *The function u is of class C^2 and satisfies*

$$\Delta u + Vu = 0 \qquad on\ A(0,6). \tag{3.1}$$

(b) *There exists a constant C independent of ρ, n, and k such that*

$$|V(r,\theta)| \le Cr^{-\epsilon} \qquad on\ A(0,6). \tag{3.2}$$

(c) *For a constant $a > 0$ we have*

$$u(r,\theta) = \begin{cases} r^{-n}\exp(-in\theta) & on\ A(0,0.1), \\ ar^{-n-k}\exp i(-n-k)\theta & on\ A(5.9,6). \end{cases}$$

Therefore $V \equiv 0$ on the annuli $A(0,0.1)$ and $A(5.9,6)$.

(d) *Let $m(r) = \max\{|u(r,\theta)|, 0 \le \theta \le 2\pi\}$. Then*

$$\log m(r) - \log m(\rho) \le \log 2 - \frac{1}{6}\int_\rho^r t^\delta dt,$$

for $\rho \le r \le \rho + 6\rho^{(1-\delta)/2}$.

Proof of Theorem 2. First we fix a large $\rho_1 > 0$ and set $\rho_{j+1} = \rho_j + 6\rho_j^{(1-\delta)/2}$ for $j = 1, 2, \ldots$. Then we set $n_j = [\rho_j^{1+\delta}]$, where $[x] = \max\{n \in \mathbf{Z} : n \le x\}$, and $k_j = n_{j+1} - n_j$. For $j = 1, 2, \ldots$ we have $n_j = \rho_j^{1+\delta} - \gamma_j$, with $0 \le \gamma_j < 1$, and

$$\begin{aligned} k_j &= \rho_{j+1}^{1+\delta} - \rho_j^{1+\delta} + \gamma_j - \gamma_{j+1} \\ &= \rho_j^{1+\delta}(1 + 6\rho_j^{-(1+\delta)/2})^{1+\delta} - \rho_j^{1+\delta} + \gamma_j - \gamma_{j+1} \\ &= 6(1+\delta)\rho_j^{(1+\delta)/2} + 18\delta(1+\delta) + O(\rho_j^{-(1+\delta)/2}) + \gamma_j - \gamma_{j+1}. \end{aligned}$$

Therefore if ρ_1 is large we may assume that $|k_j - 6(1+\delta)\rho_j^{(1+\delta)/2}| \le 1 + 20\delta(1+\delta)$. For $j = 1, 2, \ldots$, let a_j be constants and u_j and V_j be functions constructed on $\rho_j \le r \le \rho_{j+1}$ as in Lemma 3.1. Then $u_j(\rho_j,\theta) = \rho_j^{-n_j}\exp(-in_j\theta)$ and $u_j(\rho_{j+1},\theta) = a_j\rho_{j+1}^{-n_{j+1}}\exp(-in_{j+1}\theta)$. Since $\rho_j \to \infty$ as $j \to \infty$, then for $r > \rho_1$ we set $V(r,\theta) = V_j(r,\theta)$ and $\Psi(r,\theta) = A_ju_j(r,\theta)$ for $\rho_j \le r \le \rho_{j+1}$, where we define $A_j = a_0a_1\ldots a_{j-1}$, for $j = 1, 2, \ldots$, and $a_0 = 1$. Clearly V satisfies (1.2) and Ψ is of class C^2 and satisfies $-\Delta\Psi + V\Psi = 0$ on Ω_{ρ_1}. To prove that Ψ satisfies (1.4) we set $\mu(r) = \max\{|\Psi(r,\theta)| : 0 \le \theta \le 2\pi\}$ for $r > \rho_1$ and pick l such that $\rho_l \le r \le \rho_{l+1}$. Then

$$\log\mu(r) = (\log m_l(r) - \log m_l(\rho_l) + \ldots + (\log m_1(\rho_2) - \log m_1(\rho_1)) + \log m_1(\rho_1),$$

where $m_j(r)$ is as in (d) of Lemma 3.1. Using (d) of this lemma we find that

$$\log \mu(r) \leq l \log 2 - \frac{1}{6}\int_{\rho_1}^{r} t^{\delta} dt + \log m(\rho_1).$$

Thus if $\delta > 1$ then $\log \mu(r) \leq Cr^{(1+\delta)/2} - cr^{1+\delta}$, and if $0 < \delta \leq 1$ then $\log \mu(r) \leq Cr - cr^{1+\delta}$, where C and c are positive constants. Therefore, since $\delta > 0$, for r sufficiently large we have

$$0 < \mu(r) \leq C \exp(-\beta r^{1+\delta}),$$

for some $\beta > 0$. The functions V and Ψ defined above can easily be extended to $\mathbf{R}^2$ in a way that Theorem 2 is satisfied.

Proof of Lemma 3.1. We will smoothly modify in four steps the function $u_1 = r^{-n} \exp(-in\theta)$ into a function u on $A(0, 6)$ that satisfies (a), (b), (c), and (d). In this proof C denotes a positive constant independent of ρ, k, and n.

I. The annulus $A(0, 2)$. For $m = 0, 1, \ldots, 2n + 2k - 1$ we set $\theta_m = mT$, where $T \equiv \pi/(n+k)$. Let f be a smooth T-periodic function on $\mathbf{R}$ such that $\int_0^T f(\theta)d\theta = 0$, $f(\theta) = -4k$ on $[0, T/5] \cup [4T/5, T]$, and $-4k \leq f(\theta) \leq 5k$ and $|f'(\theta)| \leq Ck/T$, for $0 \leq \theta \leq T$. Set

$$\Phi(\theta) = \int_0^{\theta} f(t)dt.$$

Clearly Φ is T- and 2π-periodic, and $\Phi(\theta_m) = 0$. In addition, for $\theta \in \mathbf{R}$ we have

$$|\Phi(\theta)| \leq 5k/(n+k), \quad |\Phi'(\theta)| \leq 5k, \quad \text{and} \quad |\Phi''(\theta)| \leq Ckn, \tag{3.3}$$

and

$$\Phi(\theta) = -4k(\theta - \theta_m) \equiv -4k\theta + b_m, \qquad \text{for } |\theta - \theta_m| \leq T/5. \tag{3.4}$$

Set $F(\theta) = (n + 2k)\theta + \Phi(\theta)$, $b = (\rho + \rho^{(1-\delta)/2})^{-2k}$, and $u_2 = -br^{-n+2k} \exp(iF(\theta))$. Note that $|u_1(r, \theta)| = |u_2(r, \theta)|$ for $r = \rho + \rho^{(1-\delta)/2}$; in addition, it follows from (3.4) that $u_2 = -br^{-(n-2k)} \exp(i(n - 2k)\theta + ib_m)$ on the sectors

$$S_m \equiv \{(r, \theta) : |\theta - \theta_m| \leq T/5\}, \qquad m = 0, 1, \ldots, 2n + 2k - 1.$$

On $A(0, 1/3)$ we have

$$\begin{aligned} \frac{|u_2(r,\theta)|}{|u_1(r,\theta)|} &= \frac{r^{2k}}{(\rho + \rho^{(1-\delta)/2})^{2k}} \\ &\leq \left(1 + \frac{2}{3}\frac{1}{\rho^{(1+\delta)/2} + \frac{1}{3}}\right)^{-2k} \end{aligned}$$

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Hence using the assumptions on k and ρ we obtain

$$|u_2(r,\theta)| \leq \exp(-8)|u_1(r,\theta)| \qquad \text{on } A(0,1/3). \tag{3.5}$$

Similarly

$$|u_2(r,\theta)| \geq \exp(8)|u_1(r,\theta)| \qquad \text{on } A(5/3,2). \tag{3.6}$$

Let $\psi_i(r)$, $i=1,2$, be C^∞-functions taking values between 0 and 1 such that ψ_1 vanishes for $r \geq \rho + 1.9\rho^{(1-\delta)/2}$ and equals 1 for $r \leq \rho + (5/3)\rho^{(1-\delta)/2}$, ψ_2 vanishes for $r \leq \rho + 0.1\rho^{(1-\delta)/2}$ and equals 1 for $r \geq \rho + (1/3)\rho^{(1-\delta)/2}$, and

$$|\psi_i^{(p)}(r)| \leq C\rho^{-p(1-\delta)/2}, \qquad r \geq 0; \quad i=1,2;\ p=1,2. \tag{3.7}$$

Define $u = \psi_1 u_1 + \psi_2 u_2$. Clearly u is harmonic in $S \equiv A(1/3,5/3) \cap (\cup S_m)$. Now set

$$V(r,\theta) = \begin{cases} 0 & (r,\theta) \in S, \\ \Delta u/u & \text{otherwise.} \end{cases}$$

Clearly (3.1) holds on $A(0,2)$. Next we show that $|u| > 0$ on $A(0,2)\backslash S$ and that (3.2) holds on $A(0,2)$.

On $A(0,1/3)$ we have

$$\Delta u = \psi_2 \Delta u_2 + 2\psi_2' \partial_r u_2 + (\psi_2'/r + \psi_2'')u_2 \tag{3.8}$$

and, using (3.5),

$$|u| \geq |u_1| - |u_2| \geq \frac{1}{2}|u_1| \geq \exp(7)|u_2| > 0. \tag{3.9}$$

Similarly on $A(5/3,2)$ we have

$$\Delta u = \Delta u_2 + 2\psi_1' \partial_r u_1 + (\psi_1'/r + \psi_1'')u_1 \tag{3.10}$$

and, using (3.6),

$$|u| \geq |u_2| - |u_1| \geq \exp(7)|u_1| > 0. \tag{3.11}$$

A short calculation shows that on $A(0,2)$ we have

$$\Delta u_2 = \left[\frac{(4k+2n+\Phi')\Phi' - 8kn}{r^2} + \frac{i\Phi''}{r^2}\right]u_2.$$

Using (3.3) we obtain

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$$|\Delta u_2| \leq \frac{Ckn}{r^2}|u_2|.$$

Thus, by the assumptions on k, n, ϵ, and δ, we have

$$|\Delta u_2| \leq Cr^{-\epsilon}|u_2| \qquad \text{on } A(0,2). \tag{3.12}$$

We also have

$$|\psi_2'\partial_r u_2| = |\psi_2'\frac{n-2k}{r}u_2| \leq C\frac{n}{r}\rho^{-(1-\delta)/2}|u_2| \leq Cr^{-\epsilon}|u_2|$$

and

$$|(\psi_2'/r + \psi_2'')u_2| \leq C(\rho^{-(1-\delta)/2}/r + \rho^{-(1-\delta)})|u_2| \leq Cr^{-\epsilon}|u_2|.$$

Combining these three last estimates, (3.8), and (3.9) we find that (3.2) holds on $A(0,1/3)$. Analogously, using (3.10), (3.11), and (3.12), we have that (3.2) holds on $A(5/3,2)$. It remains to show that $|u|>0$ and that (3.2) holds on

$$P_m = \left\{(r,\theta) : \theta_m + \frac{T}{5} \leq \theta \leq \theta_m + \frac{4T}{5}\right\} \cap A(1/3, 5/3), \quad m = 0, \ldots, 2n+2k-1.$$

For this purpose we set $G(\theta) = F(\theta) + n\theta$. On the annular sectors P_m we have

$$|u| = |u_1 + u_2| = |u_2||\exp(iG(\theta)) - \frac{1}{br^{2k}}|. \tag{3.13}$$

We will show now that for some $\eta > 0$ we have

$$|\exp(iG(\theta)) - \frac{1}{br^{2k}}| \geq \eta, \quad (r,\theta) \in P_m, \quad m = 0, \ldots, 2n+2k-1. \tag{3.14}$$

Using this last, (3.12), and the fact that $\Delta u = \Delta u_2$ on P_m we obtain $|u| > \eta|u_2|$ and therefore (3.2) holds on P_m. To prove (3.14) note that $G(\theta) = 2(n+k)\theta + \Phi(\theta)$ and $G'(\theta) = 2(n+k) + f(\theta)$. Hence by the assumptions of f, k, and n we may assume that $G'(\theta) > n > 0$. Since $G(\theta_m) = 2\pi m$ and $G(\theta_{m+1}) = 2\pi(m+1)$ we conclude that

$$2\pi m + \frac{nT}{5} \leq G(\theta) \leq 2\pi(m+1) - \frac{nT}{5} \qquad \text{for} \quad \theta_m + \frac{T}{5} \leq \theta \leq \theta_m + \frac{4T}{5}.$$

Using the definition of T and the assumptions on k and n we find that

$$2\pi m + \frac{\pi}{7} \leq G(\theta) \leq 2\pi(m+1) - \frac{\pi}{7} \qquad \text{for} \quad \theta_m + \frac{T}{5} \leq \theta \leq \theta_m + \frac{4T}{5}.$$

It follows from this last estimate and (3.13) that (3.14) holds with $\eta = \sin(\pi/7)$.

II. On $A(2,3)$ we deform u_2 into $u_3 = -br^{-n+2k}\exp i(-n+2k)\theta$. Let $\psi(r)$ be a C^∞-function which takes values between 0 and 1, equals 1 for $r \le \rho + (7/3)\rho^{(1-\delta)/2}$, vanishes for $r \ge \rho + (8/3)\rho^{(1-\delta)/2}$, and satisfies (3.7). On $A(2,3)$ we set $u = -br^{-n+2k}\exp i(\psi(r)\Phi(\theta) + (n+2k)\theta)$ and $V = \Delta u/u$. A short calculation shows that

$$\Delta u = \left[\left(\frac{-n+2k}{r} + i\psi'\Phi\right)^2 + i\Phi\left(\frac{\psi'}{r} + \psi''\right) - \frac{(n+2k+\psi\Phi')^2}{r^2} + \frac{i\psi\Phi''}{r^2}\right]u.$$

Using (3.3), the assumptions on $\psi^{(p)}$, and the assumptions on k, n, ϵ, and δ we find that $\psi'\Phi = O(1/r)$, that $\psi\Phi' = O(k)$, that $\psi\Phi'' = O(kn)$, and that $\Phi(\psi'/r + \psi'') = O(r^{-\epsilon})$. Hence

$$\Delta u = \left[\frac{O(nk)}{r^2} + O(r^{-\epsilon})\right]u.$$

Using again the assumptions on k, n, ϵ, and δ we find that (3.2) holds on $A(2,3)$.

III. On $A(3,4)$ we deform u_3 into $u_4 = -br^{-(n+2k)}\exp i(n+2k)\theta$, where b is as in I and $d \equiv (\rho + 3\rho^{(1-\delta)/2})^{4k}$. Let ψ be a C^∞-function which takes values between 0 and 1, equals 1 for $r \le \rho + (10/3)\rho^{(1-\delta)/2}$, vanishes for $r \ge \rho + (11/3)\rho^{(1-\delta)/2}$, and satisfies (3.7). Next we define $h(r) = \psi(r) + (1-\psi(r))dr^{-4k}$. It is easily verified using the assumptions on ψ, k, and δ that h satisfies (3.7) and that

$$h(r) \ge dr^{-4k} \ge \left(1 + \frac{1}{\rho^{(1+\delta)/2} + 1}\right)^{-4k} \ge \exp(-25(1+\delta)).$$

Now we set $u = u_3 h$ and $V = \Delta u/u$, and verify as above that (3.2) holds on $A(3,4)$. In addition, on $A(11/2, 4)$ we have $u = -bdr^{-(n+2k)}\exp i(n+2k)\theta$.

IV. Finally on $A(4,6)$ we deform u_4 into $u_5 = ar^{-n-k}\exp i(-n-k)\theta$, where $a \equiv bd(\rho + 5\rho^{(1-\delta)/2})^{-k}$ and b and d are as in III. Note that a has been chosen so that $|u_4(r,\theta)| = |u_5(r,\theta)|$ for $r = \rho + 5\rho^{(1-\delta)/2}$. Let $\psi_i(r)$, $i = 1, 2$, be C^∞-functions taking values between 0 and 1 and satisfying (3.7), such that ψ_1 vanishes for $r \ge \rho + 5.9\rho^{(1-\delta)/2}$ and equals 1 for $r \le \rho + (17/3)\rho^{(1-\delta)/2}$, and ψ_2 vanishes for $r \le \rho + 4.1\rho^{(1-\delta)/2}$ and equals 1 for $r \ge \rho + (13/3)\rho^{(1-\delta)/2}$. Now on $A(4,6)$ we set $u = \psi_1 u_4 + \psi_2 u_5$. It is clear that u is harmonic on $A(13/3, 17/3)$. Therefore we set $V = 0$ on this annulus. We verify as in I that $V = \Delta u/u$ satisfies (3.2) on the remaining points of $A(4,6)$.

To finish this proof we set $m(r) = \max\{|u(r,\theta)|, 0 \le \theta \le 2p\}$ and

$$M(r) = \begin{cases} r^{-n} & \rho \le r \le \rho + \rho^{(1-\delta)/2}, \\ br^{-n+2k} & \rho + \rho^{(1-\delta)/2} \le r \le \rho + 3\rho^{(1-\delta)/2}, \\ br^{-n+2k}h(r) & \rho + 3\rho^{(1-\delta)/2} \le r \le \rho + 4\rho^{(1-\delta)/2}, \\ bdr^{-n-2k} & \rho + 4\rho^{(1-\delta)/2} \le r \le \rho + 5\rho^{(1-\delta)/2}, \\ ar^{-n-k} & \rho + 5\rho^{(1-\delta)/2} \le r \le \rho + 6\rho^{(1-\delta)/2}. \end{cases}$$

where a, b, and d are as in IV. It is clear that $M(r)$ is a continuous piecewise smooth function on $[\rho, \rho + 6\rho^{(1-\delta)/2}]$ that satisfies $m(r) \leq 2M(r)$, $m(\rho) = M(\rho)$, and

$$\frac{d}{dr}\log M(r) = \frac{-n + O(k)}{r} \leq -\frac{\rho^{1+\delta}}{2r} \leq -\frac{1}{6}(\rho + 6\rho^{(1-\delta)/2})^{\delta} \leq -\frac{1}{6}r^{\delta}.$$

Therefore

$$\log m(r) - \log m(\rho) \leq \log 2 + \int_{\rho}^{r} \frac{d}{dr}\log M(t)dt \leq \log 2 - \frac{1}{6}\int_{\delta}^{r} dt,$$

which proves Lemma 3.1.

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