# A NOTE ON COMMUTATIVE *l*-GROUPS

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(Received 25 June 1969)

Communicated by B. Mond

## Introduction

Let $G$ be a commutative lattice ordered group. Theorem 1 gives necessary and sufficient conditions under which $a^{\perp}$ with $a \in G$ is a maximal $l$-ideal. A wide family of $l$-groups $G$ having the property that the orthogonal complement of each atom is a maximal $l$-ideal is described. Conditionally $\sigma$-complete and hence conditionally complete vector lattices belong to the family. It follows immediately that if $a$ is an atom in a conditionally complete vector lattice then $a^{\perp}$ is a maximal vector lattice ideal. This theorem has been proved in [7] by Yamamuro. Theorem 2 generalizes another result contained in [7]. Namely we prove that if $M$ is a closed maximal $l$-ideal of an archimedean $l$-group $G$ then there exists an atom $a \in G$ such that $M = a^{\perp}$.

## 1. Notations and supplementary results

In a commutative $l$-group $G$ with $a \in G$, we write $G^+$ for the set of positive elements; $(a)$ for the $l$-ideal generated by $a$, i.e. $(a) = \{g \in G : |g| \leqq n|a| \text{ for some } n\}$; $(A)$ will denote the $l$-ideal generated by a subset $A$ of $G$. Two elements $g_1$, $g_2 \in G$ are said to be disjoint (written $g_1 \perp g_2$) if $|g_1| \wedge |g_2| = 0$. We put $a^{\perp} = \{g \in G : |g| \wedge |a| = 0\}$. It is well known that $a^{\perp}$ is an $l$-ideal. It follows easily that $(a) \cap a^{\perp} = \{0\}$. Further if $A$ is a subset of $G$ then $A^{\perp}$ is defined by $A^{\perp} = \cap \{a^{\perp} : a \in A\}$ and $A^{\perp\perp}$ means $(A^{\perp})^{\perp}$. In case $G = A^{\perp} \oplus A^{\perp\perp}$ the following properties of projections $p_1$ and $p_2$ onto $A^{\perp}$ and $A^{\perp\perp}$ respectively are easily proved:

(i) if $g \geqq 0$ then $p_1(g) \geqq 0$ and $p_2(g) \geqq 0$,

(ii) $p_i(a+b) = p_i(a)+p_i(b)$ for $i = 1, 2$

and

(iii) $p_i(na) = np_i(a)$, $i = 1, 2$.

To obtain these results it is sufficient to bear in mind that any $l$-group is a distributive lattice and that $g_1+g_2 = g_1 \vee g_2 + g_1 \wedge g_2$. For other concepts used and not defined we refer to Birkhoff [1].

## 2. Discrete archimedean elements and maximal *l*-ideals

Definition 1. An element $a \in G$ is said to be discrete [6] if the conditions $0 \leqq g_1 \leqq |a|$, $0 \leqq g_2 \leqq |a|$, $g_1 \perp g_2$ imply that at least one of the elements $g_1$ and $g_2$ equals zero.

Definition 2. A non-zero element $a \in G$ is said to be archimedean if for any $0 \neq g \in G^+$ there exist natural numbers $n_1$ and $n_2$ (depending on $g$) such that $n_1 g \nless |a|$ ($n_1 g$ is not less than $|a|$) and $n_2|a| \nless g$.

Remark. It is quite obvious that an *l*-group $G$ is archimedean if and only if each of its non-zero elements is archemidean.

Lemma 1. *The following statements are equivalent:*

(i) *$a \in G$ is discrete,*

(ii) *if $g_1 \perp g_2$ then at least one of them belongs to $a^\perp$.*

Proof. Let a be discrete and let $g_1 \perp g_2$. In this case $b_1 = |a| \wedge |g_1|$ and $b_2 = |a| \wedge |g_2|$ are disjoint positive elements dominated by $|a|$. Thus, by definition 1, at least one of them equals zero.

Conversely, suppose that $0 \leqq g_1 \leqq |a|$, $0 \leqq g_2 \leqq |a|$ and that $g_1 \perp g_2$. According to (ii), we may assume that e.g. $g_1 \in a^\perp$, i.e. $g_1 \wedge |a| = 0$. But $g_1 \wedge |a| = g_1$ since $g_1 \leqq |a|$. Thus $g_1 = 0$.

Lemma 2. *If $a \in G$ is discrete then the l-ideal $(a)$ generated by $a$ is totally ordered.*

Proof. If $g \in (a)$ then $g^+$ and $g^-$ belong also to $(a)$. But $g^+ \perp g^-$ and so, by lemma 1, at least one of the elements $g^+$ and $g^-$ belongs to $a^\perp$. If e.g. $g^- \in a^\perp$, then $g^- \in (a) \cap a^\perp$ and hence $g^- = 0$. Thus, in this case $g = g^+ - g^- = g^+ \geqq 0$.

Lemma 3. *If $a \in G$ is a discrete archimedean element then $(a)$ is generated by any of its non-zero elements.*

Proof. Let $g \in (a)$ and $g \neq 0$. Since $a$ is archimedean and $|g| > 0$, there exists $n$ such that $n|g| \nless |a|$. Since $n|g| \in (a)$ and $(a)$ is totally ordered, by lemma 2, $|a| \leqq n|g|$. So $(a) \subseteq (g) \subseteq (a)$ and thus $(g) = (a)$.

Lemma 4. *If $a \in G$ is archimedean and discrete then*

$$G = (a) \oplus a^\perp.$$

Proof. Since $a$ is archimedean, for any $g \in G^+$ there exists $n$ such that $n|a| \nleqq g$. Consider the elements $b_1 = (n|a| - g)^+$ and $b_2 = (n|a| - g)^-$. Since $n|a| \nleqq g$, it follows that $b_1 > 0$. On the other hand $b_1 \leqq n|a|$ and hence $b_1 \in (a)$. Now $b_1 \in (a)$ and $b_1 \neq 0$ imply that $b_1 \notin a^\perp$. Taking into account that $b_2 \perp b_1$ and that $a$ is discrete, by lemma 1, we infer that $b_2 \in a^\perp$. Thus

$$n|a| - g = b_1 - b_2 \in (a) \oplus a^{\perp}.$$

But $$n|a| \in (a) \subseteq (a) \oplus a^{\perp},$$

and so $$g \in (a) \oplus a^{\perp}.$$

For an arbitrary $g \in G$ we have $g = g^+ - g^-$ with $g^+, g^- \in (a) \oplus a^{\perp}$. Thus $g \in (a) \oplus a^{\perp}$ and so $(a) \oplus a^{\perp} = G$.

THEOREM 1. *For an element a belonging to a commutative l-group G the following statements are equivalent*:

(i) *a is archimedean and discrete,*

(ii) *$a^{\perp}$ is a maximal l-ideal.*

PROOF OF (i) ⇒ (ii). $a \neq 0$, by definition 2, so $a \notin a^{\perp}$ and hence $a^{\perp}$ is a proper $l$-ideal. Suppose that $M$ is an $l$-ideal of $G$ property containing $a^{\perp}$. Let $b \in M \backslash a^{\perp}$. Then $b_1 = |b| \in M \backslash a^{\perp}$. Since $b_1 \notin a^{\perp}$, $c = b_1 \wedge |a| > 0$. So, $0 < c \leqq |a|$ and thus, by lemma 3 and 4,

$$G = (a) \oplus a^{\perp} = (c) \oplus a^{\perp} \subseteq M.$$

Consequently, $M = G$ and therefore $a^{\perp}$ is maximal.

PROOF OF (ii) ⇒ (i). Since $a^{\perp}$ is maximal and thus a proper ideal, it follows immediately that $a \neq 0$. Assume that $g_1 \perp g_2$, $0 < g_1 \leqq |a|$ and $0 \leqq g_2 \leqq |a|$. In this case the $l$-ideal $J = (g_2, a^{\perp})$ generated by $g_2$ and $a^{\perp}$ is proper because $g_1 \notin J$. Since $a^{\perp}$ is maximal and $a^{\perp} \subseteq J$, it follows that $J = a^{\perp}$. Consequently, $g_2 \in a^{\perp}$. Hence $g_2 \in (a) \cap a^{\perp}$ and so $g_2 = 0$. Thus a is discrete whenever $a^{\perp}$ is maximal.

Let us assume now that there exists an element $0 < g \in G^+$ such that $ng < |a|$ for each natural $n$. It is easy to see that the ideal $J = (g, a^{\perp})$ generated by $g$ and $a^{\perp}$ is a proper ideal ($a \notin J$) properly containing $a^{\perp}$ ($g \in J$, but $g \notin a^{\perp}$). This is impossible, since $a^{\perp}$ is maximal.

Finally, suppose that there exists $g \in G^+$ such that $n|a| < g$ for all natural $n$. In this case again we obtain a contradiction because the ideal $(a, a^{\perp})$ generated by $a$ and $a^{\perp}$ is a proper ideal properly containing $a^{\perp}$. Hence $a$ is archimedean whenever $a^{\perp}$ is maximal.

## 3. Applications

DEFINITION 3. A commutative $l$-group $G$ is said to be Stone if $G = g^{\perp} \oplus g^{\perp\perp}$ for any $g \in G$.

DEFINITION 4. An element $a \in G$ is said to be an atom [7] if the conditions: $|a| = g_1 + g_2$, $g_1 \perp g_2$, $g_1, g_2 \in G^+$ imply that one of elements $g_1$, $g_2$ equals zero.

REMARK (i). Observe that the element 0 satisfies both the definitions of a discrete element and of an atom – this seems unnecessary, but we do not wish to cause confusion by deviating from the definitions in [6] and [7].

REMARK (ii). Comparing definitions 1 and 4 we conclude that every discrete element $a \in G$ is an atom. The converse is in general not true (see example 1 in the last part of the paper). Nevertheless if $G$ is Stone then the following holds:

LEMMA 5. *An element a of a Stone l-group G is an atom if and only if a is discrete.*

PROOF. According to the preceding remark it suffices to prove that if $a$ is atomic and $G$ is Stone then a is discrete. Since $a$ is discrete whenever $|a|$ is discrete, we may restrict ourselves to the case when $a > 0$.

Suppose that $g_1, g_2 \in G^+$, $g_1 \perp g_2$, $g_1 > 0$ and both are dominated by an atomic element $a$. $G$ is Stone, and so, by definition 3, $G = g_1^\perp \oplus g_1^{\perp\perp}$. Let $p_1$ and $p_2$ denote the projections on $g_1^{\perp\perp}$ and $g_1^\perp$ respectively. We have then $a = p_1(a) + p_2(a)$ with $p_1(a) \perp p_2(a)$ and since $a > 0$, $p_1(a), p_2(a) \in G^+$. Thus definition 4 implies that either $p_1(a) = 0$ or $p_2(a) = 0$. But $0 < g_1 \leqq a$ and thus, by the properties of projections, $0 < g_1 = p_1(g_1) \leqq p_1(a)$. Therefore $p_2(a) = 0$. On the other hand in view of $g_2 \perp g_1$ we obtain $0 \leqq g_2 = p_2(g_2) \leqq p_2(a) = 0$. So $g_2 = 0$. Consequently, $a$ is a discrete element as required.

As a consequence of lemma 5 and theorem 1 we obtain

THEOREM 2. *If a is a non-zero archimedean atom of a Stone l-group G then $a^\perp$ is a maximal l-ideal of G.*

THEOREM 3. *Every σ-complete (and a fortiori every complete) l-group G is archimedean Stone l-group.*

PROOF. A direct proof of Theorem 3 will be given soon in [5]. It can also be easily deduced from known results.

Combining theorems 2 and 3 we obtain

COROLLARY. *If a is a non-zero atom of a complete vector lattice E then $a^\perp$ is a maximal l-ideal.*

This proposition has been proved by S. Yamamuro in [7]. Lemma 3 of the same paper states that if $M$ is a closed maximal ideal of a complete vector lattice $E$, then there exists an atomic element $a \in E$ such that $M = a^\perp$. This statement may be essentially generalised. Namely we are able to prove:

THEOREM 4. *If M is a closed maximal l-ideal of an archimedean l-group G then there exists an atom $a \in G$ such that $M = a^\perp$.*

PROOF. The fact that $M$ is closed $l$-ideal in an archimedean $l$-group implies, by Johnson and Kist [3] (see also Conrad and McAllister [2]) that $M = M^{\perp\perp}$.

https://doi.org/10.1017/S1446788700008314 Published online by Cambridge University Press

Thus $M^{\perp} \neq \{0\}$ and there is an $a > 0$ in $M^{\perp}$. For this $a$ we have $a^{\perp} \supseteq M^{\perp\perp} = M$ with $a \notin a^{\perp}$. Thus the maximality of $M$ implies $a^{\perp} = M$. So, by Theorem 1 $a$ is discrete and hence $a$ is an atom.

Repeating the reason from [7], we obtain

COROLLARY. *If G is an archimedean Stone l-group then G is atomic (the set of atoms is dense in G) if and only if the intersection of all closed maximal l-ideals of G equals zero, and G is non-atomic (there exist no atoms in G) if and only if there exist no closed maximal l-ideals in G.*

## 4. Examples

1. Let $E = C[0, 1]$. The function $a \in E$:

$$a(t) = \begin{cases} 0 & \text{for } 0 \leqq t \leqq \frac{1}{2}, \\ t - \frac{1}{2} & \text{for } \frac{1}{2} < t \leqq 1 \end{cases}$$

is an atom but it is not a discrete element. Thus, according to theorem 1, $a^{\perp}$ is not maximal. Since $C[0, 1]$ is archimedean, theorem 2 implies that $C[0, 1]$ is not Stone.

2. Consider $R^2$ 'lexicographically' ordered, e.g. $(x, y) \geqq 0$ iff (i) $x > 0$ or (ii) $x = 0$, $y \geqq 0$. This space is totally ordered and hence every element $a \in R^2$ is an atom. On the other hand for any $0 \neq a \in R^2$ we have $a^{\perp} = \{0\}$ and thus for no atom $a$ of $R^2$ is $a^{\perp}$ maximal. This is so since no $a \in R^2$ is archimedean. The space in question is a Stone (non-archimedean) $l$-group. The ideal $M = \{(x, y) \in R^2 : x = 0\}$ is a maximal closed $l$-ideal, but as it was mentioned there exists no atom $a \in R^2$ such that $M = a^{\perp}$. This example shows that the condition that $G$ is archimedean is essential in theorem 4.

3. Let $E = C[0, 1] \times R \times R^2$ with $R^2$ ordered as in example 2. An element $(x, y, z) \in E$ (with $x \in C[0, 1]$, $y \in R$ and $z \in R^2$) is said to be positive iff $x \geqq 0$, $y \geqq 0$, $z \geqq 0$. $E$ is non-Stone and non-archimedean vector lattice. Nevertheless the element $(0, a, 0)$ with $a > 0$ is an atom of $E$ and $(0, a, 0)^{\perp} = \{(x, 0, z) : x \in C[0, 1], z \in R^2\}$ is a maximal $l$-ideal of $E$. This is so because $(0, a, 0)$ is a discrete archimedean element of $E$.

4. Let $S$ be the $l$-group (in fact vector lattice) of all equivalence classes of simple functions defined on a totally $\sigma$-finite measure space $(X, \mathscr{X}, \mu)$. Then results of Masterson [4] pp. 469–470 imply that $S$ is an archimedean Stone $l$-group which is not $\sigma$-complete.

This example shows that theorem 2 is an essential generalization of theorem 2 in [7].

## References

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