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GEOMETRIC MAPPINGS ON GEOMETRIC LATTICES

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1. Introduction. It is a classical result of mathematics that there is an intimate connection between linear algebra and projective or affine geometry. Thus, many algebraic results can be given a geometric interpretation, and geometric theorems can quite often be proved more easily by algebraic methods. In this paper we apply topological ideas to geometric lattices, structures which provide the framework for the study of abstract linear independence, and obtain affine geometry from the mappings that preserve the closure operator that is associated with these lattices. These mappings are closely connected with semi-linear transformations on a vector space, and thus linear algebra and affine geometry are derived from the study of a certain closure operator and mappings which preserve it, even if the "space" is finite.

2. Preliminary notions. We present here some basic definitions and ideas without proof. See [**2–5**] for more details. Let S be a non-empty set. By a closure operator on S we mean a mapping $X \to \bar{X}$, where X and $\bar{X}$ are subsets of S, satisfying the following properties:

(1) $X \subseteq \bar{X}$,

(2) If $X \subseteq Y$, then $\bar{X} \subseteq \bar{Y}$,

(3) $(\bar{X})^- = \bar{X}$.

Observe that $\overline{X \cup Y}$ and $\bar{X} \cup \bar{Y}$ are not necessarily equal. We say that X is *closed* if and only if $X = \bar{X}$. The closed sets of a closure space form a complete lattice with meet corresponding to intersection. The closure space $(S, -)$ is said to be a *combinatorial geometry* if it satisfies the following extra conditions:

(4) For any elements $a, b \in S$ and for any subset $X \subseteq S$, if $a \in \overline{X \cup b}$ and $a \notin \bar{X}$, then $b \in \overline{X \cup a}$ (Exchange property),

(5) $\bar{\emptyset} = \emptyset$, and $\bar{x} = x$ for every $x \in X$,

(6) If $y \in \bar{X}$, then $y \in \bar{X}_F$, where X_F is a finite subset of X.

The geometry will be said to have a *finite basis* if and only if:

(7) Any subset $X \subseteq S$ has a finite subset $X_F \subseteq X$ such that $\bar{X}_F = \bar{X}$.

Observe that (7) implies (6). If $y \in \bar{X}$, then we say that *y depends* on X.

We shall denote a lattice by $(L, +, \cdot)$, with 0 and I the minimum and maximum elements, respectively, if they exist. The element *b covers c* ($b \succ c$) if and only if $b > c$ and there is no x with $b > x > c$. A *point* or *atom* is an element which covers 0, and a *hyperplane* or *coatom* or *copoint* is an element covered by I. A lattice L is *geometric* if and only if it has the following properties:

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(8) L is complete,
(9) Every element in L is the join of points,
(10) If p is a point and $p \nleqq b$, then $p + b$ covers b,
(11) If $p \leqq \sum_{\alpha \in A} p_\alpha$, where p and the p_α are points, then there exists a finite subset B of A such that $p \leqq \sum_{\beta \in B} p_\beta$.

The lattice is of *finite length* or *dimension* if and only if it has no infinite chains.

We have the following basic results. The lattice of closed sets of a combinatorial geometry is geometric, and conversely, if L is a geometric lattice, then the equation $\bar{A} = \{p \in P | \, p \leqq \sum A\}$ defined for all subsets A of the set P of points of L defines a geometry $G(P)$ on P. The lattice of closed sets of $G(P)$ is isomorphic to L, and L is of finite length if and only if $G(P)$ has a finite basis.

In a geometric lattice, if one maximal chain between elements is finite, then all maximal chains are finite and have the same length. This enables us to define an integer-valued dimension function $D(x)$ on a geometric lattice of finite length which has the following properties:

(12) $D(0) = 0$,
(13) If $a > b$, then $D(a) > D(b)$,
(14) $D(a) = D(b) + 1$ if a covers b,
(15) $D(a) + D(b) \geqq D(a + b) + D(ab)$.

We write $(a, b)M$ and say that the pair (a, b) is *modular* if and only if $(c + a)b = c + ab$ for every $c \leqq b$. In a geometric lattice this relation is symmetric, and in a modular lattice it is universal. Its significance is that it implies that the mapping $x \rightarrow x + a$, where $x \in [ab, b]$, is one-to-one and preserves the covering condition. In a geometric lattice of finite length, equality in (15) is equivalent to $(a, b)M$. If $(a, b)M$ for all b, then we write aM and say that a is a modular element (M-element). The hyperplane h is modular if and only if for every $z \in L$, z covers or is equal to hz. A geometric lattice is said to be *special* if and only if $ab \neq 0$ implies that $(a, b)M$. We write $(a, b)\perp$ if and only if $(a, b)M$ and $ab = 0$ and say that a and b are *independent*. For a geometric lattice, if $y \leqq z \leqq x$, then there exists w such that $(z, w)M$, $zw = y$, $z + w = x$, and we say that w is an independent complement of z within $[y, x]$. A set B of points is said to be independent if and only if $b \notin \overline{B - b}$ for every $b \in B$, and B is a basis if and only if it is independent and its closure is all of S. Every two bases have the same cardinality, and any independent set can be extended to a basis. An independent set of points generates a sublattice which is isomorphic to the Boolean algebra on the set of points.

As examples of geometries, we have the one obtained from a vector space V by throwing away 0, identifying two vectors which differ by a non-zero multiple (so that points are 1-dimensional subspaces), and using linear dependence as our closure operator. The resulting lattice is the lattice of subspaces of the vector space V, also known as projective geometry. We get another example by using *all* of the vectors and defining dependence as affine

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dependence, i.e., $x = \sum_i \alpha_i x_i$, where $\sum_i \alpha_i = 1$. This gives us the lattice of flats, i.e., points, lines, planes, etc., (not necessarily through the origin) and is known as affine geometry. The aforementioned references have many more examples.

3. Geometric mappings.

Definition 1. Let L_1 and L_2 be geometric lattices with S_1 and S_2 their corresponding geometries. A mapping f from the points of S_1 *onto* the points of S_2 is said to be *geometric* when and only when it sends closed sets onto closed sets and the inverse image of a closed set is closed. Thus f is a special kind of "continuous" function.

PROPOSITION 1. *A mapping f from S_1 onto S_2 is geometric if and only if $\overline{f(X)} = f(\bar{X})$ for every $X \subseteq S_1$.*

Proof. Suppose that f is geometric. Since $\bar{X}$ is a closed set, $f(\bar{X})$ is closed; thus since $f(X) \subseteq f(\bar{X})$, $\overline{f(X)} \subseteq \overline{f(\bar{X})} = f(\bar{X})$. If we let $\overline{f(X)} = W$, then $f^{-1}(W) = T$ is closed because W is closed. Since W contains $f(X)$, $X \subseteq T$ and thus $\bar{X} \subseteq \bar{T} = T$. Hence $f(\bar{X}) \subseteq f(T) = W = \overline{f(X)}$.

Conversely, suppose that $\overline{f(X)} = f(\bar{X})$ for every $X \subseteq S_1$. If X is closed, then $\overline{f(X)} = f(X)$ which implies that $f(X)$ is closed. Suppose now that W is a closed set of points in S_2. Consider $f^{-1}(W) = Y$. Now $\overline{f(Y)} = f(\bar{Y})$ and since $f(Y) = W$, $\bar{W} = f(\bar{Y})$. But since W is closed, we have $W = f(\bar{Y})$ which from the definition of Y implies that $\bar{Y} \subseteq Y$, and so Y is closed.

The mapping $f: S_1 \to S_2$ can be extended in a natural way to a mapping from L_1 onto L_2 as follows.

Definition 2. If $x \in L_1$ and P is the set of points $\leqq x$, then $f(x) \equiv \sum f(P)$.

PROPOSITION 2. $f(\sum_\alpha x_\alpha) = \sum_\alpha f(x_\alpha)$, *$f$ a geometric mapping*, $\alpha \in \Lambda$.

Proof. It is obvious that $f(\sum_\alpha x_\alpha) \geqq \sum_\alpha f(x_\alpha)$. Let A_α be the closed set corresponding to x_α. Define $x = \sum_\alpha x_\alpha$ and $A = \overline{\bigcup A_\alpha}$. A is the closed subset corresponding to x. Let B be the closed subset corresponding to $\sum_\alpha f(x_\alpha)$. The set $f^{-1}(B)$ is a closed subset of S_1. Since $f(x_{\alpha'}) \leqq \sum_\alpha f(x_\alpha)$ for each α', $f(A_{\alpha'}) \subseteq B$. Thus $A_\alpha \subseteq f^{-1}(B)$ for each α and therefore $\overline{\bigcup A_\alpha} \subseteq f^{-1}(B)$. Thus $f(A) \subseteq B$, and so $\sum f(A) \leqq \sum B$ or $f(x) \leqq \sum_\alpha f(x_\alpha)$.

PROPOSITION 3. *Under a geometric mapping, the image of a basis spans the image lattice.*

Proof. Let X be a basis for S_1. Then $S_1 = \bar{X}$. Since $f(\bar{X}) \subseteq \overline{f(X)}$,

$$f(S_1) = S_2 \subseteq \overline{f(X)}.$$

COROLLARY 1. *The dimension of the image lattice under a geometric mapping is $\leqq$ the dimension of the preimage.*

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COROLLARY 2. *If the dimension of the image lattice is the same as that of the preimage, each being of finite dimension, then the two lattices are isomorphic.*

Proof. It suffices to show that the mapping is one-to-one. Let p' and p'' be distinct points of L_1. Since the set $\{p', p''\}$ is independent, we can extend it to a basis B which is finite. By Proposition 3, $f(B)$ spans S_2, and since the dimension of S_2 is the same as that of S_1, $f(B)$ must have the same cardinality as that of B and must be independent. Hence $f(p') \neq f(p'')$, and thus f is one-to-one.

Propositions 2 and 3 are true merely if $f(\bar{X}) \subseteq \overline{f(X)}$, i.e., the inverse image of a closed set is closed. A mapping with this latter property is called a strong map (see [**3**]). The notion of a geometric mapping is much stronger than that of a strong map. A geometric lattice is a complete, atomic Boolean algebra if and only if all sets are closed in the corresponding geometry. Thus an image of such a Boolean algebra under a geometric mapping must be a Boolean algebra. But every geometric lattice is the image of a Boolean algebra under a strong map: merely consider the given geometry as an image of the geometry where every set is closed.

We know from the theory of vector spaces that a homomorphic mage of a vector space is isomorphic to a subspace. A similar result holds for geometric mappings. We first need a lemma.

LEMMA 1. *If B is a basis for S_1, $f(B)$ is a basis of S_2, and f is one-to-one on B, then f is one-to-one on S_1.*

Proof. Let $p \in S_1$, $p \notin B$. Suppose that $f(p) = f(b_0)$ for some $b_0 \in B$. There exists a finite set of points $\{b_0, b_1, \ldots, b_n\} \in B$ such that $p \in \overline{\{b_0, b_1, \ldots, b_n\}}$ but $p \notin \overline{\{b_0, b_1, \ldots, b_{n-1}\}}$. Thus $b_n \in \overline{\{p, b_0, \ldots, b_{n-1}\}}$, and therefore $f(b_n) \in \overline{\{f(p), f(b_0), \ldots, f(b_{n-1})\}}$. But since $f(p) = f(b_0)$, this says that $f(b_n) \in \overline{\{f(b_0), \ldots, f(b_{n-1})\}}$ which is impossible since $f(B)$ is independent. Hence $f(p) \neq f(b)$ for any $b \in B$.

We can replace some $b \in B$, say b_i, by p to obtain a new basis B' of S_1. If $f(p) \in \overline{f(B - b_i)}$, then $\overline{f(B - b_i)} = \overline{f(B')} \supseteq f(\overline{B'}) = f(S_1) = S_2$. Thus $f(b_i) \in \overline{f(B - b_i)}$ which is false since $f(B)$ is independent. Hence $f(B')$ is independent, and thus it is a basis since B' spans S_1. Moreover, f is one-to-one on B'. Thus if $q \in S_1$, $q \notin B'$, with $f(q) = f(p)$, then we obtain a contradiction by applying the preceding proof to B' instead of B.

Remark 1. The above proof shows that Lemma 1 holds if f is merely a strong map. In the finite-dimensional case, Lemma 1 is an immediate consequence of Corollary 2.

THEOREM 1. *Let f be a geometric mapping from L_1 onto L_2. Then L_1 contains an interval $[0, t]$ which is isomorphic to L_2 under the mapping f.*

Proof. Let B_1 be a basis for L_1. By Proposition 3, $f(B_1)$ spans L_2. We choose a basis B_2 for L_2 contained in $f(B_1)$. There is a subset $A \subseteq B_1$ such that $f(A) = B_2$ with f one-to-one on A. Evidently, A is an independent set of

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points because B_1 is independent. If we consider $\bar{A}$ which determines an interval $[0, t]$ in L_1, then we have $f(\bar{A}) = \overline{f(A)} = \bar{B}_2 = S_2$. But A is a basis for $\bar{A}$, $f(A)$ is a basis of S_2, and f is one-to-one on A. By Lemma 1, f is one-to-one on $\bar{A}$, and thus $[0, t]$ and L_2 are isomorphic.

Remark 2. The discussion about Boolean algebras preceding Lemma 1 shows that Theorem 1 cannot hold, in general, if f is merely strong.

We shall now consider some geometric mappings on geometric lattices which are non-trivial. As we have already pointed out, Boolean geometric lattices have many geometric mappings because *any* function onto a subinterval is a geometric mapping. If L is a geometric lattice with two elements a and b such that $a + b = I, ab = 0$, and every point is $\leqq a$ or b, then we can define a geometric mapping by mapping all points under a and b, respectively, onto two different points. Thus the image lattice is the Boolean algebra on a two-element set. L can be irreducible, i.e., not a direct union, for example, the affine geometry on GF(2). Any linear transformation on a vector space defines a geometric mapping on the corresponding affine space. In particular, projections are geometric mappings. As another example we have the lattice consisting of the points lying on the planes $x = 0$, $y = 0$, $y = x$, $y = -x$ and all of the flats generated by these points by joining. The projection of these points upon the points lying in the plane $z = 0$ constitutes a geometric mapping of the lattice upon one of its intervals. Notice that this lattice is not special because two planes can meet in a point, and that every line contains at least three points.

Definition 3. By the *kernel* of a geometric mapping we mean the partition determined by the mapping, that is, two elements are equivalent if they have the same image.

Observe that in the case of linear transformations the blocks of the kernel have the same dimension, but this need not be the case in our example dealing with the complementary elements a and b. We analyse this situation further by proving the following theorem which is fundamental to the results that follow.

THEOREM 2. *Let L be a geometric lattice such that every two hyperplanes have a common complement. Let f be a geometric mapping from L onto L' with r' a point of L'. If $m \neq I$ is the maximum element in L such that $f(m) = r'$, then there exists a set $\{p_i\}$ of points in L such that $\{p_i\}$ is independent, f is one-to-one on $\{p_i\}$, $\{f(p_i)\}$ is independent, $(m, \sum_i p_i)\perp$, $(f(m), \sum_i f(p_i))\perp$, and $m + \sum_i p_i = I$.*

Note. Recall that if two hyperplanes have a common complement which is not an atom, then they must have another common complement which is an atom.

Proof. Let $\{p_k\}$ be a maximal set of points in L such that $\{p_k\}$ is independent, f is one-to-one on $\{p_k\}$, $\{f(p_k)\}$ is independent, $(m, \sum_k p_k)\perp$, and $(f(m),$

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$\sum_k f(p_k))\perp$. If $m + \sum p_k = I$, then the proof is complete. Suppose then that $m + \sum p_k \neq I$. Let w be the maximum element such that

$$f(w) = f(\sum p_k) = \sum f(p_k).$$

It is clear that $w \neq I$ since $(f(m), \sum f(p_k))\perp$. By hypothesis, there exists a point $q \in L$ such that $q \nleqq w$, $m + \sum p_k$, and since m and w are maximum elements that are mapped onto closed sets, $f(q) \nleqq f(w)$, $f(q) \neq f(m)$. If $f(q) \leqq f(m) + \sum f(p_k)$, then $f(m) \leqq f(q) + \sum f(p_k) = f(q + \sum p_k)$ because $f(q) \nleqq \sum f(p_k) = f(w)$ (recall that $f(m)$ and $f(q)$ are points). Since f maps closed sets onto closed sets, there exists a point $t \leqq q + \sum p_k$ such that $f(t) = f(m)$. From the definition of m, $t \leqq m$. But since $(q, m + \sum p_k)\perp$ and $(m, \sum p_k)\perp$, $m(q + \sum p_k) = 0$ which contradicts the fact that

$$t \leqq m(q + \sum p_k).$$

Thus $m + \sum p_k = I$, and the proof is complete.

Remark 3. The proof of the above theorem combined with Lemma 1 shows that $[0, p + \sum p_i]$ is isomorphic to L', where p is any point in m.

COROLLARY 3. *If L is of finite length, then* $D(I) - D(m) = D(I') - 1$.

Remark 4. This result is the analogue of the result in vector space theory that the rank + nullity = dimension of space. The equation reads slightly differently because our dimension for a flat is one higher than the dimension for a flat in vector space theory.

COROLLARY 4. *If L is of finite length, then any two maximum elements which are mapped onto points have the same dimension.*

A stronger result will be proved below.

COROLLARY 5. *Suppose that m is a maximum element mapped onto a point and that p is a point contained in m. If n is an independent complement of m, then f is one-to-one on* $[0, n + p]$, *and* $[0, n + p]$ *is isomorphic to* L'.

Proof. We have $f(n + p) = f(n) + f(p) = f(n) + f(m) = f(I) = I'$. Thus f carries $[0, n + p]$ onto L'. By Theorem 1, $[0, n + p]$ contains an interval $[0, k]$ on which f is one-to-one and onto L', and k can be chosen as $k = s + p$ with $s \leqq n$. Let $p \cup \{p_i\}$ be a basis for $[0, s + p]$ with $\sum p_i = s$. Now $\{p_i\}$ is independent, f is one-to-one on $\{p_i\}$, $\{f(p_i)\}$ is independent, $(m, \sum p_i)\perp$, and $(f(m) = f(p), \sum f(p_i))\perp$. If we extend $\{p_i\}$ to a maximal set $\{q_i\}$ having these properties, then the proof of Theorem 2 shows that $m + \sum q_i = I$, and by Remark 3, f is one-to-one on $[0, p + \sum q_i]$ onto L'. But since f is one-to-one on $[0, p + \sum p_i]$ onto L', $p + \sum p_i = p + \sum q_i$ and therefore

$$m + s = m + \sum p_i = m + p + \sum p_i = m + p + \sum q_i = I.$$

Since $(m, n)\perp$, this implies that $s = n$, and the proof is complete.

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Remark 5. In the finite-dimensional case we can simplify the proof by merely showing that $n + p$ and I' have the same dimension.

COROLLARY 6. *Suppose that m is a maximum element which is mapped onto a point, and k is independent of m. Then f is one-to-one on $[0, k]$.*

Proof. There exists an independent complement of m which contains k. We now apply Corollary 5.

COROLLARY 7. *Suppose that m_1 and m_2 are maximum elements which are mapped onto distinct points. Then $m_1m_2 = 0$ and $m_1 + m_2 > m_1, m_2$.*

Proof. If $m_1m_2 \geqq p$, then $f(m_1) = f(p) = f(m_2)$. Hence $m_1m_2 = 0$. Let p_1 be a point with $m_1 \geqq p_1, f(m_1) = f(p_1)$. Suppose that there exists a point $p_3 \leqq m_1$ such that $(p_3, p_1 + m_2)\perp$. Then $(p_1 + p_3, m_2)\perp$ and therefore f is one-to-one on $[0, p_1 + p_3]$ which is impossible since $f(p_1) = f(p_3)$. Thus $m_1 \leqq p_1 + m_2$; hence, $m_1 + m_2 = p_1 + m_2$. Thus $m_1 + m_2 > m_2$ and similarly $m_1 + m_2 > m_1$.

Remark 6. Note that $(m_1, m_2)M'$ if they are not points. This shows that projective geometry has only trivial or one-to-one geometric mappings defined on it.

We shall now show that if m_1 and m_2 are maximum elements which are mapped onto distinct points, then the intervals $[0, m_1]$ and $[0, m_2]$ are isomorphic. First, we need a lemma.

LEMMA 2. *Let L be a geometric lattice in which every two hyperplanes have a common complement. If $a + b > a, b$, then a and b have a common independent complement.*

Proof. Let z be an independent complement of $a + b$. Then $z + a$ and $z + b$ are distinct hyperplanes. By hypothesis, $z + a$ and $z + b$ have a common complement p which is an atom. Then $z + p$ is a common independent complement of a and b.

THEOREM 3. *Let L be a geometric lattice in which every two hyperplanes have a common complement, and let f be a geometric mapping defined on L with L' its image. If m_1 and m_2 are maximum elements that are mapped onto points, then the intervals $[0, m_1]$ and $[0, m_2]$ are isomorphic.*

Proof. By Lemma 2, there exists an element n which is a common independent complement of m_1 and m_2. Suppose that p_1 is a point, $p_1 \leqq m_1$. Consider $m_2(n + p_1)$. Now f is one-to-one on the interval $[0, n + p_1]$ and maps it onto the image lattice L'. Hence there is exactly one point p_2 in $[0, n + p_1]$ for which $f(p_2) = f(m_2)$. Thus $p_2 \leqq m_2(n + p_1)$. But $m_2(n + p_1)$ is either a point or 0. Hence $p_2 = m_2(n + p_1)$. This also implies that $(m_2, n + p_1)M$. Now if $p_3 \leqq m_1$ and $m_2(n + p_1) = m_2(n + p_3)$, then $n + m_2(n + p_1) = n + m_2(n + p_3)$ which implies that $n + p_1 = n + p_3$ (since $(m_2, n + p_3)M$ also) and thus $m_1(n + p_1) = m_1(n + p_3)$ or $p_1 = p_3$. Hence the mapping $p_i \rightarrow m_2(n + p_i)$ is

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one-to-one. If now q_1 is a point with $q_1 \leqq m_2$, then $m_1(n + q_1)$ is a point $\leqq m_1$, and $[(q_1 + n)m_1 + n]m_2 = q_1$. Thus the mapping $p_i \to m_2(n + p_i)$ is onto; in fact, the mappings $p_i \to m_2(n + p_i)$ and $q_i \to m_1(n + q_i)$ are inverses, and if $p_i \to q_i$, then and only then is $n + p_i = n + q_i$. Thus we have a one-to-one mapping of the points of m_1 onto the points of m_2. To complete the proof we shall show that independent sets of points correspond to independent sets of points. If $\{p_1, \ldots, p_k\}$ is independent, then the length of a maximal chain between $n + \sum p_i$ and n is equal to $k + 1$. But since $n + p_i = n + q_i$, the length of a maximal chain between $n + \sum q_i$ and n is equal to $k + 1$. Thus $D(\sum q_i) = k$, and this implies that $\{q_1, \ldots, q_k\}$ is independent. We can obviously go in the reverse direction, so that finite independent sets correspond to finite independent sets. But since an infinite set of points is independent if and only if every finite subset is independent (by definition), independent sets correspond to independent sets, and the proof is complete.

In the next theorem we assume that every line in L has at least three points. This guarantees that in every interval $[0, x]$, any two elements covered by x have a common complement.

THEOREM 4. *Let every line in L have at least three points, and let n be an independent complement of a maximum element m mapping onto a point under the non-trivial geometric mapping f. Then the elements $x \geqq n$ can be divided up into two disjoint sets: the set of xs for which $f(x) = f(I)$ and the set of xs for which $f(x) = f(n)$. The elements of the former are of the form $n + q$ where $0 < q \leqq m$, and every element of this form lies in that set. The latter elements exhaust an interval $[n, h]$, where h is a hyperplane of the form $m_1 + n$, m_1 being a maximum element which meets n in a point p. Moreover, h is an M-element relative to $[n, I]$, and $[n, h]$ is isomorphic to $[p, m_1]$.*

Proof. Let p be a point contained in n. There exists a maximum element m_1 for which $f(m_1) = f(p)$. Since $m_1 n \neq 0, f(p) = f(m_1 n)$, and therefore $p = m_1 n$ because f is one-to-one on $[0, n]$. Now $m + m_1 > m$, and so we have $m + m_1 = m + p$; and because $(n, m)M$, we have $(n, m + p)M$; thus $(n, m + m_1)M$ and $n(m + m_1) = p$. Since $(n, m + m_1)M$ and $n(m + m_1) = p$, $(n, m_1)M$; and because $m + m_1 > m_1$, $m_1 + n$ is a hyperplane h. Furthermore,

$$f(h) = f(m_1 + n) = f(m_1) + f(n) = f(p) + f(n) = f(n).$$

If $x \geqq n$ and $f(x) = f(n)$, then $f(x + h) = f(n)$. Since h is a hyperplane, this implies that $x \leqq h$: if $n \leqq x \leqq h$, then it is obvious that $f(x) = f(n)$.

Let $x \geqq n$ be such that $f(x) = f(n)$ so that $x \leqq h$. We choose n_1 to be an independent complement of p within n. Since $(m_1, n)M$, $(m_1, n_1)\perp$. Since $n > n_1$ and f is one-to-one on $[0, n]$, $f(x) > f(n_1)$. Now xm_1 is a maximal element within x mapping onto a point, and $(n_1, xm_1)\perp$. But n_1 is a maximal element within x on which f is one-to-one and which is independent of xm_1, because if $x \geqq t > n_1$ and f is one-to-one on $[0, t]$, then

$$f(t) = f(n) \geqq f(p) = f(m_1)$$

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and so t contains a point within m_1; hence $txm_1 = tm_1 = 0$ is impossible. Thus by Corollary 6 applied to $[0, x]$, n_1 is an independent complement of xm_1 within $[0; x]$; and therefore since $n_1 + xm_1 = x$, $p + n_1 + xm_1 = x$ or $n + xm_1 = x$. This implies that $(x, m_1)M$, and since this is true for every x in $[n, h]$, $[n, h]$ is isomorphic to $[p, m_1]$.

Now let $x > n$ be such that $f(x) = f(I)$. Since f maps $[0, x]$ onto L', $xm \neq 0$. Now $(xm, n)\perp$ and xm is a maximum element within x mapping onto a point. If $xm + n \neq x$, then by applying Corollary 6 to the interval $[0, x]$, we see that there exists an element r, $x > r > n$, such that $(xm, r)\perp$ with f one-to-one on $[0, r]$. This implies that $f(r) = f(I)$ since $f(I) > f(n)$. But then r contains a point q for which $f(q) = f(m)$. This is impossible since $rm = 0$. Hence $x = n + xm$. If $x = n + q$, where $0 < q \leqq m$, then

$$f(x) = f(n+q) = f(n) + f(q) = f(n) + f(m) = f(n+m) = f(I).$$

Finally, let $w > n$ be such that $f(w) = f(I)$. If we consider f acting on $[0, w]$, then we find that there exists an interval $[n, h']$ with $w > h'$ which consists precisely of those xs with $n \leqq x \leqq w$ for which $f(x) = f(n)$. But all such elements x must be $\leqq h$. Hence $h' \leqq h$, and $wh = h'$. In other words, if $h \ngeqq w$, where $w \in [n, I]$, then $w > wh$. This implies that h is an M-element relative to $[n, I]$. The proof is complete.

Corollary 8. *If L is special, then the results of Theorem 4 are true if we merely assume that every two hyperplanes have a common complement.*

Proof. The fact that $[n, h]$ is isomorphic to $[p, m_1]$ and that h is an M-element relative to $[n, I]$ follows immediately from the modularity of $[p, I]$ and $[n, I]$. The result that $[n, h]$ consists precisely of those elements $x \geqq n$ for which $f(x) = f(n)$ was deduced from the hypothesis that every two hyperplanes have a common complement. As was pointed out, if $x > n$ and $f(x) = f(I)$, then $xm \neq 0$; therefore $(m, x)M$ and $x = n + mx$. Finally, the proof of the Theorem 4 shows that if $x = n + q$, where $0 < q \leqq m$, then $f(x) = f(I)$.

Corollary 9. *If L is special, every two hyperplanes have a common complement, and $[0, m]$ has length $\geqq 3$, then $[0, m]$ is an affine geometry or an affine line.*

Proof. Any interval $[k, I]$ has the property that every two hyperplanes have a common complement. Hence, if $k \neq 0$, then since $[k, I]$ is modular and irreducible, it is a projective geometry or a projective line. Thus $[n, I]$ is a projective geometry or a projective line. Since $[0, m]$ is isomorphic to a subsystem which is obtained from $[n, I]$ by a deletion of all elements under a hyperplane h except n, $[0, m]$ is an affine geometry or an affine line.

Corollary 10. *If m is a hyperplane, then it is an M-element relative to $[r, I]$, where r is any point $\leqq m$.*

Proof. Since $mm_1 = 0$, r is an independent complement of m_1. The proof of the theorem then applies since $m + r = m$.

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PROPOSITION 4. *Let L be a geometric lattice of length $\geqq 4$ in which every two hyperplanes have a common complement, and let f be a geometric mapping with image L'. If the maximum element m mapped onto a point is a hyperplane, then L' is a line with at least three points.*

Proof. L' is evidently a line, and it contains at least three points because there must exist at least three distinct maximum elements which are mapped onto points.

COROLLARY 11. *If in addition to the hypothesis of Proposition 4 we assume that every line in m has at least three points, then every line in L contains at least three points.*

Proof. Any line lying within a maximum element which is mapped onto a point will have at least three points, and any other line is mapped one-to-one onto L' which also has at least three points.

COROLLARY 12. *If in addition to the hypothesis of Proposition 4 we assume that L is special and that $[0, m]$ is not an affine geometry with precisely two points on each line, then every line in L contains at least three points.*

4. Affine geometry. Theorem 4 and its corollaries (Corollaries 8–10) show that parts of L resemble an affine geometry, indeed sometimes are an affine geometry, if L has a single non-trivial geometric mapping defined upon it. We shall now investigate conditions that are sufficient to guarantee that L is an affine geometry. It will be seen that if L has "enough" geometric mappings, then it must be affine.

THEOREM 5. *Let L be a geometric lattice of length $\geqq 4$ in which each hyperplane h_i is a maximum element mapped onto a point under a geometric mapping f_i. If in addition*

(a) *every two hyperplanes in L have a common complement, or*

(b) *at least one line in L contains at least three points,*

then L is the lattice of flats of an affine geometry.

Proof. We first show that either (a) or (b) implies that every line in L has at least three points so that the conditions are equivalent in the presence of the other assumptions. Suppose that (a) holds. If l is a line, there exists a hyperplane h which does not contain l but does meet it in a point and a non-trivial geometric mapping f which sends h onto a point. The image of f must be a line with at least three points, and since $l \nleqq h$ but does meet it in a point, f maps l onto this line so that l contains at least three points. Now suppose instead that (b) holds and that l is a line with at least three points. Let h be a hyperplane which does not contain l but which does meet it in a point. There exists a non-trivial geometric mapping f which sends h onto a point. Since l contains at least three points, so does the image of f which must be a line, and so does every line which is not contained in a maximum element mapped onto a point. If k is a

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line lying on a maximum element w which maps onto a point, then there exists a line t, meeting k in a point p_1, which contains at least two other points r_1 and r_2, and which is mapped by f onto a line. Let p_2 be another point of k. There exists a hyperplane z which contains the line $p_2 + r_1$ but not the point p_1 and therefore not the point r_2. By assumption there exists a non-trivial geometric mapping g which maps z onto a point. The lines $t = p_1 + r_1 + r_2$ and $k = p_1 + p_2$ are both mapped one-to-one onto the same line, and thus k has at least three points. Hence every line has at least three points if (b) holds.

Let p be a point of L and let m be a hyperplane containing p. By hypothesis, there exists a geometric mapping f for which $f(p) = f(m)$. By Corollary 10, m is an M-element relative to $[p, I]$. Since m can be any hyperplane containing p, every hyperplane containing p is an M-element relative to $[p, I]$. The results of [**4**, p. 287] then show that $[p, I]$ is a modular lattice. Thus L is special.

If L is of length 4, then it is automatically special. In this case if the line l and the point p are not incident, then by hypothesis there exists a geometric mapping f which maps l onto a point. Thus, there exists a line l' which contains p but does not intersect l. Any line containing p which is not a maximum element mapped onto a point will be mapped onto the image of f and so will have to intersect l. Thus l' is the unique line containing p which is parallel to l, and so Euclid's parallel postulate is satisfied. Thus L is an affine plane.

If L is of length $\geqq 5$, then by Corollary 9, every interval $[0, h]$, where h is a hyperplane, is an affine geometry. Thus L is an affine geometry because Euclid's parallel postulate will hold in any plane since L has length $\geqq 5$ (see [**4**, Part 3, Chapter 4]).

The above proof used a great many of the mappings to show that L was special. If we assume that L is special, then the number of mappings required can be drastically reduced. Before we look further into this problem we shall prove a theorem that will be useful for other problems as well as the present task.

THEOREM 6. *Let L be a geometric lattice in which every two hyperplanes have a common complement. Suppose that f_1 and f_2 are geometric mappings with $f_1(m) = f_1(p), f_2(m) = f_2(p)$, m is a maximum element of both mappings, and p is a point. If $f_1(w) = f_1(q)$, where w is a maximum element of f_1 and q is a point $\neq p$, then $f_2(w) = f_2(q)$.*

Proof. If $w = m$, the proof is trivial. Thus assume that $w \neq m$. Then $wm = 0, w + m \succ w, m$. Suppose now that z is a maximum element of f_2 which contains q and that $z \neq w$. We have $zm = 0$ and $z + m \succ z, m$. Thus $p + w = q + m = m + w$ and $p + z = q + m = m + z$ so that they are all equal. But $f_1(p + z) = f_1(p) + f_1(z)$, and since $p + z = p + w$ and $f_1(w) = f_1(q), f_1(p + z) = f_1(p + w) = f_1(p) + f_1(q)$. Now $f_1(z) > f_1(q)$ since $w \ngeqq z$, and since $f_1(p + z) = f_1(p) + f_1(q)$, this implies that

$$f_1(z) = f_1(p) + f_1(q)$$

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which is a line. Hence there is a point r contained in z such that $f_1(r) = f_1(p)$. This implies that $r \leqq m$ which means that $mz \neq 0$, a contradiction. Thus $z = w$ and $f_2(w) = f_2(q)$.

COROLLARY 13. *If two geometric mappings f_1 and f_2 defined on a lattice L in which every two hyperplanes have a common complement have a common block in their kernels, then their kernels are identical, i.e., a single maximum element which is mapped onto a point essentially determines the geometric mapping.*

Remark 7. Theorem 6 and Corollary 13 need not be true in a Boolean algebra.

THEOREM 7. *Let L be a special geometric lattice of length $\geqq 5$ in which every line has at least three points. If L has two distinct, non-trivial geometric mappings which map a hyperplane onto a point, then L is an affine geometry.*

Proof. Let f be one such geometric mapping. By Corollary 9, the points of L are partitioned into a family $\mathscr{F}$ of hyperplanes each of which is an affine geometry. Any other hyperplane h of L is modular with every member of the family $\mathscr{F}$ since f maps h onto the image of L and L is special. To complete the proof we must show that $[0, h]$ is an affine geometry. The interval $[0, h]$ is mapped one-to-one in an order-preserving fashion onto the interval $[p, I]$, where p is an atomic complement of h, under the mapping $x \rightarrow x + p$. Hence $[0, h]$ can be viewed as a subgeometry of a projective geometry since $[p, I]$ is a projective geometry. Moreover, this projective geometry is the projective geometry determined by the affine geometries of $\mathscr{F}$. The intersection of h with the various members of $\mathscr{F}$ partitions $[0, h]$ into a family $\mathscr{F}'$ of copoints relative to $[0, h]$. Let k be the element in $\mathscr{F}'$ which lies in the member of $\mathscr{F}$ that contains p. Since $[0, h]$ is a subgeometry of a projective geometry, these elements in $\mathscr{F}'$ can be thought of as copoints in a projective geometry. The results of Theorem 4 applied to the mapping $x \rightarrow x + p$ and the elements of $\mathscr{F}'$ not equal to k show that any point contained in them in the projective geometry lies in $[0, h]$ except that for each of them there is a unique coline of points in the projective geometry that is "missing". The same is true for k because the element in $\mathscr{F}$ containing p is an affine geometry. We shall show that the elements in $\mathscr{F}'$ can actually be viewed as copoints in an affine geometry. Let us consider the element k in $\mathscr{F}'$. Every element in $\mathscr{F}'$ not equal to k meets k in 0 relative to $[0, h]$ and in an element covered by k relative to the projective geometry within which $[0, h]$ is embedded. Now k is itself an affine geometry, and thus the only hyperplane relative to $[0, k]$ "missing" is the one at "infinity". Thus in the projective geometry within which $[0, h]$ lies, all of the elements of $\mathscr{F}'$ meet in a common element c covered by each, i.e., in the "missing" hyperplane of $[0, k]$. Two cases present themselves:

(a) $\mathscr{F}'$ does not contain all of the copoints in the projective geometry which contain c,

(b) $\mathscr{F}'$ does contain all of the copoints in the projective geometry which contain c.

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In case (a), there is some copoint in the projective geometry which contains no points of $[0, h]$, and thus $[0, h]$ can be thought of as being embedded in an affine geometry. The members of $\mathscr{F}'$ are complete copoints in this affine geometry, i.e., any point contained in them in the affine geometry lies in $[0, h]$. In case (b), $[0, h]$ must be isomorphic to the subgeometry obtained by removing all of the points of c. In this case, only the elements of $\mathscr{F}'$ will be copoints which are affine geometries.

The second geometric mapping g defined on L produces a family $\mathscr{G}'$ of copoints relative to $[0, h]$ which are also affine geometries. Theorem 6 shows that $\mathscr{F}'$ and $\mathscr{G}'$ are either identical or disjoint. If they are disjoint, case (b) cannot occur because of an excess of copoints which are affine geometries, and $[0, h]$ must be embedded in an affine geometry. We can consider the family $\mathscr{F}'$ to be a family of copoints with equations X = const. If we take a single member of $\mathscr{G}'$, then its equation can be taken to be Y = const. Since any X values are allowed in this copoint, we must have the complete family of parallel copoints X = const in the affine geometry so that $[0, h]$ is the affine geometry. If $\mathscr{F}'$ and $\mathscr{G}'$ are equal, then we choose some copoint in $[0, h]$ which does not lie in $\mathscr{F}'$. This element b lies in a hyperplane h' which is not in $\mathscr{F}$ or $\mathscr{G}$. But if $\mathscr{F}''$ and $\mathscr{G}''$, the families of parallel copoints in $[0, h']$, are identical, then $\mathscr{F} = \mathscr{G}$ which is false. Thus $[0, h']$ is an affine geometry and so is $[0, b]$. Since $[0, b]$ is an affine geometry, case (b) cannot thus occur. Now we could have chosen p to be a common complement for both h and h'; therefore, since $[0, b]$ is obtained from the projective geometry into which $[0, h]$ and $[0, h']$ are both embedded by the deletion of the points lying under a unique coline of this projective geometry, b is a complete copoint of the affine geometry in which $[0, h]$ lies. Thus as before we can show that $[0, h]$ is an affine geometry.

Remark 8. Wilcox (see [**6**]) in a long, unpublished manuscript has shown that a special, geometric lattice L of length $\geqq 6$ in which each line has at least three points can be embedded in a projective geometry (PG) in such a way that every element in the PG, except possibly some hyperplanes, is a meet of a pair of elements in L, and moreover, intervals of the form $[p, I]$, where p is a point, remain unchanged. Assuming this result, we can see that if L has a single geometric mapping with a hyperplane as a maximum element, then it is a family of parallel hyperplanes in an affine geometry or a projective geometry with all elements $\leqq$ a single coline removed.

5. Geometric mappings and semi-affine transformations. We shall now study the connection between geometric mappings and semi-affine transformations (semi-linear + a constant) in affine geometry and vector spaces. It is well known [**1**] that collineations in affine geometries are determined by non-singular semi-affine transformations if the affine geometry is two-dimensional or greater. It is easily seen that any semi-affine transformation on a vector space induces a geometric mapping on the corresponding affine space. We have the following converse.

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THEOREM 8. *Let f be a geometric mapping from L_1 onto L_2, where L_1 and L_2 are affine lattices over a division ring* $\neq$ GF(2) *and L_2 is at least two-dimensional (length* $\geqq 4$). *Then there exists a semi-affine transformation between the corresponding vector spaces which induces this geometric mapping.*

Proof. Let vector addition be denoted by $\oplus$ and let θ be the zero vector. Let E be the maximum flat for which $f(E) = f(\theta)$. Then E is a subspace of V_1. Let F be a complementary subspace of E so that $E \oplus F = V_1$, $E \cap F = \theta$. Let G be a relative complement of θ within F so that $G + \theta = F$, $G\theta = 0$. Then $G + E = G + \theta + E = F + E = I$, and $GE = GFE = G\theta = 0$; and since $(E, F)M$, if $X \leqq G$, then $(X + E)G = X$ so that $(E, G)M$. By Corollary 5, f is one-to-one on $[0, F]$ and onto L_2. Thus f on $[0, F]$ is induced by a non-singular semi-affine transformation $A(x) \oplus b$, where $A(x)$ is semi-linear, from F onto V_2. Let v be any vector in V_1. Then $v = x \oplus y$, where $x \in F$ and $y \in E$, and this expression is unique. Define $K(x) = A(x) \oplus b$, $K(y) = \theta$. Thus $K(v) = A(x) \oplus b$, and K is a semi-affine transformation from V_1 onto V_2. The geometric mapping induced by K and the mapping f have a common block in their kernels, and thus their kernels are identical by Corollary 13. But since F is mapped onto L_2 by f, and since by the definition of K, the geometric mapping induced by K on F is identical to f on F, it follows that the geometric mapping induced by K and the mapping f are identical.

Remark 9. If L_2 is of length 3, then f merely maps a family of parallel hyperplanes in L_1 onto the points of L_2. If we define any one-to-one mapping from a family of parallel hyperplanes in L_1 onto the points of L_2, then we obtain a geometric mapping, and we get all geometric mappings from L_1 onto L_2 in this manner.

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