BULL. AUSTRAL. MATH. SOC. MOS 2205
VOL. 1 (1969), 47-58

Adjunction semigroups

J. T. Borrego

Let S and T be compact (topological) semigroups, A be a closed subsemigroup of S, and f be a continuous homomorphism of A onto T. It is a natural question to ask is there a compact semigroup Z containing T and a continuous homomorphism ϕ of S onto Z such that ϕ restricted to A is f?

In the category of compact Hausdorff spaces, the answer to the analogous question may be given by the following construction due to Borsuk. Let X and Y be compact Hausdorff spaces, A be a closed subspace of X, and f a continuous function of A onto Y. Then $R(f, X) = \{(x, y) | f(x) = f(y) \text{ or } x = y\}$ is a closed equivalence relation on X. The adjunction space $Z(f, x)$ is $X/R(f, X)$. The purpose of this note is to investigate this construction in semigroups.

Using the previous notation, we say A is agreeable with respect to S if and only if $R(f, S)$ is a congruence of S for all epimorphisms defined on A.

For $x, y \in A$ let $F(x, y)$ be the smallest congruence of A containing (x, y).

THEOREM: *A is agreeable with respect to S if and only if $F(x, y) \cup$ (diagonal of S) is a congruence of S for all $x, y \in A$.*

Using the above theorem, a set of necessary and sufficient conditions for the minimal ideal to be agreeable are obtained. Various sufficient conditions for A to be agreeable are also given.

Received 21 March 1969. Received by J. Austral. Math. Soc. 30 January 1968. Communicated by G.B. Preston

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

1. Introduction

Let X and Y be normal Hausdorff spaces, A be a closed subspace of X, and let f be a continuous function from A onto Y. Define the closed equivalence relation $R(f, X) = \{(x, y) \mid f(x) = f(y) \text{ or } x = y\}$ and let $Z(f, X) = X/R(f, X)$. This construction was defined by Borsuk [2] and $Z(f, X)$ is called the *adjunction space* of Y to X by f. The purpose of this note is to investigate this construction in semigroups.

The author wishes to thank Professor A.D. Wallace for his many valuable suggestions.

1.1 THEOREM [2]. *If X, Y, A, and f are as above and if ϕ is the natural projection of X onto $Z(f, X)$, then*

(1) $\phi \mid (X \setminus A)$ is one-to-one, and

(2) there exists a unique homeomorphism $k : Y \to Z(f, X)$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Z(f, X) \\ \cup & & \uparrow k \\ A & \xrightarrow{f} & Y \end{array} \quad .$$

2. Preliminaries

A *semigroup* is a Hausdorff space together with a continuous associative multiplication which will be denoted by juxtaposition here. Since normality is needed in the construction of adjunction spaces, and for simplicity, we shall make the blanket hypothesis that all semigroups are compact. The definitions which are used here and general information concerning semigroups may be found in [3], [4], and [5].

Notation. Throughout S and T will be compact normal semigroups. If $A, B \subset S$, then $AB = \{ab \mid a \in A, b \in B\}$ and $A^2 = AA$. The *closure of a set* A will be written A^*. The *diagonal of* A, i.e., $\{(a, a) \mid a \in A\}$ will be written ΔA or simply as Δ.

2.1 DEFINITION. A *congruence* C of S is an equivalence relation on S such that $(\Delta S)C \cup C(\Delta S) \subset C \subset S \times S$. (The multiplication in $S \times S$ is coordinatewise.)

2.2 THEOREM. *Let E be an equivalence relation which is defined on*

S. There exists a continuous, associative multiplication on S/E such that the natural projection of S onto S/E is a homomorphism if and only if E is a congruence. Moreover, if such a multiplication exists, then it is unique.

This theorem is well-known.

2.3 REMARK. Let $A = A^* \subset S$, f be a continuous function of A onto T, ϕ be the natural projection of S onto $Z(f, S)$, and let k be the unique homeomorphism such that $kf = \phi|A$. If $R(f, S)$ is a congruence of S, then

(1) If $x, y, xy \in A$, then $kf(xy) = kf(x)kf(y)$,

and (2) k is a homomorphism if and only if A is a subsemigroup and f is a homomorphism.

Proof. (2) necessity. Since $kf(A) = \phi(A)$ and $\phi^{-1}\phi(A) = A$, $A^2 \subset A$. If $x, y \in A$, then

$$f(x)f(y) = k^{-1}\phi(x)k^{-1}\phi(y) = f(xy) .$$

Because of 2.3 (2) we shall restrict our attention to epimorphisms (continuous, onto homomorphism) defined on subsemigroups in constructing adjunction semigroups.

2.4 DEFINITION. Let $A^2 \subset A = A^* \subset S$. Then A is *agreeable* with respect to S if and only if $R(f, S)$ is a congruence for all continuous epimorphisms f defined on A.

Thus, the problem which is studied here is to find sufficient conditions on A to insure that A is agreeable with respect to S.

2.5 PROPOSITION. *Let $A^2 \subset A = A^* \subset S$. Then the following are equivalent:*

(1) A is agreeable with respect to S.

(2) If C is a closed congruence of A, then $C \cup \Delta S$ is a congruence of S.

(3) If $x, y \in A$ and if $F(x, y)$ is smallest closed congruence of A containing (x, y), then $F(x, y) \cup \Delta S$ is a congruence of S.

Proof. (1) $\Longleftrightarrow$ (2) follows immediately from the one-to-one

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

correspondence between the closed congruences of A and the epimorphisms defined on A.

(2) $\Longleftrightarrow$ (3) follows from the fact that $C = \bigcup \{F(x , y) \mid (x , y) \in C\}$ for all closed congruences C of A.

3. Semi-ideals

In this section we shall develop a necessary condition for agreeability.

Notation. If $B \subset S$ then $B^{(-1)}B = \{x \subset S \mid Bx \cap B \neq \emptyset\}$ and $B^{[-1]}B = \{x \in S \mid Bx \subset B\}$. $BB^{(-1)}$ and $BB^{[-1]}$ are defined dually. Note that I is an ideal of S if and only if $II^{[-1]} = S = I^{[-1]}I$.

3.1 DEFINITION. Let A be a non-empty subset of S. Then A is a *semi-ideal* of S if and only if:

(1) A is closed,

(2) if $x \in S$, then $\mathrm{card}[xA \cap (S \setminus A)] \leqq 1$ and $\mathrm{card}[Ax \cap (S \setminus A)] \leqq 1$,

(3) $A^{(-1)}A \subset A^{[-1]}A$, and $AA^{(-1)} \subset AA^{[-1]}$.

The following remark gives some properties of semi-ideals.

3.2 REMARK. (1) Let A be a non-void closed subset of S, then A is a semi-ideal of S if and only if:

(i) if $x \in S$, then $\mathrm{card}\, xA = 1$ or $x \in AA^{[-1]}$,

(ii) if $x \in S$, then $\mathrm{card}\, Ax = 1$ or $x \in A^{[-1]}A$.

(2) A closed ideal of S is a semi-ideal of S.

(3) If A is a semi-ideal of S and if A contains an ideal of S, then A is an ideal of S.

(4) Semi-ideals are preserved under closed epimorphisms.

Proof. (1) If A is a semi-ideal of S and if $x \in S \setminus AA^{[-1]}$, then $x \in S \setminus AA^{(-1)}$ so that $xA \subset S \setminus A$ and $\mathrm{card}(xA) = 1$.

Suppose (i) and (ii) are satisfied. If $x \in AA^{(-1)}$, then

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

card $xA = 1$ or $x \in AA^{[-1]}$. If card $xA = 1$, then $x \in AA^{[-1]}$.

(2), (3) and (4) are straightforward computations.

3.3 PROPOSITION. *If A is a closed subset of S, then $(A \times A) \cup (\Delta S)$ is a congruence of S if and only if A is a semi-ideal of S.*

Proof. Let $x \in S$; $t , y \in A$. If $x \in A^{(-1)}A$, then $xt , xy \in A$ and if $x \in S \setminus A^{(-1)}A$, then card $xA = 1$ so that $xt = xy$. Thus, $(\Delta S)[(A \times A) \cup \Delta S] \subset (A \times A) \cup \Delta S$, and by using a dual argument, it may be seen that $(A \times A) \cup \Delta S$ is a right congruence.

3.4 COROLLARY. *If A is an agreeable subsemigroup of S, then A is a semi-ideal of S.*

Proof. Let $T = \{e\}$ and f be the constant epimorphism of S onto T .

3.5 LEMMA. *Let A be a subsemigroup and semi-ideal of S and let f be a continuous epimorphism from A onto T. Then $R(f , S)$ is a congruence of S if and only if*

(1) if $x \in [(AA^{[-1]}) \setminus A]$ and if $a , a' \in A$ such that $f(a) = f(a')$, then $f(xa) = f(xa')$, and

(2) if $x \in [(A^{[-1]}A) \setminus A]$ and if $a , a' \in A$ such that $f(a) = f(a')$, then $f(ax) = f(a'x)$.

Proof. Let $a , a' \in A$ such that $f(a) = f(a')$. If $x \in A$, then $f(xa) = f(xa')$. If $x \in S \setminus (A^{[-1]}A)$, then card $xA = 1$ and $xa = xa'$. Thus, $R(f , S)$ is a left congruence and by a dual argument, $R(f , S)$ is a right congruence.

3.6 PROPOSITION. *If A is an agreeable subsemigroup of S, then every semi-ideal of A is a semi-ideal of S.*

Proof. If P is a semi-ideal of A , then $(P \times P) \cup (\Delta A)$ is a congruence of A so that $(P \times P) \cup (\Delta S) = (P \times P) \cup (\Delta A) \cup (\Delta S)$ is a congruence of S (2.5). Thus, P is a semi-ideal of S (3.4).

The following example will show that the converse of the above proposition is in general false.

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

3.7 EXAMPLE. Let $T = \left\{\begin{bmatrix} x & 0 \\ 0 & 1\end{bmatrix} \mid 0 \leqq x \leqq 1\right\} \quad \left\{\begin{bmatrix} 0 & y \\ 0 & 1\end{bmatrix} \mid 0 \leqq y \leqq 1\right\}$ under the usual matrix multiplication and let Y = the unit interval with the multiplication given by $xy = y$. (Both with the usual topologies). Let $S = T \times Y$ with coordinatewise multiplication. Then the minimal ideal K of S is $\left\{\begin{vmatrix} 0 & y \\ 0 & 1\end{vmatrix} \mid 0 \leqq y \leqq 1\right\} \times Y$.

If A is a semi-ideal of K then card $A = 1$ or $A = K$.

Proof. Suppose card $A > 1$. Then there are two distinct points (x,y) and (u,w) in A. There are two cases, (1) $x \neq u$ and (2) $y \neq w$. Since (2) is similar to (1), we shall only consider (1) here.

(1) If $z \in Y$, then $(x,y)(x,z) \neq (u,w)(x,z)$ and $(x,z) \in A^{[-1]}A$. In particular, $(u,z),(x,z) \in A$, so that $(\{x\} \times Y) \cup (\{u\} \times Y) \subset A$.

Let $0 \leqq y \leqq 1$ and $z = \begin{bmatrix} 0 & y \\ 0 & 1\end{bmatrix}$ and t, t' be distinct points of Y. Since $(x,t'),(x,t)$ A and $(z,t)(x,t) \neq (z,t)(x,t')$, (z,t) and (z,t') are in A. Thus, $K = A$.

It is now clear that every semi-ideal of K is a semi-ideal of S. It remains only to show that A is not agreeable with respect to S. Let $f : K \quad K$ be defined by

$$f\left(\begin{bmatrix} 0 & x \\ 0 & 1\end{bmatrix}, y\right) = \begin{cases} \left(\begin{bmatrix} 0 & 2x \\ 0 & 1\end{bmatrix}, 2y\right) & \text{if } x \leqq \frac{1}{2} \text{ and } y \leqq \frac{1}{2}, \\ \left(\begin{bmatrix} 0 & 1 \\ 0 & 1\end{bmatrix}, 2y\right) & \text{if } x > \frac{1}{2} \text{ and } y \leqq \frac{1}{2} \\ \left(\begin{bmatrix} 0 & 2x \\ 0 & 1\end{bmatrix}, 1\right) & \text{if } x \leqq \frac{1}{2} \text{ and } y > \frac{1}{2} \\ \left(\begin{bmatrix} 0 & 1 \\ 0 & 1\end{bmatrix}, 1\right) & \text{if } x > \frac{1}{2} \text{ and } y > \frac{1}{2}. \end{cases}$$

It is clear that f is an epimorphism.

It remains to show that $R(f,S)$ is not a congruence. Let $x = \begin{bmatrix} 0 & 1/2 \\ 0 & 1\end{bmatrix}$, $y = \begin{bmatrix} 0 & 1 \\ 0 & 1\end{bmatrix}$, $z = \begin{bmatrix} 1/2 & 0 \\ 0 & 1\end{bmatrix}$, and $t = \begin{bmatrix} 0 & 1/4 \\ 0 & 1\end{bmatrix}$. Then $f(x,0) = f(y,0)$ but $(z,0)(y,0) = (x,0)$ and $(z,0)(x,0) = (t,0)$ and $((x,0),(t,0)) \notin R(f,S)$.

4. Sufficient conditions

In this section some sufficient conditions for agreeability are given.

4.1 LEMMA. *If* $A^2 \subset A \subset S$ *, then* $AA^{[-1]}$ *and* $A^{[-1]}A$ *are subsemigroups of* S *, which contain* A *as a left ideal and as a right ideal, respectively.*

4.2 PROPOSITION. *If* A *is a subsemigroup and a semi-ideal of* S *and if* A *is a homomorphic retract* [1] *of* $AA^{[-1]}$ *and* $A^{[-1]}A$ *, then* A *is agreeable with respect to* S *.*

Proof. It is enough to show that 3.5 is satisfied for an arbitrary epimorphism f defined on A .

Let r be a homomorphic retract of $A^{[-1]}A$ onto A . If $x \in [(A^{[-1]}A) \setminus A]$ and a' , $a \in A$ such that $f(a) = f(a')$, then $f(ax) = f(r(ax)) = f(a)fr(x) = f(a'x)$. The other part of 3.5 may be verified in a similar fashion.

4.3 COROLLARY. *If the minimal ideal of* S *is a group, then it is agreeable with respect to* S *.*

This follows immediately from the fact that in this case the minimal ideal is a homomorphic retract of S [1].

4.4 PROPOSITION. *If* A *is a subsemigroup and a semi-ideal and if* A *satisfies:*

(1) *if* $x \in [(AA^{[-1]}) \setminus A]$ *, then* card $xA = 1$ *or* $xa = a$ *for all* $a \in A$ *, and*

(2) *if* $x \in [(A^{[-1]}A) \setminus A]$ *, then* card $Ax = 1$ *or* $ax = a$ *for all* $a \in A$ *,*

then A *is agreeable with respect to* S *.*

Proof. It is enough to verify 3.5 for an arbitrary epimorphism f defined on A .

If $x \in [(A^{[-1]}A) \setminus A]$ and if a , $a' \in A$ such that $f(a) = f(a')$, then card $Ax = 1$ or $ax = a$ and $ax = a'$. If card $Ax = 1$, then $ax = a'x$. If $ax = a$ and $a'x = a'$, then $f(ax) = f(a'x)$.

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

5. The case in which A consists of left zeros

In this section A will be a subsemigroup of S. We shall be interested in the case where A consists of left zeros of S, i.e., $ax = a$ for all $x \in S$ and $a \in A$. However, a weaker hypothesis will give the results in this section and the next definition gives a name to that hypothesis.

5.1 DEFINITION. A subsemigroup A is said to be 2-*simple* if and only if every subset of A with at least two elements is a semi-ideal of A and card $A \geqq 2$.

5.2 REMARK. If A consists of left zeros (or right zeros) of S then A is 2-simple.

5.3 THEOREM. *If A is 2-simple, then A is agreeable with respect to S if and only if every semi-ideal of A is a semi-ideal of S.*

Proof. The necessity follows from 3.6.

If $x, y \in A$, then $F(x, y) = (\{x, y\} \times \{x, y\}) \cup (\Delta A)$ (see 2.5) so that $\{x, y\}$ is a semi-ideal of S which means that $F(x, y) \cup \Delta S$ is a congruence of S and the conclusion follows from 2.5.

Next we investigate the relationship between Proposition 4.4 and Theorem 5.3.

5.4 LEMMA. *Let A be a 2-simple subsemigroup of S.*

(1) If there are $x \in AA^{[-1]}$ and $a \in A$ such that $xa \neq a$ and card $xA > 2$, then there exists a semi-ideal of A which is not a semi-ideal of S.

(2) If there are $x \in A^{[-1]}A$ and $a \in A$ such that $ax \neq a$ and card $Ax > 2$, then there exists a semi-ideal of A which is not a semi-ideal of S.

Proof. (1) Since card $xA > 2$, there is $a' \in xA$ such that $xa \neq a'$ and $a' \neq a$. From the fact that $a' \in xA$, it is seen that there is a $a'' \in A$ such that $xa'' = a'$. Let $B = \{a, a''\}$; B is a semi-ideal of A.

Suppose B is a semi-ideal of S. Since $a', xa \in xB$, card $xB > 1$ so that $x \in BB^{[-1]}$. It follows that $xa = a''$ because

$xa = a' \neq a$. But $xa = a''$, so that $xa = a'' = a'$ which is a contradiction.

(2) is dual to (1).

5.5 LEMMA. *Let A be a 2-simple subsemigroup with cardinality greater than 2.*

(1) If there exists $x \in AA^{[-1]}$ such that card $xA = 2$ *, then there is a semi-ideal of A which is not a semi-ideal of S .*

(2) If there exists $x \in A^{[-1]}A$ such that card $Ax = 2$ *, then there exists a semi-ideal of A which is not a semi-ideal of S .*

Proof. (1) Let $a \in A \setminus xA$, let $xa = a''$, and let $a'' \in xA \setminus \{a'\}$. Then there are three cases to be considered: (a) $xa' = a''$; (b) $xa' = a'$, $xa'' = a''$; and (c) $xa' = a''$, $xa'' = a'$.

(a) The set $B = \{a , a''\}$ is a semi-ideal of A . If B is a semi-ideal of S , then $x \in BB^{[-1]}$ so that $a'' = xa' \in B$ which is a contradiction.

(b) The set $B = \{a , a''\}$ is a semi-ideal of A . Since $xa'' \in B$, $x \in BB^{(-1)}$ but $xa = a' \notin B$ so that $BB^{(-1)}$ is not a subset of $BB^{[-1]}$ and B is not a semi-ideal of S .

(c) Let $a''' \in A$ such that $xa''' = a''$. Let $B = \{a , a'''\}$. Then $xB = \{a' , a''\} \subset S \setminus B$. Thus, B is a semi-ideal of A but not of S .

5.6 PROPOSITION. *Let A be a 2-simple subsemigroup of S such that* card $A > 2$ *. Then every semi-ideal of A is a semi-ideal of S if and only if the following conditions are satisfied:*

(1) If $x \in AA^{[-1]}$, then card $xA = 1$ *or $xa = a$ for all $a \in A$.*

(2) If $x \in A^{[-1]}A$, then card $Ax = 1$ *or $ax = a$ for all $a \in A$.*

(3) A is a semi-ideal of S .

Proof. The necessity follows from Lemmas 3.4 and 3.5. The sufficiency follows from Propositions 3.6 and 4.4.

5.7 COROLLARY. *Let A be a closed 2-simple subsemigroup of S such*

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

that card $A > 2$. *Then* A *is agreeable with respect to* S *if and only if 5.6 (1), 5.6 (2), and 5.6 (3) are satisfied.*

5.8 COROLLARY. *Let* A *be a closed subsemigroup of* S *which consists of left zeros of* S *with* card $A > 2$. *Then* A *is agreeable with respect to* S *if and only if 5.6 (1) and 5.6 (3) are satisfied.*

5.9 COROLLARY. *Let* A *be a closed subsemigroup of* S *which consists of right zeros of* S *with* card $A > 2$. *Then* A *is agreeable with respect to* S *if and only if 5.6 (2) and 5.6 (3) are satisfied.*

6. The minimal ideal

In this section we shall use the results of the previous sections to compute necessary and sufficient conditions for the minimal ideal of S to be agreeable with respect to S . Throughout K will be the minimal ideal of the compact semigroup S . For information concerning the structure of K see [3], [4], and [5].

6.1 LEMMA. *Let* A *be an agreeable subsemigroup of* S *and let* f *be an epimorphism defined on* A . *Then* $\phi(A)$ *is agreeable with respect to* $Z(f, S)$, *where* ϕ *is the natural projection of* S *onto* $Z(f, S)$.

The proof of this lemma is a straightforward computation.

Notation. $E = \{x \in S \mid x^2 = x\}$. The continuous function u [5] from K onto $K \cap E$ is defined by $u(x)$ if and only if $x \in eS \cap Se$ ($eS \cap Se$ is a group [3]).

6.2 PROPOSITION. *If* K *is agreeable with respect to* S *and if there are at least three minimal right ideals of* S , *then* K *satisfies:*

(1) *If* $x \in S$, *then* $xtS = tS$ *for* $t \in K$ *or there exists* $t' \in K$ *such that* $xK = t'S$.

Proof. Let $e \in E$ and define $f : K \to P = Se \cap E$ by $f(x) = u(xe)$. Then f is a homomorphic retraction [1]. From 6.1 we see that if ϕ is the natural projection of S onto $Z(f, S)$, then $\phi(K)$ is agreeable with respect to S . Since P is 2-simple so is $\phi(K)$ and we may apply 5.7. Thus, since $\phi(K)(\phi(K)^{[-1]}) = Z(f,S)$, if $x \in S$ then card $\phi(x)\phi(K) = 1$ or $\phi(x)\phi(t) = \phi(t)$ for all $t \in K$. But $\phi^{-1}(t) = tS$ for $t \in K$ so that card $\phi(x)\phi(K) = 1$ implies $xK = t'S$ for some $t' \in K$ and $\phi(x)\phi(t) = t$ for $t \in K$ implies $xtS = tS$.

Now we state the dual of Proposition 6.2.

6.3 PROPOSITION. *If* K *is agreeable with respect to* S *and if there are at least three minimal left ideals of* S *, then* K *satisfies:*

(1) If $x \in S$ *, then* $Stx = St$ *for all* $t \in K$ *or there exists* $t' \in K$ *such that* $Kx = St'$.

Next we turn our attention to the converse of the previous two propositions.

6.4 PROPOSITION.

(1) If K *satisfies 6.2 (1) and 6.3 (1), then* K *is agreeable with respect to* S .

(2) If S *has two minimal left ideals, and if* K *satisfies 6.2 (1), then* K *is agreeable with respect to* S .

(3) If S *has two minimal right ideals, and if* K *satisfies 6.3 (1), then* K *is agreeable with respect to* S .

(4) If S *has two minimal left ideals and two minimal right ideals, then* K *is agreeable with respect to* S .

Proof. Let f be a continuous epimorphism defined on K . Note that $f(K)$ is a compact simple semigroup. Let k , $k' \in K$ such that $f(k) = f(k')$, and let $x \in S$. It should be noted that $f(u(k)) = f(u(k'))$, that $f(K)$ is simple, and that $fu = uf$.

(1) Since the proof that $R(f , S)$ is a right congruence is dual to the proof that $R(f , S)$ is a left congruence, we shall do only that proof. There are two cases: (a) $xkS = kS$ and $xk'S = k'S$, and (b) $xkS = xk'S$.

(a) Since $u(xk) = u(k)$ and $u(xk') = u(k')$, $f(u(xk)) = f(u(xk'))$ so that $f(xk) = f(u(xk)x)f(k') = f(u(xk'))f(xk') = f(xk')$.

(b) Since $f(u(xk)) = f(u(xu(k')k)) = u(f(xu(k'))f(k) = u(f(xk'))$ $= f(u(xk'))$, this case follows in a manner similar to (a).

(2) The fact that $R(f , S)$ is a right congruence follows in a manner similar to (1). It remains only to show that $R(f , S)$ is a left congruence. There are four cases, (a) $xkS = kS$ and $xk'S = k'S$, (b) $xkS = kS = xk'S$, (c) $xkS = k'S = xk'S$, and (d) $xkS = k'S$ and

https://doi.org/10.1017/S0004972700041289 Published online by Cambridge University Press

$xk' = kS$.

The cases (a), (b), and (c) follow in the same manner as (1).

(d) Since $f(u(xk)) = fu(k'k) = uf(k'k) = f(u(xk'))$, the case follows in a manner similar to (1 , a).

(3) This is entirely dual to (2).

(4) The fact that $R(f , S)$ is a left congruence follows in the same manner as in (2) and the right congruency follows in the same manner as in (3).

6.5 COROLLARY. *The minimal ideal of a compact semigroup S is agreeable with respect to S if and only if (1), (2), (3), or (4) of Proposition 6.4 are satisfied.*

References

[1] J.T. Borrego, "Homomorphic retractions in semigroups", *Proc. Amer. Math. Soc.* 18 (1967), 716-719.

[2] Karol Borsuk, "Quelques retracts singulaires", *Fund. Math.* 24 (1935), 249-258.

[3] A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups* (Math. Surveys 7 (1), Amer. Math. Soc., Providence, 1962).

[4] Aida Beatrijs Paalman-de Miranda, *Topological semigroups* (Mathmatisch Centrum, Amsterdam, 1964).

[5] A.D. Wallace, "The structure of topological semigroups", *Bull. Amer. Math. Soc.* 61 (1955), 95-112.

[6] A.D. Wallace, "Relative ideals II", *Acta. Math. Acad. Sci. Hungar.* 14 (1963), 137-148.

University of Massachusetts
Amherst, Massachusetts, USA.