# Quantitative proofs of certain Algebraic Inequalities.

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1. By a quantitative proof of an inequality I mean one which exhibits the difference between the two magnitudes compared in a form which shows at a glance whether the difference is positive or negative. Such a proof not merely establishes the existence of the inequality, but also gives a measure of its amount.

The formula $$a^2 + b^2 - 2ab = (a - b)^2,$$
as a proof that $a^2 + b^2 > 2ab$, is a characteristic example.

Another is the proof of the important theorem that when $p$ is a positive integer, and $x$ any positive number,

$$\frac{x^{p+1} - 1}{p+1} > \frac{x^p - 1}{p}$$

which is contained in the obvious identity

$$\begin{aligned} & p(x^{p+1} - 1) - (p+1)(x^p - 1) \\ &= (x-1)^2(px^{p-1} + \overline{p-1}\,x^{p-2} + \dots + 1). \end{aligned}$$

Dr Muirhead, in two papers in Vols. XIX. and XXI. of these *Proceedings*, has made some interesting applications of the method; in the present communication I give a few additional developments suggested by a perusal of these papers.

The inequalities with which the paper deals admit of very simple *qualitative* proofs depending on the Theory of Equations, and they have been discussed from that point of view by Euler, Fort, and Schlömilch in the manner indicated in next article.

2. Taking $n$ real positive numbers $a_1, a_2, \dots, a_n$, form the product $$(x + a_1 y)(x + a_2 y) \dots\dots (x + a_n y)$$

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and write its expansion in the form

$$A_0x^n + nA_1x^{n-1}y + \frac{n(n-1)}{1.2}A_2x^{n-2}y^2 + \ldots + nA_{n-1}xy^{n-1} + A_ny^n.$$

If we differentiate $n-r-1$ times as to $x$, and $r-1$ times as to $y$, then by a fundamental theorem the resulting quadratic function, which to a constant factor is

$$A_{r-1}x^2 + 2A_rxy + A_{r+1}y^2$$

has its linear factors real.

Hence $A_r^2 - A_{r-1}A_{r+1}$ is positive. - - - - - - $(a)$

We have therefore

$$\frac{A_1}{A_0} > \frac{A_2}{A_1} > \frac{A_3}{A_2} > \ldots > \frac{A_n}{A_{n-1}}.$$

In particular, if $s \nless r$

$$\frac{A_r}{A_{r-1}} > \frac{A_{s+1}}{A_s}$$

and $A_rA_s - A_{r-1}A_{s+1}$ is positive. - - - - - - $(b)$

Also $$\frac{A_r}{A_{r-1}} \cdot \frac{A_{r-1}}{A_{r-2}} \cdots \frac{A_{r-p+1}}{A_{r-p}} > \frac{A_{s+1}}{A_s}\,\frac{A_{s+2}}{A_{s+1}} \cdots \frac{A_{s+p}}{A_{s+p-1}},$$

that is $$\frac{A_r}{A_{r-p}} > \frac{A_{s+p}}{A_s}$$

or $A_rA_s - A_{r-p}A_{s+p}$ is positive. - - - - - - $(c)$

Again from $(a)$ we have $A_r^{2r} > A_{r-1}^r\,A_{r+1}^r$, and therefore

$$\frac{A_r^{r+1}}{A_{r+1}^r} > \frac{A_{r-1}^r}{A_r^{r-1}} > \frac{A_{r-2}^{r-1}}{A_{r-1}^{r-2}} > \ldots\ldots > \frac{A_1^2}{A_2} > 1$$

and $A_r^{r+1} - A_{r+1}^r$ is positive. - - - - - - $(d)$

Generally, if we raise the inequalities

$$A_1^2 > A_0A_2$$
$$A_2^2 > A_1A_3$$
$$A_3^2 > A_2A_4$$
$$\ldots\ldots\ldots\ldots$$

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to the positive integral powers $a_1$, $a_2$, $a_3$, ... and multiply, we get

$$A_1^{2a_1}A_2^{2a_2}A_3^{2a_3}\ldots\ldots > A_0^{a_1}A_1^{a_2}A_2^{a_1+a_3}A_3^{a_2+a_4}\ldots$$

or say, when the common factors are removed

$$A_r^{a}A_s^{\beta}\ldots\ldots > A_p^{\gamma}A_q^{\delta}\ldots\ldots$$

so that $$A_r^{a}A_s^{\beta}\ldots\ldots - A_p^{\gamma}A_q^{\delta}\ldots\ldots \text{ is positive.} \qquad (e)$$

Dr Muirhead, in Vol. XXI., proves (*a*) by expressing $A_r^2 - A_{r-1}A_{r+1}$ as a sum of terms each of which is manifestly positive. The problem proposed here is to do the same thing for the general form (*e*) and in particular for the special cases (*b*), (*c*), (*d*). Moreover we stipulate that the functions as the sum of which the difference (*e*) is expressed, besides being patently positive, shall be *integral* functions of the *a*'s. This restriction aside, the problem would be extremely simple. In fact, given a number of inequalities between positive quantities, such as

$$x_1 > y_1$$
$$x_2 > y_2$$
$$x_3 > y_3$$
$$\ldots\ldots\ldots$$

from which we can deduce $x_1x_2x_3\ldots > y_1y_2y_3\ldots$, nothing is easier than to write the difference

$$x_1x_2x_3\ldots\ldots - y_1y_2y_3\ldots\ldots$$

in such a form as to show its essentially positive character.

For example, we may write

$$x_1 = y_1 + (x_1 - y_1)$$

$$x_2 = y_2 + (x_2 - y_2), \text{ etc.,}$$

and then the extended product of the binomial factors

$$(y_1 + \overline{x_1 - y_1})(y_2 + \overline{x_2 - y_2})\ldots\ldots$$

contains $y_1y_2\ldots$ together with obviously positive terms.

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Or better, we may write

$$x_1x_2 \quad - y_1y_2 \quad = x_2(x_1 - y_1) + y_1(x_2 - y_2)$$

$$x_1x_2x_3 - y_1y_2y_3 = (x_1x_2 - y_1y_2)x_3 + y_1y_2(x_3 - y_3)$$

$$= x_2x_3(x_1 - y_1) + y_1x_3(x_2 - y_2) + y_1y_2(x_3 - y_3)$$

the general result corresponding to which is obvious.

The form ($e$) *before common factors have been removed* from the two terms, could be treated in this way and expressed in terms of the differences ($a$). But the simplified form of ($e$) and in particular the forms ($b$), ($c$), ($d$) would thus be given as a sum of terms which, while plainly positive, would in general involve powers of the A's in the denominators,

$$e.g., \quad A_1A_2 - A_0A_3 = \frac{A_1}{A_2}(A_2^2 - A_1A_3) + \frac{A_3}{A_2}(A_1^2 - A_0A_2).$$

It will be shown, however, that by making use in the way thus suggested of forms of the type ($b$) in addition to those of the type ($a$) the difficulty about fractions can be surmounted.

What we do, then, is this:—*First*, we investigate expressions for $A_rA_s - A_{r-1}A_{s+1}$ where $s \nless r$, as a sum of evidently positive terms. The method used is different from Muirhead's, but some of the expressions found reduce to those given by him when $r = s$. *Second*, we express the differences ($c$), ($d$), ($e$) as linear functions of forms of the types ($a$) and ($b$), the coefficients of these functions being products of integral powers of the A's, or sums of such.

3. Given $n$ letters $a$, $b$, $c$, ..., which in the applications to inequalities we shall suppose to denote positive numbers;

$$\text{let} \quad P_r \equiv \Sigma abc \ldots\ldots \text{ to } r \text{ factors};$$

also let $P_0 = 1$, and when the integer $r$ is negative or greater than $n$, $P_r = 0$.

In $P_r$ let A = sum of the terms containing $a$,
B = " " " " " $b$, etc.

In $P_s$ let A′ = sum of the terms *not* containing $a$,
B′ = " " " " " " $b$, etc.

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Then we have

$$\begin{aligned} A + B + C + \ldots\ldots &= rP_r, \\ A' + B' + C' + \ldots\ldots &= (n-s)P_s, \\ \text{and} \quad \frac{A}{a} + \frac{B}{b} + \frac{C}{c} + \ldots\ldots &= (n-r+1)P_{r-1}; \\ aA' + bB' + cC' + \ldots\ldots &= (s+1)P_{s+1} \end{aligned}$$

as we see at once by counting the number of times any particular term occurs in the expressions on the left.

Thus $r(n-s)P_rP_s - (n-r+1)(s+1)P_{r-1}P_{s+1}$

$$= (A+B+\ldots)(A'+B'+\ldots) - \left(\frac{A}{a}+\frac{B}{b}+\ldots\right)(aA'+bB'+\ldots)$$

$$= \Sigma\{AB'(1-b/a) + A'B(1-a/b)\}$$

$$= \Sigma(a-b)\left(\frac{A}{a}B' - \frac{B}{b}A'\right). \quad - \quad - \quad - \quad - \quad - \quad - \quad (1)$$

Now we may write

$$P_r = abT + (a+b)S + R,$$
$$P_s = abT' + (a+b)S' + R'$$

where T, S, R, T′, S′, R′ do not contain $a$ or $b$.

With this notation

$$\begin{aligned} A &= a(bT + S), \\ B &= b(aT + S) \\ \text{and} \quad A' &= bS' + R', \\ B' &= aS' + R'. \end{aligned}$$

The typical term on the right of (1) is then

$$(a-b)\{(bT+S)(aS'+R') - (aT+S)(bS'+R')\}$$
$$= (a-b)^2(SS' - TR')$$
$$= (a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab} - P_{r-2}^{ab}P_s^{ab}) \quad - \quad - \quad - \quad - \quad - \quad (2)$$

where we use the symbol $P_t^{ab}$ to denote $\Sigma cd\ldots$ to $t$ factors taken from the $n-2$ letters left when $a$, $b$ are excluded from the original $n$.

Hence from (1)

$$r(n-s)P_rP_s - (n-r+1)(s+1)P_{r-1}P_{s+1}$$
$$= \Sigma(a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab} - P_{r-2}^{ab}P_s^{ab}). \quad - \quad - \quad - \quad - \quad (3)$$

On this formula the whole of the succeeding work is based.

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4. If the cofactor of $(a-b)^2$ in (2) be fully expanded, the numerical coefficient of every term will be positive, provided $s$ is not less than $r$, as we shall always suppose to be the case.

For consider $P_rP_s - P_{r-1}P_{s+1}$, involving $t$ letters $a_1, a_2, \ldots, a_t$.

The product $P_rP_s$, when the number of letters is sufficient, involves terms of the types

$$a_1a_2 \ldots\ldots a_r . a_{r+1} \ldots a_{r+s}\ , \text{ coefficient} = (r+s)!/(r!s!)$$
$$a_1^2a_2 \ldots\ldots a_r . a_{r+1} \ldots a_{r+s-1}, \text{ coefficient} = (r+s-2)!/(r-1)!(s-1)!$$
$$a_1^2a_2^2a_3 \ldots a_r . a_{r+1} \ldots a_{r+s-2}, \text{ coefficient} = (r+s-4)!/(r-2)!(s-2)!$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_1^2a_2^2 \ldots\ldots a_r^2 . a_{r+1} \ldots a_s\ , \text{ coefficient} = 1.$$

When $r+s>t$, some of the earlier terms will not occur, but those terms which do occur have the coefficient stated.

$P_{r-1}P_{s+1}$ will involve terms of the same types, except the last, and the coefficients will be

$$(r+s)!/(r-1)!(s+1)!,\quad (r+s-2)!/(r-2)!s!,\quad \text{etc.}$$

In $P_rP_s - P_{r-1}P_{s+1}$ the coefficient of the terms containing $p$ squares will therefore be

$$(r+s-2p)!\{1/(r-p)!(s-p)! - 1/(r-p-1)!(s-p+1)!\}$$
$$= (s-r+1)(r+s-2p)!/(r-p)!(s-p+1)!$$

which is positive.

We may put this result in the form

$$P_rP_s - P_{r-1}P_{s+1}$$
$$= \Sigma a_1^2a_2^2 \ldots a_r^2 a_{r+1} \ldots a_s$$
$$+ (s-r+1)\Sigma a_1^2a_2^2 \ldots a_{r-1}^2 a_r \ldots a_{s+1} + \ldots$$
$$+ (s-r+1)\frac{(r+s-2p)!}{(r-p)!(s-p+1)!}\Sigma a_1^2a_2^2 \ldots a_p^2 a_{p+1}a_{p+2} \ldots a_{r+s-p} + \ldots \quad (4)$$

When $r+s \not> t$, the last term will be

$$\frac{(s-r+1)(r+s)!}{r!(s+1)!}\Sigma a_1a_2 \ldots a_{r+s},$$

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but when $r+s>t$, the series will stop when $r+s-p=t$, that is for $p=r+s-t$. If we substitute from (4) in the right of (3) we obtain a formula for the left of (3) as a sum of positive terms. For the case $r=s$ this is Muirhead's formula (33), Vol. XXI., page 156.

5. From the equation (3) itself we can derive other interesting expansions of $P_r P_s - P_{r-1} P_{s+1}$ serving like (4) to show that this function contains only positive terms.

Consider first the form which (3) takes when in addition to the $n$ letters $a, b, c, \ldots$, there are other $\nu$ letters $\alpha, \beta, \gamma, \ldots$. This may be written

$$r(n+\nu-s)P_r P_s - (n+\nu-r+1)(s+1)P_{r-1}P_{s+1}$$
$$= \Sigma(a-b)^2(P_{r-1}^{ab} P_{s-1}^{ab} - P_{r-2}^{ab} P_s^{ab})$$
$$+ \Sigma(a-\alpha)^2(P_{r-1}^{a\alpha} P_{s-1}^{a\alpha} - P_{r-2}^{a\alpha} P_s^{a\alpha})$$
$$+ \Sigma(\alpha-\beta)^2(P_{r-1}^{\alpha\beta} P_{s-1}^{\alpha\beta} - P_{r-2}^{\alpha\beta} P_s^{\alpha\beta}).$$

Now put $\alpha, \beta, \gamma, \ldots$ all equal to 0, and this becomes

$$r(n+\nu-s)P_r P_s - (n+\nu-r+1)(s+1)P_{r-1}P_{s+1}$$
$$= \Sigma(a-b)^2(P_{r-1}^{ab} P_{s-1}^{ab} - P_{r-2}^{ab} P_s^{ab})$$
$$+ \quad \nu\Sigma a^2 \quad (P_{r-1}^{a} P_{s-1}^{a} - P_{r-2}^{a} P_s^{a})$$

where, of course, only the $n$ letters $a, b, c, \ldots$ are now involved.

$\nu$ being arbitrary, we must have

$$rP_r P_s - (s+1)P_{r-1}P_{s+1} = \Sigma a^2(P_{r-1}^{a} P_{s-1}^{a} - P_{r-2}^{a} P_s^{a}) \quad \cdot \quad (5)$$

which may be written in either of the forms

$$(s+1)(P_r P_s - P_{r-1}P_{s+1}) = (s-r+1)P_r \;\; P_s \;\; + \Sigma a^2(P_{r-1}^{a} P_{s-1}^{a} - P_{r-2}^{a} P_s^{a}), \quad (6)$$

$$r(P_r P_s - P_{r-1}P_{s+1}) = (s-r+1)P_{r-1}P_{s+1} + \Sigma a^2(P_{r-1}^{a} P_{s-1}^{a} - P_{r-2}^{a} P_s^{a}). \quad (6)'$$

(6) or (6)′ may be used as a reduction formula for expressing $P_r P_s - P_{r-1}P_{s+1}$ as a sum of positive terms. Taking (6) we get

$$P_{r-1}^{a} P_{s-1}^{a} - P_{r-2}^{a} P_s^{a} = \frac{s-r+1}{s} P_{r-1}^{a} P_{s-1}^{a} + \frac{1}{s}\Sigma b^2(P_{r-2}^{ab} P_{s-2}^{ab} - P_{r-3}^{ab} P_{s-1}^{ab})$$

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and therefore by substitution in (6)

$$P_r P_s - P_{r-1} P_{s+1} = \frac{s-r+1}{s+1} P_r P_s + \frac{s-r+1}{s \cdot s+1} \Sigma a^2 P^a_{r-1} P^a_{s-1}$$
$$+ \frac{2}{s \cdot s+1} \Sigma a^2 b^2 (P^{ab}_{r-2} P^{ab}_{s-2} - P^{ab}_{r-3} P^{ab}_{s-1}).$$

We can apply (6) again to the last term, and so on, and obtain finally

$$P_r P_s - P_{r-1} P_{s+1} = (s-r+1)\frac{1}{s+1} P_r P_s + (s-r+1)\frac{1}{s+1 \cdot s} \Sigma a^2 P^a_{r-1} P^a_{s-1}$$
$$+ (s-r+1)\frac{1 \cdot 2}{s+1 \cdot s \cdot s-1} \Sigma a^2 b^2 P^{ab}_{r-2} P^{ab}_{s-2} + \ldots\ldots$$
$$+ \frac{r\,!}{s+1 \cdot s \ldots\ldots (s-r+2)} \Sigma(a^2 b^2 \ldots,\ r \text{ letters}) P^{ab \ldots,\ r \text{ letters}}_{s-r}. \quad (7)$$

A similar formula can be derived from (6)′.

If we use the formula (7) to develop (3) we get

$$\frac{s}{s-r+1}\{r(n-s)P_r P_s - (n-r+1)(s+1)P_{r-1}P_{s+1}\}$$
$$= \Sigma(a-b)^2 P^{ab}_{r-1} P^{ab}_{s-1} + \frac{1}{s-1} \Sigma(a-b)^2 c^2 P^{abc}_{r-2} P^{abc}_{s-2}$$
$$+ \frac{1 \cdot 2}{s-1 \cdot s-2} \Sigma(a-b)^2 c^2 d^2 P^{abcd}_{r-3} P^{abcd}_{s-3} + \ldots\ldots$$
$$+ \frac{1 \cdot 2 \ldots (r-1)}{s-1 \cdot s-2 \ldots (s-r+1)} \Sigma(a-b)^2 (c^2 d^2 \ldots,\ \overline{r-1} \text{ letters}) P^{ab \ldots,\ (r+1) \text{ letters}}_{s-r} \quad (8)$$

which when $r = s$ becomes Muirhead's (38), page 157, Vol. XXI.

6. By a simple transformation we can derive from (8) another formula which is perhaps simpler and in certain cases will contain fewer terms.

One special case we may note at once. Put $s = n-1$ in (3) and we have immediately

$$rP_r P_{n-1} - (n-r+1)nP_{r-1}P_n = \Sigma(a-b)^2 P^{ab}_{r-1} P^{ab}_{n-2},$$

since $P^{ab}_{n-1}$, referring to only $n-2$ letters, is zero. For this case (8) is clearly a much longer formula.

For the general case, apply (8) not to $a$, $b$, $c$, ... themselves, but to their reciprocals $1/a$, $1/b$, $1/c$, ... .

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Obviously $P_r$ becomes $P_{n-r}/P_n$; $P^{ab}_{r-1}$ becomes $P^{ab}_{n-r-1}/P^{ab}_{n-2}$, that is, $ab\,P^{ab}_{n-r-1}/P_n$, and so on. Hence, multiplying every term by $P_n^2$,

$$\frac{s}{s-r+1}\{r(n-s)P_{n-r}P_{n-s}-(n-r+1)(s+1)P_{n-r+1}P_{n-s-1}\}$$

$$=\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2 a^2b^2P^{ab}_{n-r-1}P^{ab}_{n-s-1}+\frac{1}{s-1}\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2\frac{1}{c^2}\cdot a^2b^2c^2\,P^{abc}_{n-r-1}\,P^{abc}_{n-s-1}$$

$$+\frac{1\,.\,2}{s-1\,.\,s-2}\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2\frac{1}{c^2d^2}\cdot a^2b^2c^2d^2\,P^{abcd}_{n-r-1}\,P^{abcd}_{n-s-1}+\ldots\ldots$$

$$+\frac{1\,.\,2\ldots(r-1)}{s-1\,.\,s-2\ldots(s-r+1)}\Sigma\left(\frac{1}{a}-\frac{1}{b}\right)^2\frac{1}{c^2d^2\ldots,\,(r-1)\text{ letters}}(a^2b^2c^2d^2\ldots,\,\overline{r+1}\text{ letters})(P_{n-r-1}P_{n-s-1})^{abc\ldots,\,r+1\text{ letters}}$$

or, changing $r$ into $n-s$ and $s$ into $n-r$,

$$\frac{n-r}{s-r+1}\{r(n-s)\,P_rP_s-(n-r+1)(s+1)\,P_{r-1}\,P_{s+1}\}$$

$$=\Sigma\,(a-b)^2P^{ab}_{r-1}P^{ab}_{s-1}+\frac{1}{n-r-1}\Sigma(a-b)^2P^{abc}_{r-1}P^{abc}_{s-1}+\frac{1\,.\,2}{n-r-1\,.\,n-r-2}\Sigma(a-b)^2P^{abcd}_{r-1}P^{abcd}_{s-1}+\ldots\ldots$$

$$+\frac{1\,.\,2\ldots(n-s-1)}{n-r-1\,.\,n-r-2\ldots(s-r+1)}\Sigma(a-b)^2(P_{r-1}\,P_{s-1})^{abc\ldots,\,\overline{n-s+1}\text{ letters}}$$



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7. By a slightly different method of manipulating (3) we can obtain still another expression for $r(n-s)P_rP_s-(n-r+1)(s+1)P_{r-1}P_{s+1}$ as a sum of positive terms.

We can write (3) in the form

$$(n-r+1)(s+1)(P_rP_s-P_{r-1}P_{s+1})$$
$$=(n+1)(s-r+1)P_rP_s+\Sigma(a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab}-P_{r-2}^{ab}P_s^{ab})$$

and use this as a formula of reduction to expand the last term on the right.

We thus find

$$r(n-s)P_rP_s-(n-r+1)(s+1)P_{r-1}P_{s+1}$$
$$\{=\Sigma(a-b)^2(P_{r-1}^{ab}P_{s-1}^{ab}-P_{r-2}^{ab}P_s^{ab})\}$$
$$=\frac{(n-1)(s-r+1)}{s(n-r)}\Sigma(a-b)^2P_{r-1}^{ab}P_{s-1}^{ab}+\frac{1\,.\,2\,.\,n-3\,.\,s-r+1}{s\,.\,s-1\,.\,n-r\,.\,n-r-1}\Sigma(a-b)^2(c-d)^2(P_{r-2}P_{s-2})^{abcd}$$
$$+\ldots+\frac{q\,!\,(n+1-2q)(s-r+1)}{s.s-1..s-q+1.n-r.n-r-1..n-r-q+1}\Sigma(a-b)^2(c-d)^2..,\ 2q\text{ letters }(P_{r-q}P_{s-q})^{abcd\ldots,\text{ these }2q\text{ letters}}$$
$$+\ldots\ldots\ldots\ldots \qquad (10)$$

For the last term $q=$ the smaller of $r$, $n-s$.

The transformation of Art. 6 when applied to (10) simply reproduces the formula.

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8. The preceding formulæ, viz., (8), (9), (10) and the result of substituting (4) in (3) furnish various expressions for the difference

$$r(n-s)P_rP_s-(n-r+1)(s+1)P_{r-1}P_{s+1}$$

as a sum of terms obviously positive when $a$, $b$, $c$, etc., are positive and not all equal, and $r$ is less than or equal to $s$.

If as in Art. 2 we put ${}_nC_r \cdot A_r$ for $P_r$, and so on, this difference takes the form $\quad r(n-s)_nC_r \cdot {}_nC_s(A_rA_s-A_{r-1}A_{s+1})$.

We have thus obtained expressions evidently positive for

$$A_r^2-A_{r-1}A_{r+1}$$

$$\text{and}\quad A_rA_s-A_{r-1}A_{s+1}\quad(r<s)$$

thus accounting for cases ($a$) and ($b$) of Art. 2.

In accordance with the statement at the end of that Article, the general problem now before us is to throw any difference of the type ($e$) into the form of a sum of terms such as $K(A_rA_s-A_{r-1}A_{s+1})$ where $r \not> s$ and K is a positive product of the powers of the A's; a problem of some interest even apart from the application to inequalities.

9. The forms ($c$) and ($d$) are particular cases of ($e$), but as they are specially simple and interesting, we shall deal with them here individually before proceeding to a method applicable to the general form.

We have immediately

$$\begin{aligned}A_rA_s-A_{r-p}A_{s+p}&=(A_rA_s-A_{r-1}A_{s+1})+(A_{r-1}A_{s+1}-A_{r-2}A_{s+2})+\ldots\\&\quad+(A_{r-p+1}A_{s+p-1}-A_{r-p}A_{s+p}). \qquad (11)\end{aligned}$$

$$\begin{aligned}A_r^{r+1}-A_0A_{r+1}^r &= (A_r^{r+1}-A_{r-1}A_r^{r-1}A_{r+1})+(A_{r-1}A_r^{r-1}A_{r+1}-A_{r-2}A_r^{r-2}A_{r+1}^2)\\&\quad+(A_{r-2}A_r^{r-2}A_{r+1}^2-A_{r-3}A_r^{r-3}A_{r+1}^3)+\ldots\ldots\\&\quad+(A_2A_r^2A_{r+1}^{r-2}-A_1A_rA_{r+1}^{r-1})+(A_1A_rA_{r+1}^{r-1}-A_0A_{r+1}^r)\\&= A_r^{r-1}(A_r^2-A_{r-1}A_{r+1})+A_r^{r-2}A_{r+1}(A_{r-1}A_r-A_{r-2}A_{r+1})\\&\quad+A_r^{r-3}A_{r+1}^2(A_{r-2}A_r-A_{r-3}A_{r+1})+\ldots\ldots\\&\quad+A_rA_{r+1}^{r-2}(A_2A_r-A_1A_{r+1})+A_{r+1}^{r-1}(A_1A_r-A_0A_{r+1}). \qquad (12)\end{aligned}$$

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10. The general difference is what is obtained from

$$A_1^{2a_1} A_2^{2a_2} \ldots A_t^{2a_t} - A_0^{a_1} A_1^{a_2} A_2^{a_1+a_3} A_3^{a_2+a_4} \ldots A_{t-1}^{a_{t-2}+a_t} A_t^{a_{t-1}} A_{t+1}^{a_t}, \quad (13)$$

when all factors, in the form of powers of A's, which are common to the two terms are removed.

Some of the $a$'s may not occur, that is, they may be zero; but none of them are to be negative; when this condition is fulfilled we shall call (13) or (13) simplified a *proper* difference.

Even after simplification, note that if $a_p$ and $a_t$ are the $a$'s of lowest and highest order, then $A_{p-1}$ and $A_{t+1}$ are the lowest and highest A's which occur, and they occur in the *negative* term in the form $A_{p-1}^{a_p}$, $A_{t+1}^{a_t}$.

Any difference being given, even in its simplified form, its $a$'s are determinate and can easily be found. Thus suppose that a given difference is

$$A_1^{x_1} A_2^{x_2} \ldots A_t^{x_t} - A_0^{y_0} A_1^{y_1} A_2^{y_2} \ldots A_t^{y_t} A_{t+1}^{y_{t+1}} \quad \cdot \quad (14)$$

where, if the difference has been simplified, one of $x_r$, $y_r$ is zero.

If this is derivable from (13) by mere elision of common factors, then

$$\left.\begin{array}{lll} \text{considering} & A_0, \text{ we get} & a_1 = y_0 \\ & A_1, & a_2 - 2a_1 = y_1 - x_1 \\ & A_2, & a_3 - 2a_2 + a_1 = y_2 - x_2 \\ & \ldots\ldots\ldots\ldots & \\ & A_{t-1}, & a_t - 2a_{t-1} + a_{t-2} = y_{t-1} - x_{t-1} \\ & A_t, & a_{t-1} - 2a_t = y_t - x_t \\ & A_{t+1}, & a_t = y_{t+1} \end{array}\right\} \quad \cdot \quad \cdot \quad (15)$$

The first $t$ of these equations give in succession $a_1, a_2, \ldots, a_t$. The remaining two equations give

$$y_0 + (y_1 - x_1) + (y_2 - x_2) + \ldots + (y_t - x_t) + y_{t+1} = 0$$

and $\quad (y_1 - x_1) + 2(y_2 - x_2) + 3(y_3 - x_3) + \ldots + t(y_t - x_t) + (t+1)y_{t+1} = 0.$

These are the conditions that (14) should be derivable from a form of the type (13), and express the facts that the two terms of (14) are homogeneous in the A's and also in $a, b, c, \ldots$.

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The values of the $a$'s are easily found explicitly from (15). Thus

$$a_1 = y_0,$$
$$a_2 - a_1 = y_0 + (y_1 - x_1),$$
$$a_3 - a_2 = y_0 + (y_1 - x_1) + (y_2 - x_2), \text{ etc.}$$

so that

$$\left.\begin{aligned} a_1 &= y_0, \\ a_2 &= 2y_0 + (y_1 - x_1), \\ a_3 &= 3y_0 + 2(y_1 - x_1) + (y_2 - x_2), \\ a_4 &= 4y_0 + 3(y_1 - x_1) + 2(y_2 - x_2) + (y_3 - x_3), \text{ etc.} \end{aligned}\right\} \quad (16)$$

For example, for $\quad A_r A_s - A_{r-p} A_{s+p} \quad (s \nless r)$

the $a$'s that occur are

$$\left.\begin{aligned} a_{r-p+1} &= 1, \; a_{r-p+2} = 2, \; a_{r-p+3} = 3, \; \ldots, \; a_r = p; \\ a_{r+1} &= a_{r+2} = \ldots = a_s = p; \\ a_{s+1} &= p - 1, \; a_{s+2} = p - 2, \; \ldots\ldots, \; a_{s+p-1} = 1. \end{aligned}\right\}$$

In particular, for $\quad A_r A_s - A_{r-1} A_{s+1}$

$$a_r = a_{r+1} = a_{r+2} = \ldots = a_s = 1.$$

Again for $\quad A_r^{r+1} - A_0 A_{r+1}^r$

$$a_1 = 1, \; a_2 = 2, \; a_3 = 3, \; \ldots, \; a_r = r.$$

11. It is now easy to explain a method of reducing any given proper difference (14) to the shape we desire.

For brevity, denote the given difference by $X - Y$.

Then the process we will give is a step-by-step one, the first step being to put $X - Y$ in the form

$$(X - Y_1) + (Y_1 - Y)$$

where (i) $Y_1 - Y$ is the product of powers of A's by a proper difference of the elementary type $A_r A_s - A_{r-1} A_{s+1}$

and (ii) $X - Y_1$ is a proper difference with some of its $a$'s smaller, and none larger, than those of $X - Y$.

Suppose that $A_{r-1} A_{s+1}$ occurs in $Y$ as a factor.

For $Y_1$ take what $Y$ becomes when this factor is replaced by $A_r A_s$.

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Then (i) $Y_1 - Y$ is clearly $(A_r A_s - A_{r-1} A_{s+1}) \times$ a product of powers of A's,

and (ii) $X - Y_1$ is a difference in which the values of all the $\alpha$'s from $\alpha_r$ to $\alpha_s$ inclusive are reduced by 1 from their values in $X - Y$, the other $\alpha$'s remaining unchanged; as may easily be verified from equations (16).

More generally, if $(A_{r-1} A_{s+1})^m$ occurs in Y as a factor, and we form $Y_1$ by replacing this by $(A_r A_s)^m$, then

(i) $Y_1 - Y$ will be $(A_r^m A_s^m - A_{r-1}^m A_{s+1}^m) \times$ a product of powers of A's, that is, $(A_r A_s - A_{r-1} A_{s+1}) \times$ an integral function of A's with positive coefficients;

and (ii) $X - Y_1$ will have all the $\alpha$'s from $\alpha_r$ to $\alpha_s$ inclusive reduced by $m$ from their values in $X - Y$.

For the success of this process of reduction based on replacing $A_{r-1} A_{s+1}$ in Y by $A_r A_s$ it is necessary

(i) that $A_{r-1} A_{s+1}$ should occur in Y;

(ii) that $X - Y_1$ should still be a *proper* difference, which requires that none of the new $\alpha$'s should be negative, or that in $X - Y$ all the $\alpha$'s from $\alpha_r$ to $\alpha_s$ inclusive should occur.

Both these conditions will be fulfilled if we take for $r$ the index of the lowest $\alpha$ that occurs in $X - Y$, in which case $A_{r-1}$ certainly occurs in Y, as pointed out near the beginning of Art. 10; and if at the same time we choose $s$ so that $\alpha_{s+1}$ is the first $\alpha$ after $\alpha_r$ which does *not* occur in Y, in which case $A_{s+1}$ certainly occurs in Y, viz., raised to the power $\alpha_s + \alpha_{s+2}$, as we see from (13) since $A_{s+1}$ does not occur in X.

More generally, we can replace at one step $(A_{r-1} A_{s+1})^m$ by $(A_r A_s)^m$ provided, with $r$ and $s$ chosen as just explained, every $\alpha$ from $\alpha_r$ to $\alpha_s$ inclusive is equal to $m$ at least.

Having thus got $X - Y = (X - Y_1) + (Y_1 - Y)$ we proceed in the same way with $X - Y_1$ and finally obtain

$$X - Y = (X - Y_n) + (Y_n - Y_{n-1}) + \ldots + (Y_2 - Y_1) + (Y_1 - Y)$$

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where each of the differences in brackets is of the type

$$(A_r A_s - A_{r-1} A_{s+1}) \times$$

an integral function of A's with positive coefficients.

The reader may take the results of Art. 9 as easy examples of this process.

As another example consider the case

$$X = A_2^7 A_4^4 A_6^4\,; \quad Y = A_0 A_1^4 A_3^3 A_5^4 A_7^3.$$

The scheme of $a$'s, originally, and after each step, is

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $X - Y$. | 1 | 6 | 4 | 5 | 2 | 3 |
| $X - Y_1$. | | 5 | 3 | 4 | 1 | 2 |
| $X - Y_2$. | | 4 | 2 | 3 | | 1 |
| $X - Y_3$. | | 2 | | 1 | | 1 |
| $X - Y_4$. | | | | 1 | | 1 |
| $X - Y_5$. | | | | | | 1 |

Therefore

$$Y_1 = A_1^5 A_3^3 A_5^4 A_6 A_7^2$$

$$Y_2 = A_1^4 A_2 A_3^3 A_5^4 A_6^2 A_7$$

$$Y_3 = A_1^2 A_2^3 A_3^3 A_4^2 A_5^2 A_6^2 A_7$$

$$Y_4 = A_2^7 A_3 A_4^2 A_5^2 A_6^2 A_7$$

$$Y_5 = A_2^7 A_4^4 A_5 A_6^2 A_7$$

and

$$\begin{aligned} X - Y = &\ (X - Y_5) + (Y_5 - Y_4) + (Y_4 - Y_3) + (Y_3 - Y_2) + (Y_2 - Y_1) + (Y_1 - Y) \\ = &\ A_2^7 A_4^4 A_6^2 (A_6^2 - A_5 A_7) \\ &+ A_2^7 A_4^2 A_5 A_6^2 A_7 (A_4^2 - A_3 A_5) \\ &+ A_2^3 A_3 A_4^2 A_5^2 A_6^2 A_7 (A_2^4 - A_1^2 A_3^2) \\ &+ A_1^2 A_2 A_3^3 A_5^2 A_6^2 A_7 (A_2^2 A_4^2 - A_1^2 A_5^2) \\ &+ A_1^4 A_3^3 A_5^4 A_6 A_7 (A_2 A_6 - A_1 A_7) \\ &+ A_1^4 A_3^3 A_5^4 A_7^2 (A_1 A_6 - A_0 A_7), \end{aligned}$$

where we may replace

$$A_2^4 - A_1^2 A_3^2 \quad \text{by} \quad (A_2^2 + A_1 A_3)(A_2^2 - A_1 A_3)$$

and $$A_2^2 A_4^2 - A_1^2 A_5^2 \quad \text{by} \quad (A_2 A_4 + A_1 A_5)(A_2 A_4 - A_1 A_5).$$

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12. The preceding process for decomposing a "proper difference" is simple and straightforward, but it is not to be inferred that the decomposition obtained is the only one possible which would serve the purpose. Take for example

$$A_3^4 - A_0A_4^3.$$

As in Art. 9 this is

$$A_3^2(A_3^2 - A_2A_4) + A_3A_4(A_2A_3 - A_1A_4) + A_4^2(A_1A_3 - A_0A_4).$$

But it may also be put in the form

$$(A_3^2 + A_2A_4)(A_3^2 - A_2A_4) + A_4^2(A_2^2 - A_1A_3) + A_4^2(A_1A_3 - A_0A_4).$$

The latter form, however, contains a term more than the other, if we count $(A_rA_s - A_{r-1}A_{s+1})$ as one term, but $(A_r^mA_s^m - A_{r-1}^mA_{s+1}^m)$, *i.e.*, $(A_rA_s - A_{r-1}A_{s+1})(A_r^{m-1}A_s^{m-1} + \dots)$ as $m$ terms.

So, generally, it may happen that it is possible to make two or more steps of the single step by which in the above process we change $Y_a$ into $Y_b$ by replacing $A_{r-1}A_{s+1}$ by $A_rA_s$. For if an A, say $A_{x+1}$, intermediate to $A_{r-1}$ and $A_{s+1}$ occur in $Y_a$, as it may very well do, we may first change $Y_a$ into $Y_c$, say, by replacing $A_{r-1}A_{x+1}$ by $A_rA_x$, and then change $Y_c$ into $Y_b$ by replacing $A_xA_{s+1}$ by $A_{x+1}A_s$. The only effect of this is to split up a single term of the original final result into two, and therefore to increase the total number of terms by one.

It would therefore seem that the decomposition we have given is that which leads to the minimum number of terms.

13. An inequality of perennial interest is that which holds between the Arithmetic and Geometric Means of a set of positive numbers, and we may conclude by examining what the preceding methods make of this important example.

The Arithmetic Mean of the $n$ numbers $a$, $b$, $c$, ... is

$$(a + b + c + \dots)/n \text{ or } A_1;$$

their Geometric Mean is $(abc \dots)^{1/n}$, *i.e.*, $A_n^{1/n}$.

We prove $A_1 > A_n^{1/n}$ by exhibiting $A_1^n - A_n$ in an explicitly positive form.

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To look for a moment at a more general case, note that by $(d)$

$$A_r^{1/r} > A_{r+1}^{1/\overline{r+1}},$$

and therefore $\qquad A_r^{1/r} > A_s^{1/s}$ if $s > r$;

hence $\qquad A_r^s - A_s^r$ or $A_r^s - A_0^{s-r}A_s^r$ is positive,

is, in fact, a *proper* difference when $s > r$.

The $a$'s of this difference are by (16)

$$a_1 \;= s - r, \qquad a_2 \;= 2(s-r),\; a_3 = 3(s-r), \ldots,\; a_r \;= r(s-r);$$

$$a_{r+1} = r(s-r-1),\; a_{r+2} = r(s-r-2), \ldots\ldots \qquad ,\; a_{s-1} = r.$$

If $r$ and $s$ are given numerically, the reduction by the method of Art. 11 is very simple; for, since the $a$'s diminish steadily from $a_r$ towards both ends of the series, it is clear that at every step we have merely to replace in the negative term the lowest A by the A one higher and the highest A by the A one lower. But it would be somewhat difficult to state a *general* formula for the result, the form of which in fact depends on the relative magnitudes of the various multiples of $r$ and $s$.

For the special case $A_1^n - A_0^{n-1}A_n$ the difficulty does not exist.

The $a$'s are $a_1 = n-1,\; a_2 = n-2,\; a_3 = n-3,\; \ldots,\; a_n = 1$;

and the formula, which may be verified at a glance without reference to the general method, is

$$A_1^n - A_n = A_1^{n-2}(A_1^2 - A_2) + A_1^{n-3}(A_1A_2 - A_3) + A_1^{n-4}(A_1A_3 - A_4) + \ldots$$
$$+ A_1(A_1A_{n-2} - A_{n-1}) + (A_1A_{n-1} - A_n).$$

Now (3) of Art. 3 gives

$$(n-s)P_1P_s - n(s+1)P_{s+1} = \Sigma(a-b)^2P_{s-1}^{ab}$$

$$i.e., \quad n^2\frac{(n-1)(n-2)\ldots(n-s)}{1\,.\,2\ldots\ldots s}(A_1A_s - A_{s+1}) = \Sigma(a-b)^2P_{s-1}^{ab}.$$

Hence

$$A_1^n - A_n = \frac{1}{n^2}\Sigma(a-b)^2\left\{\frac{1}{n-1}A_1^{n-2} + \frac{1\,.\,2}{n-1\,.\,n-2}A_1^{n-3}P_1^{ab} + \frac{1\,.\,2\,.\,3}{n-1\,.\,n-2\,.\,n-3}A_1^{n-4}P_2^{ab} + \ldots + \frac{1}{n-1}A_1P_{n-3}^{ab} + P_{n-2}^{ab}\right\}.$$

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