Canad. Math. Bull. Vol. **49** (2), 2006 pp. 237–246

Approximation by Rational Mappings, via Homotopy Theory

P. M. Gauthier and E. S. Zeron

Abstract. Continuous mappings defined from compact subsets K of complex Euclidean space $\mathbb{C}^n$ into complex projective space $\mathbb{P}^m$ are approximated by rational mappings. The fundamental tool employed is homotopy theory.

1 Introduction

Let $f\colon K \to \mathbb{C}$ be a continuous function defined on a compact subset K of $\mathbb{C}^n$ (resp. an n-complex manifold N), with $n \geq 1$. We say that $f\colon K \to \mathbb{C}$ can be uniformly approximated by rational (resp. holomorphic) functions if for any given $\delta > 0$ there exists a rational (resp. holomorphic) function H well defined on a neighbourhood of K and such that $|H(z) - f(z)| < \delta$ holds for $z \in K$. In particular, a rational function $H = \frac{P}{Q}$ is well defined on a neighbourhood of K whenever the zero locus of Q does not meet K. Moreover, it is a known consequence of the Hartogs Rosenthal theorem that f can be uniformly approximated by rational functions whenever $K \subset \mathbb{C}^n$ has zero two-dimensional Hausdorff measure (see for example [1] or [5]).

On the other hand, suppose we are given a rational function $\frac{P}{Q}$ in irreducible form; that is, such that the pair of holomorphic polynomials P and Q does not have common factors. We automatically have that $\frac{P}{Q}$ is a holomorphic mapping defined from $\mathbb{C}^n \setminus E$ into the Riemann sphere $\mathcal{S}^2 := \mathbb{C} \cup \{\infty\}$, where the critical set E is the intersection of the zero loci of P and Q. The previous statement immediately drives us to consider whether a continuous mapping $g\colon K \to \mathcal{S}^2$ can be uniformly approximated by rational functions, following the ideas presented in [6]. We endow the Riemann sphere with the natural distance

$$d[w,z] := \frac{|w-z|}{(1+w\bar{w})^{1/2}(1+z\bar{z})^{1/2}}, \quad \text{for all } w, z \in \mathcal{S}^2. \tag{1}$$

Thus $d[w,z] = d[\frac{1}{w}, \frac{1}{z}]$ and $d[0,\infty] = 1$. The main objective of this paper is to deduce several sufficient conditions for the possibility of rational approximation of g. The following is related to work of Chirka.

Theorem 1 *Let K be a compact subset of $\mathbb{C}^n$ with zero two-dimensional Hausdorff measure, and $n \geq 1$. Given a continuous mapping $g\colon K \to \mathcal{S}^2$ and any $\delta > 0$, there*

Received by the editors June 2, 2004.
Research supported by CRSNG(Canada), Cinvestav(México) and Conacyt(México).
AMS subject classification: 32E30, 32C18.
Keywords: Rational approximation, homotopy type, null-homotopic.

exists a rational function $\frac{P}{Q}$ defined on $\mathbb{C}^n$, whose critical set does not meet K, and such that $d[\frac{P}{Q}(z), g(z)] < \delta$ for $z \in K$.

Proof First, we have that the Riemann sphere $\mathbb{S}^2$ is an absolute neighbourhood retract, for any compact smooth manifold is homeomorphic to a compact polyhedron, see [10, pp. 332 and 339]. Thus, there exists a continuous mapping $g_2 \colon U \to \mathbb{S}^2$ defined on an open neighbourhood U of K such that $g_2(z) = g(z)$ for every $z \in K$. On the other hand, since the pair $(\mathbb{S}^2, d)$ is a compact metric space, there exists a smooth mapping $g_3 \colon U \to \mathbb{S}^2$ such that $d[g_2(z), g_3(z)] < \frac{\delta}{2}$ for every $z \in U$ (see for example [2, p. 97]).

The image $g_3(K)$ in $\mathbb{S}^2$ has zero two-dimensional Hausdorff Measure, because g_3 is smooth and K has zero two-dimensional Hausdorff measure. Whence, we may find a point β contained in $\mathbb{S}^2 \setminus g_3(K)$. We may even suppose, without lost of generality, that $\beta = \infty$. The previous calculations imply that the restriction $g_3|_K$ is continuous and well defined from K into the complex plane $\mathbb{C}$. Finally, by the Hartogs Rosenthal Theorem, there is a rational function $\frac{P}{Q}$ such that the zero locus of Q does not meet K, and the inequality

$$d\Big[\frac{P}{Q}(z), g_3(z)\Big] \leq \Big|\frac{P}{Q}(z) - g_3(z)\Big| < \frac{\delta}{2}$$

holds for $z \in K$. Thus, we automatically have that $d[\frac{P}{Q}(z), g(z)] < \delta$ on K, and we finish the proof by recalling that the critical set of $\frac{P}{Q}$ is contained in the zero locus of Q. ∎

Thus, the condition of having zero two-dimensional Hausdorff measure is strong enough to assure that we can uniformly approximate continuous mappings whose image is contained in the Riemann sphere rather than the complex plane. The main theorem of this paper asserts that we may also approximate mappings in *fat* compact sets with non-zero or even infinite two-dimensional Hausdorff measure; we only need to add an extra homotopical condition. The second and third sections of this paper present and prove the main theorem of this paper (and a partial converse theorem). Finally, the last part of this paper presents some interesting examples related to null-homotopic functions.

2 Main Theorem

Let $\mathbb{P}^m$ be the m-complex projective space, composed of all the complex lines in $\mathbb{C}^{m+1}$ which pass through the origin. It is well known that $\mathbb{P}^m$ is a m-complex manifold, and that there exists a natural holomorphic projection ρ_m defined from $\mathbb{C}^{m+1} \setminus \{0\}$ onto $\mathbb{P}^m$, which sends any point $(z_0, \dots, z_m) \neq 0$ to the complex line

$$\rho_m(z_0, \dots, z_m) = [z_0, \dots, z_m] := \{(z_0 t, \dots, z_m t) : t \in \mathbb{C}\}.$$

In particular, we have that the one-dimensional complex projective space $\mathbb{P}^1$ is the Riemann sphere $\mathbb{S}^2$, and the natural holomorphic projection ρ_1 is given by $\rho_1(w, z) =$

https://doi.org/10.4153/CMB-2006-024-4 Published online by Cambridge University Press

w/z. Whence, any rational mapping $\frac{P}{Q}$ defined on $\mathbb{C}^n$ is the composition $\rho_1(P, Q)$, where (P, Q) is a holomorphic polynomial function from $\mathbb{C}^n$ into $\mathbb{C}^2$. The critical set E of $\frac{P}{Q}$ is the inverse image $(P, Q)^{-1}(0)$, and so $\frac{P}{Q}$ is a holomorphic mapping defined from $\mathbb{C}^n \setminus E$ into $\mathbb{S}^2$. We may now extend the notion of rational mapping to consider the natural projections ρ_m for $m \geq 1$.

Definition 2 A rational mapping based on $\mathbb{C}^n$, with image in $\mathbb{P}^m$, is defined as the composition $\rho_m(R)$, for a given holomorphic polynomial mapping $R\colon \mathbb{C}^n \to \mathbb{C}^{m+1}$. The critical set E of $\rho_m(R)$ is then defined as the inverse image $R^{-1}(0)$, and so $\rho_m(R)$ is a holomorphic mapping defined from $\mathbb{C}^n \setminus E$ into $\mathbb{P}^m$.

We may now present and prove the main theorem of our paper. Here we are considering rational uniform approximation of functions defined on *fat* compact sets with images in $\mathbb{P}^m$. Recall that a continuous function $g\colon K \to \mathbb{P}^m$ is null-homotopic when there exists a continuous function G defined from $K \times [0, 1]$ into $\mathbb{P}^m$ such that $G(z, 1) = g(z)$ and $G(z, 0) = [1, \dots, 1]$, for every $z \in K$.

Theorem 3 *Let K be a compact set in $\mathbb{C}^n$, with $n \geq 1$, such that every continuous function $f\colon K \to \mathbb{C}$ can be uniformly approximated by rational functions. Besides, for $m \geq 1$, let d be a metric on $\mathbb{P}^m$ which induces the topology. Then, given a null-homotopic continuous mapping $g\colon K \to \mathbb{P}^m$ and any $\delta > 0$, there exists a rational mapping $\rho_m(R)$ defined on $\mathbb{C}^n$ whose critical set does not meet K, and such that $d\big[\rho_m\big(R(z)\big), g(z)\big] < \delta$ holds for $z \in K$.*

Proof Let $I = [0, 1]$ be the unit closed interval in the real line. Since g is null-homotopic, there is a continuous mapping $G\colon K \times I \to \mathbb{P}^m$ such that $G(z, 1) = g(z)$ and $G(z, 0) = [1, \dots, 1]$, for every $z \in K$. We may then build the following commutative diagram.

$$\begin{array}{ccc} K & \xrightarrow{\ c\ } & \mathbb{C}^{m+1} \setminus \{0\} \\ \big\downarrow{\scriptstyle \jmath} & & \big\downarrow{\scriptstyle \rho_m} \\ K \times I & \xrightarrow{\ G\ } & \mathbb{P}^m \end{array}$$

where $c(z) = (1, \dots, 1)$ is a constant mapping and $\jmath(z) = (z, 0)$ is the natural inclusion. It is well known that the projection ρ_m induces a locally trivial fibre bundle on $\mathbb{C}^{m+1} \setminus \{0\}$, with base $\mathbb{P}^m$ and fibre $\mathbb{C} \setminus \{0\}$. This fibre bundle has the homotopy lifting property (see [2, pp. 450–455] or [4, pp. 27 and 67]). Hence, there exists a continuous mapping F from $K \times I$ into $\mathbb{C}^{m+1} \setminus \{0\}$ such that $\rho_m(F)$ is identically equal to G on $K \times I$.

On the other hand, Let W be an open set in $\mathbb{C}^{m+1}$ such that $F(z, 1) \in W$ for every $z \in K$. Besides, suppose that the closure $\bar{W}$ is compact and does not contains the origin. Notice that the projection ρ_m from $\mathbb{C}^{m+1} \setminus \{0\}$ into $\mathbb{P}^m$ is continuous with respect to the metric d, because d indeed induces the topology of $\mathbb{P}^m$; so ρ_m is also uniformly continuous on $\bar{W}$. Express $F(z, 1) = \big(a_0(z), \dots, a_m(z)\big)$ as a vector in $\mathbb{C}^{m+1}$. Recalling the given hypotheses and choosing small enough constants $\delta_k > 0$,

https://doi.org/10.4153/CMB-2006-024-4 Published online by Cambridge University Press

for $0 \le k \le m$, we may find rational functions $\frac{P_k}{Q_k}$ such that K meets the zero locus of no Q_k,

$$\left|\frac{P_k}{Q_k}(z) - a_k(z)\right| < \delta_k \quad \text{and} \tag{2}$$

$$\Big(\frac{P_0}{Q_0}(z), \dots, \frac{P_m}{Q_m}(z)\Big) \in \bar{W} \quad \text{whenever } z \in K.$$

Finally, consider the polynomial mapping $R\colon \mathbb{C}^n \to \mathbb{C}^{m+1}$ given by

$$R := \Big(\frac{P_0}{Q_0}\prod_{k=0}^{m} Q_k, \dots, \frac{P_m}{Q_m}\prod_{k=0}^{m} Q_k\Big).$$

It is easy to deduce that neither $(\frac{P_0}{Q_0}, \dots, \frac{P_m}{Q_m})$ nor the product $\prod_{k=0}^{m} Q_k$ can vanish on K. Whence, the compact set K does not meet the critical set $R^{-1}(0)$ of the rational function $\rho_m(R)$. We may also deduce that $\rho_m(R) = \rho_m(\frac{P_0}{Q_0}, \dots, \frac{P_m}{Q_m})$ on K. Thus, recalling that ρ_m is uniformly continuous on $\bar{W}$, we only need to choose even smaller constants $\delta_k > 0$ in equation (2) to get

$$d\big[\rho_m\big(R(z)\big), \rho_m\big(F(z,1)\big)\big] < \delta \quad \text{for every } z \in K.$$

The result that we are looking for follows after noticing that $\rho_m\big(F(z,1)\big)$ is identically equal to $G(z,1) = g(z)$ for every $z \in K$. ■

Theorem 3 was inspired by the result presented by Grauert and Kerner in [8] and [7]. We must point out that the main result used in the proof of Theorem 3 is the fact that the holomorphic mapping ρ_m defined from $\mathbb{C}^{m+1} \setminus \{0\}$ onto $\mathbb{P}^m$ has the homotopy lifting property. Thus, for each null-homotopic continuous mapping $g\colon K \to \mathbb{P}^m$ there exists a continuous function F from K into $\mathbb{C}^{m+1} \setminus \{0\}$ such that $g = \rho_m(F)$ on K. We only have to approximate the function F by a vector of rational functions, and to compose this vector with the original holomorphic mapping ρ_m. In general, we can apply the same procedure to any complex manifold M for which there exists an open subset $U \subset \mathbb{C}^m$ and a holomorphic mapping $\rho\colon U \to M$ with the homotopy lifting property. Namely, ρ has the property that every commutative square:

$$\begin{array}{ccc} K & \longrightarrow & U \\ \big\downarrow{\scriptstyle \jmath} & & \big\downarrow{\scriptstyle \rho} \\ K \times I & \longrightarrow & M \end{array}$$

has a diagonal $F\colon K \times I \to U$ which also commutes with all the sides of the square (see [4, pp. 27 and 67] or [2, pp. 450–455]). We may then deduce the following general theorem.

Theorem 4 *Let ρ be a holomorphic mapping defined from an open set $U \subset \mathbb{C}^m$ onto a complex manifold M, with $m \ge 1$. Suppose that $\rho\colon U \to M$ has the homotopy lifting*

https://doi.org/10.4153/CMB-2006-024-4 Published online by Cambridge University Press

property and that the topology of M is induced by a metric d. On the other hand, let K be a compact subset of a complex manifold N such that every continuous function $f: K \to \mathbb{C}$ can be uniformly approximated by holomorphic functions. Then, given a null-homotopic continuous mapping $g: K \to M$ and any $\delta > 0$, there exists a holomorphic mapping H defined from a neighbourhood of K into M such that $d[H(z), g(z)] < \delta$ holds for $z \in K$.

The proof of this theorem follows step by step the proof of Theorem 3, so we are omitting it. Besides, notice that we only need to prove that H is holomorphic on a neighbourhood of K; it does not have to be rational. In particular, it is well known that any Riemann surface, other than the Riemann sphere $\mathbb{S}^2$, can be holomorphically covered by the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. These covering mappings have the homotopy lifting property. Thus, we have the following.

Corollary 5 *Let M_2 be a Riemann surface, and d be a metric on M_2 which induces the topology. Besides, let K be a compact subset of a complex manifold N such that every continuous function $f: K \to \mathbb{C}$ can be uniformly approximated by holomorphic functions. Then, given a null-homotopic continuous mapping $g: K \to M_2$ and any $\delta > 0$, there exists a holomorphic mapping H defined from a neighbourhood of K into M_2 such that $d\big[H(z), g(z)\big] < \delta$ holds for $z \in K$.*

Proof The result follows from Theorem 3 when M_2 is equal to the Riemann sphere $\mathbb{S}^2$, for the one-dimensional complex projective space $\mathbb{P}^1$ is indeed the Riemann sphere. On the other hand, every Riemann surface M_2 different from $\mathbb{S}^2$ can be holomorphically covered by the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. The result follows then from the fact that these covering mappings have the homotopy lifting property (see [2, p. 140] or [4, p. 62]); we only need to apply Theorem 4. ■

Finally, we want to close this section recalling a classical result in approximation theory: Every continuous function defined between smooth compact manifolds can be uniformly approximated by smooth functions (see for example [2, p. 97]). The proof of this result uses the fact that every smooth manifold of real dimension m can be embedded inside $\mathbb{R}^{2m+1}$. We can deduce a similar result, which considers holomorphic functions instead of smooth ones, when we are working with Stein manifolds of complex dimension m, for they are naturally complex submanifolds of $\mathbb{C}^{2m+1}$.

Theorem 6 *Let M be a Stein manifold of complex dimension $m \geq 1$. Consider M as a complex submanifold of $\mathbb{C}^{2m+1}$, and set $\|\cdot\|$ to be the Euclidean norm in $\mathbb{C}^{2m+1}$. Besides, let K be a compact subset of a complex manifold N such that every function $f: K \to \mathbb{C}$, continuous on K and holomorphic on the interior, can be approximated by holomorphic functions. Then, for every mapping $g: K \to M$, continuous on K and holomorphic on the interior, and any $\delta > 0$, there exists a holomorphic mapping H defined from a neighbourhood of K into M such that $\|H(z) - g(z)\| < \delta$ holds for $z \in K$.*

Notice that we are not demanding now any homotopical hypothesis on the continuous mapping $g: K \to M$.

Proof There exists a holomorphic retraction $\pi\colon V \to M$ defined on an open neighbourhood V of M in $\mathbb{C}^{2m+1}$, because we are considering M to be a complex submanifold of $\mathbb{C}^{2m+1}$, see for example Corollary 1 in [14]. In particular, the equality $\pi(w) = w$ holds for every $w \in M$.

From now on, this proof follows the same ideas presented in [2, p. 97]. Nevertheless, we are including the details for the sake of completeness. Considering a mapping $g\colon K \to M$, continuous on K and holomorphic on the interior, let W be an open neighbourhood of $g(K)$ in $\mathbb{C}^{2m+1}$ such that the closure $\bar{W}$ is compact and contained inside V. The retraction π is then uniformly continuous on $\bar{W}$. Recalling the given hypotheses and choosing small enough constants $\delta_k > 0$, for $0 \le k \le 2m$, we may find holomorphic functions F_k defined on a neighbourhood of K such that

$$|F_k(z) - g_k(z)| < \delta_k \quad \text{and} \qquad (3)$$
$$\big(F_0(z), \dots, F_{2m}(z)\big) \in \bar{W} \quad \text{whenever } z \in K.$$

Finally, define $H := \pi(F_0, \dots, F_{2m})$. We automatically have that H is defined and holomorphic from a neighbourhood of K into M. Besides, since π is uniformly continuous on $\bar{W}$ and is the identity function on M, we only need to choose even smaller constants $\delta_k > 0$ in equation (3) in order to get $\|H(z) - g(z)\| < \delta$ for every $z \in K$. ■

3 Converse of the Main Theorem

Recalling our main Theorem 3, and its proof, we may see that the condition of being null-homotopic is a central hypothesis. We are going to prove now, in this section, that this condition of being null-homotopic is also a necessary hypothesis when we are working with CW-complexes of relatively *small* dimension. To do so, we need the following two lemmas.

Lemma 7 *Let $\rho_m(R)$ be a rational mapping from $\mathbb{C}^n$ to $\mathbb{P}^m$, with $m, n \ge 1$. Besides, let $K \subset \mathbb{C}^n$ be a compact* CW*-complex which does not meet the critical set of $\rho_m(R)$, and whose real dimension is less than or equal to $2m$. Then, the restriction $\rho_m(R)|_K$ defined from K into $\mathbb{P}^m$ is null-homotopic.*

We must point out here that any compact manifold is a CW-complex. Moreover, we strongly recommend the bibliographies of [2, 3, 4, 12] for references on homotopy theory.

Proof Consider the unit sphere $\mathbb{S}^{2m+1}$ in $\mathbb{C}^{m+1}$. It is an exercise in homotopy theory to deduce that $\mathbb{S}^{2m+1}$ is a CW-complex with exactly two cells of dimensions 0 and $2m+1$, respectively (see for example [2, p. 196] or [4, p. 70]). The cellular approximation theorem implies then that every continuous function g from K into $\mathbb{S}^{2m+1}$ is null-homotopic. Indeed, we have that g is homotopic to a cellular mapping g_2 defined from K into $\mathbb{S}^{2m+1}$ (see [2, p. 208], [3, p. 44] or [4, p. 93]). We know that every cell of K has dimension less than or equal to $2m$, because of the given hypotheses; so the cellular mapping g_2 must send K into a point, the only cell of $\mathbb{S}^2$ with dimension less

https://doi.org/10.4153/CMB-2006-024-4 Published online by Cambridge University Press

than or equal to $2m$. Hence, the mappings g_2 and g are both null-homotopic, for they are homotopic. On the other hand, it is also an exercise in homotopy theory to show that $\mathbb{C}^{m+1} \setminus \{0\}$ has the same homotopy type as $\mathbb{S}^{2m+1}$ (see [2, p. 45] or [4, pp. 19–21]). Therefore, every continuous mapping g_3 from K into $\mathbb{C}^{m+1} \setminus \{0\}$ is null-homotopic as well.

Consider now the polynomial mapping $R\colon \mathbb{C}^n \to \mathbb{C}^{m+1}$ given by the original rational mapping $\rho_m(R)$, recalling Definition 2. We obviously have that the image $R(K)$ is contained in $\mathbb{C}^{m+1} \setminus \{0\}$, for K does not meet the critical set of $\rho_m(R)$. Thus, the continuous restriction $R|_K$ defined from K into $\mathbb{C}^{m+1} \setminus \{0\}$ is indeed null-homotopic, recalling the work done in the previous paragraph. Finally, applying the natural projection ρ_m, we may conclude that $\rho_m(R)|_K$ is also null-homotopic, as we wanted. ∎

Lemma 8 *Let X and (Y, d) be two metric spaces, such that X is compact and Y is an absolute neighbourhood retract. Then, given a continuous mapping $g\colon X \to Y$, there exists a constant $\beta_g > 0$ such that: every continuous mapping $h\colon X \to Y$ is homotopic to g, whenever $d[g(x), h(x)] < \beta_g$ for each $x \in X$.*

Proof Let Y^X be the topological space composed of all the continuous mappings $h\colon X \to Y$, and endowed with the compact-open topology. Since X is compact and (Y, d) is metric, the compact-open topology of Y^X is induced by the distance

$$D(h_1, h_2) := \sup\{d[h_1(x), h_2(x)] : x \in X\},$$

for any two mappings h_1 and h_2 in Y^X (see for example [10, p. 89]). The space Y^X is locally arcwise connected and an absolute neighbourhood retract, because Y is an absolute neighbourhood retract (see [10, pp. 339–340]). Whence, there exists a fixed constant $\beta_g > 0$ such that the open ball in Y^X with centre in g and radius β_g is contained in an arcwise connected neighbourhood of g. That is, for every continuous mapping $h\colon X \to Y$ with $D(h, g) < \beta_g$, there exists an arc in Y^X whose end points are h and g. Thus, the mappings g and h are homotopic. ∎

We can now use the previous lemmas to prove that the property of being null-homotopic is also a necessary hypothesis in Theorem 3, when we work with a compact CW-complex K of dimension less than or equal to $2m$.

Theorem 9 *Given a compact* CW*-complex $K \subset \mathbb{C}^n$ of real dimension less than or equal to $2m$, for $n, m \geq 1$, and a metric d on $\mathbb{P}^m$ which induces the topology. Let $g\colon K \to \mathbb{P}^m$ be a continuous mapping such that for each $\delta > 0$ there exists a rational mapping $\rho_m(R)$ defined on $\mathbb{C}^n$ with $d[\rho_m(R), g] < \delta$ on K. Then, g is null-homotopic.*

Proof We have that g can be uniformly approximated by rational functions $\rho_m(R)$ on K. And each restriction $\rho_m(R)|_K$ is null-homotopic because of Lemma 7. Hence, g is also null-homotopic because of Lemma 8. ∎

Finally, we have to point out that the property of being null-homotopic becomes a superfluous hypothesis in Theorem 3 when the dimension of the CW-complex K is equal to one.

Lemma 10 *Let X be an arbitrary* CW*-complex of real dimension one. Then, every continuous mapping $g\colon X \to \mathbb{P}^m$ is null-homotopic, for $m \geq 1$.*

Notice that any compact one-dimensional CW-complex consists of the union of finitely many compact arcs which intersect themselves only at their end points.

Proof The complex projective space $\mathbb{P}^m$ is a finite CW-complex with one cell of each even dimension $2k$, for every $k \leq m$ (see [4, p. 71] or [11, p. 27]). The cellular approximation theorem states that g is homotopic to a cellular mapping $g_2\colon X \to \mathbb{P}^m$ (see [2, p. 208], [3, p. 44] or [4, p. 93]). We know that every cell of X has dimension less than or equal to one, so the cellular mapping g_2 must send X into a point, the only cell of $\mathbb{P}^m$ with dimension less than or equal to one. Hence, the mappings g_2 and g are both null-homotopic, for they are homotopic. ■

4 Examples

First, we should point out that there are several known compact sets $K \subset \mathbb{C}^n$ for which every continuous function defined from K into $\mathbb{C}$ can be uniformly approximated by rational functions. For example, every compact smooth submanifold $M \subset \mathbb{C}^n$ has this property when it is totally real and rationally convex. Recall that a smooth submanifold M is said to be *totally real* if for every $x \in M$, the tangent space T_xM contains no complex lines. The results presented in [13] and [9] state that for every continuous function $f\colon M \to \mathbb{C}$ and any $\delta > 0$ there exists a holomorphic function H defined in a neighbourhood of M such that $|H(z) - f(z)| < \delta$ holds for $z \in M$. The compact set K is said to be *rationally convex* if for every point z_0 outside K there exists a holomorphic polynomial Q such that $Q(z_0) = 0$ but $0 \notin Q(K)$. Thus, Exercise 4 in Chapter III of [5] states that every function holomorphic in a neighbourhood of M can be uniformly approximated by rational functions.

Moreover, every continuous function f whose image is contained in the complex plane $\mathbb{C}$ is trivially null-homotopic, for we only need to consider the function $(z,t) \mapsto t f(z)$. Also any continuous mapping g defined from $K \subset \mathbb{C}^n$ into the Riemann sphere $\mathbb{S}^2$ is null-homotopic if and only if g extends to a continuous mapping $\hat{g}\colon \mathbb{C}^n \to \mathbb{S}^2$ (see for example [2], [4, p. 25] or [12]).

On the other hand, Theorem 9 allows us to construct very interesting examples. Consider the standard two-dimensional torus T^2 in $\mathbb{C}^2$, defined by $|w| = |z| = 1$. Notice that T^2 has finite non-zero two-dimensional Hausdorff measure. Besides, every continuous function $f\colon T^2 \to \mathbb{C}$ can be approximated by rational functions, but there exists an infinite number of continuous mappings $g\colon T^2 \to \mathbb{S}^2$ which are not null-homotopic, and which cannot be uniformly approximated by rational mappings according to Theorem 9. That is, given any $\delta > 0$, the Stone Weierstrass Theorem implies the existence of a holomorphic polynomial R such that $|R(w,\bar{w},z,\bar{z}) - f(w,z)| < \delta$ for every $(w,z) \in T^2$. The rational function that we are looking for is then obtained by noticing that $\bar{w} = \frac{1}{w}$ and $\bar{z} = \frac{1}{z}$ on T^2.

Nevertheless, Hopf's classification theorem states the existence of an infinite number of continuous mappings $g\colon T^2 \to \mathbb{S}^2$ which are not null-homotopic (see [2, p. 300]). We only need to recall that the cohomology group $H^2(T^2,\mathbb{Z}) = \mathbb{Z}$, see

https://doi.org/10.4153/CMB-2006-024-4 Published online by Cambridge University Press

the calculations presented in [2, p. 347]. Theorem 9 implies that no mapping g can be uniformly approximated by rational mappings, for they are not null-homotopic and $\mathbb{S}^2 = \mathbb{P}^1$. In fact, we can directly build continuous mappings $g\colon T^2 \to \mathbb{S}^2$ which are not null-homotopic; we only need to recall that $\mathbb{S}^2$ can be obtained from T^2 by identifying both circles $\mathbb{S}^1 \times \{1\}$ and $\{1\} \times \mathbb{S}^1$ as a single point.

Finally, we should point out that, on compact subsets K of the complex plane, each mapping $f\colon K \to \mathbb{S}^2$ is null-homotopic. This result is inspired by Lemma 10, and unfortunately, it cannot be extended to consider compact subsets of $\mathbb{C}^n$, with $n \geq 2$, as we have just seen with the standard torus T_2 in $\mathbb{C}^2$.

Proposition 11 *Let K be any compact subset of the complex plane $\mathbb{C}$, and $f\colon K \to \mathbb{P}^m$ be continuous, for $m \geq 1$. Then, the mapping f is null-homotopic.*

Proof Working as in the proof of Theorem 1, there exists a continuous function $f_2\colon V \to \mathbb{P}^m$ defined on an open neighbourhood V of K such that $f_2(z) = f(z)$ for every $z \in K$, because $\mathbb{P}^m$ is an absolute neighbourhood retract. On the other hand, let us recall that any compact set in $\mathbb{C}$ has a fundamental system of open neighbourhoods, each one of which is bounded by a finite number of disjoint Jordan curves. We can then find an open set W such that $\mathbb{C} \setminus W$ has a finite number of connected components and $K \subset W \subset V$. Notice that W has the homotopy type of a compact set $\Upsilon \subset W$ consisting of the union of finitely many compact smooth arcs which intersect themselves only at their end points. That is, the compact set Υ is a one-dimensional CW-complex.

Finally, every continuous mapping $g\colon \Upsilon \to \mathbb{P}^m$ is null-homotopic, according to Lemma 10. Therefore, the restriction $f_2|_W$ defined from W into $\mathbb{P}^m$ is also null-homotopic, because W and Υ have the same homotopy type. We can deduce the result that we are looking for by noticing that $f = f_2|_K$ is null-homotopic as well, for $K \subset W$. ■

References

[1] H. Alexander and J. Wermer, *Several Complex Variables and Banach Algebras, third edition*. Graduate Texts in Math. **35**, Springer-Verlag, New York, 1998.

[2] Glen E. Bredon, *Topology and geometry*. Graduate Texts in Math. **139**, Springer-Verlag, New York, 1993.

[3] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern geometry—methods and applications. Part III. Introduction to homology theory*. Graduate Texts in Math. **124**, Springer-Verlag, New York, 1990.

[4] C. T. J. Dodson and P. E. Parker, *A user's guide to algebraic topology*. Math. Appl. **387**, Kluwer Academic Publishers, Dordrecht, 1997.

[5] T. W. Gamelin, *Uniform algebras*. Prentice-Hall, Englewood Cliffs, N. J., 1969.

[6] P. M. Gauthier, A. Roth and J. L. Walsh, *Possibility of uniform rational approximation in the spherical metric*. Canad. J. Math. **28**(1976), no. 1, 112–115.

[7] H. Grauert, *Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen*. Math. Ann. **133**(1957), 450–472.

[8] H. Grauert and H. Kerner, *Approximation von holomorphen Schnittflächen in Faserbündeln mit homogener Faser*. Arch. Math. **14**(1963), 328–333.

[9] L. Hörmander and J. Wermer, *Uniform approximation on compact sets in $\mathbb{C}^n$*. Math. Scand. **23**(1968), 5–21.

[10] K. Kuratowski, *Topology, Vol. II*. Academic Press, New York and London, 1968.

[11] J. Milnor, *Morse theory*. Ann. of Math. Studies **51**, Princeton University Press, Princeton, N.J., 1963.

[12] J. R. Munkres, *Elements of algebraic topology*. Addison-Wesley, Menlo Park, CA, 1984.
[13] R. Nirenberg and R. O. Wells Jr., *Approximation theorems on differentiable submanifolds of a complex manifold*. Trans. Amer. Math. Soc. **142**(1969), 15–35.
[14] Yum-Tong Siu, *Every Stein subvariety admits a Stein neighborhood*. Invent. Math. (1) **38**(1976), 89–100.

Département de mathématiques et statistique
Université de Montréal
Centreville, Montréal, Québec
H3C 3J7
email: gauthier@dms.umontreal.ca

Departamento de Matemáticas
CINVESTAV
Apdo. Postal 14-740, México DF
07000 México
email: eszeron@math.cinvestav.mx

https://doi.org/10.4153/CMB-2006-024-4 Published online by Cambridge University Press