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ON COPIES OF THE NULL SEQUENCE BANACH SPACE IN SOME VECTOR MEASURE SPACES

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In this note we extend a result of Drewnowski concerning copies of c_0 in the Banach space of all countably additive vector measures and study some properties of complemented copies of c_0 in several Banach spaces of vector measures.

1. PRELIMINARIES

Throughout this note (Ω, Σ) will be a measure space and X a Banach space. Our notation is standard [**2, 3**]. So $ba(\Sigma, X)$ denotes the Banach space over $\mathbb{K}$ of all X-valued bounded finitely additive measures defined on Σ provided with the semivariation norm

$$\|F\|(E) = \sup\left\{\sum_{A\in\pi} |x^* F(A)| : x^* \in X^*,\ \|x^*\| \leqslant 1,\ \pi \in \mathcal{P}(E)\right\},$$

for each $E \in \Sigma$, where $\mathcal{P}(E)$ is the class of all finite partitions of E by elements of Σ. The closed subspace of $ba(\Sigma, X)$ formed by the countably additive measures is denoted by $ca(\Sigma, X)$, while $cca(\Sigma, X)$ stands for the closed subspace of $ca(\Sigma, X)$ formed by those countably additive measures with relatively compact range. By $bvca(\Sigma, X)$ we denote the Banach space of all X-valued countably additive measures F of bounded variation defined on Σ, equipped with the variation norm $|F| = |F|(\Omega)$. It has been shown in [7] that (*a*) both $ca(\Sigma, X)$ and $bvca(\Sigma, X)$ contain a copy of c_0 if and only if either X contains a copy of c_0 or they contain a copy of ℓ_∞, and (*b*) if Σ is an infinite σ-algebra, then $ba(\Sigma, X)$ contains a copy of c_0 if and only if it contains a copy of ℓ_∞. If $ba(\Sigma, X)$ contains a complemented copy of c_0, then X contains a copy of c_0 [**8**]. Assuming that each nonzero finite positive measure $\mu \in ca(\Sigma)$ is purely atomic, then $ca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains respectively a copy of c_0 or ℓ_∞ [**5**, Theorem 2] and, with the same hypotheses, $bvca(\Sigma, X)$ contains a copy of c_0 or ℓ_∞ if and only if X contains respectively a copy of c_0 or ℓ_∞ [**6**]. On the other hand, $cca(\Sigma, X)$ contains a copy of ℓ_∞ if and only if X contains a copy of ℓ_∞ [4]. Note that $ca(\Sigma, X)$ may contain a copy of c_0 while X does not. Indeed, if $\Omega = [0, 1]$ and Σ coincides with the σ-algebra of

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all Lebesgue measurable subsets of $[0,1]$, then $ca(\Sigma, \ell_2)$ contains a copy of c_0, [**7**]. In this paper we show by a straightforward technique that, assuming $cca(\Sigma, X) = ca(\Sigma, X)$, then $ca(\Sigma, X)$ contains a copy of c_0 if and only if X contains a copy of c_0. This extends part (B) of Theorem 2 in [**5**] quoted above. On the other hand, we shall show that if $ca(\Sigma, X)$ or $bvca(\Sigma, X)$ contains a complemented copy of c_0, then X contains a copy of c_0. As background of this work we must mention [**4, 5, 6, 7, 8**].

Given a set A of a Banach space X, we represent by $[A]$ the closed linear span of A. In the sequel we shall shorten to wuC the sentence "weak unconditionally Cauchy". Let us recall that a Banach space is said to have the Schur property if each weakly convergent sequence in X is norm convergent.

2. Results

Theorem 2.1. *Assume $cca(\Sigma, X) = ca(\Sigma, X)$. Then $ca(\Sigma, X)$ contains a copy of c_0 if and only if X contains a copy of c_0.*

Proof: Let Z denote a copy of c_0 in $ca(\Sigma, X) = cca(\Sigma, X)$, and let $J : c_0 \to Z$ be a topological isomorphism from c_0 onto Z such that $F_n := Je_n$ for each $n \in \mathbb{N}$ where $\{e_n : n \in \mathbb{N}\}$ is the standard unit vector basis of c_0. We assume to the contrary that X contains no copy of c_0. Since J is a bounded linear operator, the series $\sum_{n=1}^{\infty} F_n$ is wuC in $ca(\Sigma, X)$ and hence there exists a $C > 0$ such that $\sup_{n \in \mathbf{N}} \left\| \sum_{i=1}^{n} \xi_i F_i \right\| < C \|\xi\|_\infty$ for each $\xi \in \ell_\infty$. Then, given $E \in \Sigma$, the series $\sum_{n=1}^{\infty} F_n(E)$ is wuC in X, since for each $x^* \in X^*$ the map $u : ca(\Sigma, X) \to \mathbb{K}$ defined by $u(F) = x^* F(E)$ is a continuous linear form on $ca(\Sigma, X)$ and, consequently, $\sum_{n=1}^{\infty} |x^* F_n(E)| = \sum_{n=1}^{\infty} |uF_n| < \infty$. As we are assuming that X does not contain a copy of c_0, according to a well known result of Bessaga and Pełczyński [**1**] the series $\sum_{n=1}^{\infty} F_n(E)$ is (BM)-convergent in X. So we may consider the linear operator $\varphi : \ell_\infty \to ca(\Sigma, X)$ defined by $\varphi\xi(E) = \sum_{n=1}^{\infty} \xi_n F_n(E)$. Actually $\varphi\xi \in ba(\Sigma, X)$ and φ is a bounded linear operator from ℓ_∞ into $ba(\Sigma, X)$ since, given $\xi \in \ell_\infty$ and $E \in \Sigma$, we have

$$\|\varphi\xi(E)\| \leqslant \sup_{n \in \mathbf{N}} \left\| \sum_{i=1}^{n} \xi_i F_i(E) \right\| \leqslant \sup_{n \in \mathbf{N}} \left\| \sum_{i=1}^{n} \xi_i F_i \right\| \leqslant C \|\xi\|_\infty .$$

So $\|\varphi\xi\| \leqslant 4C \|\xi\|_\infty$, which shows at the same time that $\varphi\xi \in ba(\Sigma, X)$ and that φ is a bounded linear operator from ℓ_∞ into $ba(\Sigma, X)$. Moreover, since $\left\{ \sum_{i=1}^{n} \xi_i F_i \right\}$ is a sequence of X-valued countably additive vector measures such that $\lim_{n \to \infty} \sum_{i=1}^{n} \xi_i F_i(E) = \varphi\xi(E)$ exists in X for each $E \in \Sigma$, according to the Vitali-Hahn-Saks' theorem [**3**, Chapter 1, Corollary 4.10] there exists a nonzero finite positive $\mu \in ca(\Sigma)$ such that $\varphi\xi \ll \mu$.

Consequently $\varphi\xi \in ca(\Sigma, X)$ for each $\xi \in \ell_\infty$, which shows that $\varphi(\ell_\infty) \subseteq ca(\Sigma, X)$. As $ca(\Sigma, X) = cca(\Sigma, X)$ by hypothesis, we have that $\varphi(\ell_\infty) \subseteq cca(\Sigma, X)$. Then, since

$$\|\varphi e_n\| = \|F_n\| = \|Je_n\| \geqslant \frac{1}{\|J^{-1}\|}$$

for each $n \in \mathbb{N}$, Rosenthal's ℓ_∞ theorem [**10**] yields an infinite set $M \subseteq \mathbb{N}$ such that the restriction ψ of φ to $\ell_\infty(M)$ is a topological isomorphism into $cca(\Sigma, X)$. According to the above quoted result of [**4**], this implies that X contains a copy of ℓ_∞, contradicting the fact that X does not contain a copy of c_0. ∎

Corollary 2.2. *If X has the Schur property, then $ca(\Sigma, X)$ does not contain a copy of c_0.*

Proof: Since X has the Schur property, then $cca(\Sigma, X) = ca(\Sigma, X)$. Given that X cannot contain a copy of c_0, the conclusion follows from Theorem 2.1. ∎

Corollary 2.3. [**5**, Theorem 2] *If every nonzero finite positive measure on Σ is purely atomic, then $ca(\Sigma, X)$ contains a copy of c_0 if and only if X contains a copy of c_0.*

Proof: Suppose that every nonzero finite positive measure on Σ is purely atomic and let $G \in ca(\Sigma, X)$ be nonzero. According to a theorem of Bartle-Dunford-Schwartz [**3**, Chapter 1, Corollary 2.6] there is some $\mu \in ca^+(\Sigma)$ such that $G \ll \mu$. By hypothesis μ is purely atomic, so there exists a finite or infinite pairwise disjoint sequence of atoms $\{A_n\}$ in Σ such that $\mu(A_n) > 0$ and $\bigcup_n A_n = \Omega$. Define $g : \Omega \to X$ such that $g(\omega) = \big(G(A_n)/\mu(A_n)\big)$ whenever $\omega \in A_n$ for each n. Then, as is well known, g is μ-measurable and $G(E) = (P)\int_E g(\omega)\, d\mu$ [Pettis integral] for each $E \in \Sigma$. But this implies that $G(\Sigma)$ is relatively compact in norm [**9**, Theorem 10.4.4], and consequently $G \in cca(\Sigma, X)$. Thus $ca(\Sigma, X) = cca(\Sigma, X)$ and Theorem 2.1 applies. ∎

Theorem 2.4. *If $ca(\Sigma, X)$ contains a complemented copy of c_0, then X contains a copy of c_0.*

Proof: Set $Z = [F_n]$, let $J : c_0 \to Z$ be a topological isomorphism from c_0 onto Z where $F_n := Je_n$ for each $n \in \mathbb{N}$, and let P be a bounded linear projection from $ca(\Sigma, X)$ onto Z. We assume to the contrary that X does not contain a copy of c_0 and proceed as in the proof of the previous theorem until we show that the linear operator $\varphi : \ell_\infty \to ca(\Sigma, X)$ defined by $\varphi\xi(E) = \sum_{n=1}^{\infty} \xi_n F_n(E)$ is well-defined and bounded. Consequently, $\varphi(\ell_\infty) \subseteq ca(\Sigma, X)$.

Since $PF_n = F_n$ for each $n \in M$, the bounded linear operator $Q : \ell_\infty \to c_0$ defined by $Q = J^{-1} \circ P \circ \varphi$ satisfies $Qe_n = e_n$ for each $n \in \mathbb{N}$. In fact,

$$Qe_n = J^{-1}P\varphi e_n = J^{-1}PF_n = J^{-1}F_n = J^{-1}Je_n = e_n,$$

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which implies $Q\zeta = \zeta$ for each $\zeta \in c_0$. Hence, if $\xi \in \ell_\infty$, as $Q\xi \in c_0$ one has

$$Q^2\xi = Q(Q\xi) = Q\xi.$$

But this means that Q must be a bounded projection operator from ℓ_∞ onto c_0, a contradiction. ∎

Theorem 2.5. *If $bvca(\Sigma, X)$ contains a complemented copy of c_0, then X contains a copy of c_0.*

Proof: Set $Z = [F_n]$, $J : c_0 \to Z$ a topological isomorphism from c_0 onto Z where $F_n := Je_n$ for each $n \in \mathbb{N}$, and let P be a bounded linear projection operator from $bvca(\Sigma, X)$ onto Z. By assuming to the contrary that X does not contain a copy of c_0 and reasoning as in the proof of Theorem 2.1, there is a $C > 0$ such that $\sup_{k \in \mathbb{N}} \left| \sum_{j=1}^{k} \xi_j F_j \right| \leqslant C \|\xi\|_\infty$. Define the linear operator $\varphi : \ell_\infty \to ba(\Sigma, X)$ by $\varphi\xi(E) = \sum_{n=1}^{\infty} \xi_n F_n(E)$. We only need to prove that $\varphi\xi \in bvca(\Sigma, X)$ and that φ is a bounded linear operator. So, if $\{E_i : 1 \leqslant i \leqslant n\}$ is a partition of Ω by elements of Σ then, given $\xi \in \ell_\infty$, we have

$$\begin{aligned}\sum_{i=1}^{n} \|\varphi\xi(E_i)\| &= \sum_{i=1}^{n} \left\| \sum_{j=1}^{\infty} \xi_j F_j(E_i) \right\| \leqslant \sup_{k \in \mathbb{N}} \sum_{i=1}^{n} \left\| \sum_{j=1}^{k} \xi_j F_j(E_i) \right\| \\ &\leqslant \sup_{k \in \mathbb{N}} \left| \sum_{j=1}^{k} \xi_j F_j \right| \leqslant C \|\xi\|_\infty .\end{aligned}$$

So $|\varphi\xi| \leqslant 4C \|\xi\|_\infty$, which shows at the same time that $\varphi\xi \in bvca(\Sigma, X)$ and that φ is a bounded linear operator from ℓ_∞ into $bvca(\Sigma, X)$. ∎

Added in Proof. Since $ca(\Sigma, X) \cong L_{w^*}\big(ca(\Sigma)^*, X\big)$, Theorems 2.1 and 2.4 can be viewed as particular cases of well known properties of $L_{w^*}\big(ca(\Sigma)^*, X\big)$.

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