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SOME ORDER PROPERTIES OF THE LATTICE OF VARIETIES OF COMMUTATIVE SEMIGROUPS

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1. Introduction. The most complete work on the structure of the lattice $\mathscr{L}$(Com) of varieties of commutative semigroups available to this date is [**12**]. Nevertheless, it fails to give the structure of this lattice. In the positive direction, it shows in particular that the order structure of $\mathscr{L}$(Com) is determined by the order structure of well-known lattices of integers together with the sublattice $\mathscr{L}$(Nil) of varieties of commutative nil semigroups.

In the present work, we study $\mathscr{L}$(Com) from the point of view of order. Perkins [**13**] has shown that $\mathscr{L}$(Com) has no infinite descending chains and is countable. The underlying questions we consider here arose from the results of Almeida and Reilly [**1**] in connection with generalized varieties. There, it is observed that the best-known part of $\mathscr{L}$(Com) consisting of the varieties all of whose elements are abelian groups is in a sense very wide: it contains infinite subsets of mutually incomparable elements and allows the construction of uncountably many generalized varieties and infinite descending chains of generalized varieties. On the other hand, restricting attention to the lattice $\mathscr{L}(\mathscr{N})$ of commutative nilpotent semigroups, it is shown that the corresponding lattice $\mathscr{G}(\mathscr{N})$ of generalized varieties of commutative nilpotent semigroups is countable and has no infinite descending chains. The natural question that arises, concerns how far away from groups one needs to stay in order to maintain the preceding properties. In this connection, we are naturally led to try to establish the so-called well-quasi-ordering (wqo) of various sublattices of $\mathscr{L}$(Com), for this allows us to conclude that the corresponding lattices of generalized varieties are countable. Actually, the stronger property of better-quasi ordering (bqo) is preferred because it is inherited by the lattice of generalized varieties and, in particular, it implies that every descending chain is finite. We make use of the theory of bqo as introduced by Nash-Williams (see [**9**] for a survey) to prove that the lattice of subvarieties of a proper subvariety of the variety of all commutative semigroups is bqo. This strengthens Perkins' results. We also show that a certain well-behaved sublattice of $\mathscr{L}$(Com) which had been first introduced by Schwabauer [**15**, **16**] and was again considered by Nelson [**12**] is bqo, and we prove that the lattice $\mathscr{L}(\mathscr{N})$ is wqo.

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As we do not manage to prove bqo of $\mathscr{L}$(Nil), we take an alternate approach to the study of the lattice $\mathscr{G}$(Nil) of generalized varieties of commutative nil semigroups. This is based on a special finitary form of those generalized varieties, which we obtain by carefully manipulating identities in commuting variables. We conclude that $\mathscr{G}$(Nil) is countable and has no infinite descending chains. This, coupled with the results of Nelson, allows us to answer the above mentioned question: the lattice of generalized varieties of commutative semigroups whose cyclic groups admit a uniform bound is countable and has no infinite descending chains.

In the final sections, we use the results of [**2**] relating generalized varieties to pseudovarieties of finite semigroups to give some consequences of the previous theory. In the course of this, we discover that the lattice $\mathscr{L}$(Nil) with a maximum adjoined is isomorphic to the lattice $\mathscr{G}(\mathscr{N})$, thus showing that the missing factor $\mathscr{L}$(Nil) in Nelson's results is constructed from the somewhat simpler lattice $\mathscr{L}(\mathscr{N})$. Still, $\mathscr{L}(\mathscr{N})$ is immensely complicated (cf. [**8**]).

This work was produced during the author's visit to Simon Fraser University. I wish to thank my host, Prof. N. Reilly, for bringing these questions to my attention and for many helpful discussions.

2. Preliminaries on varieties. Let

$$X = \{x, y, z, \dots, x_1, y_1, z_1, \dots\}$$

be a denumerable set of distinct variables. Let X^σ denote the free commutative semigroup on X. An element $v \in X^\sigma$ will be referred to as a *word in the commuting variables* from X; its *content* $c(v)$ is the set of variables occurring in v. An *identity of commutative semigroups* is a pair of words $(v, w) \in X^\sigma \times X^\sigma$ which we write in the form $v = w$. The identity $v = w$ is said to be *trivial* if it holds in X^σ, i.e., v and w are the same element of X^σ.

If $\mathscr{V}$ is a class of commutative semigroups and Σ is a set of identities, then by $\mathscr{V} \models \Sigma$ we mean that every semigroup in $\mathscr{V}$ satisfies every identity in Σ; when this fails, we write $\mathscr{V} \not\models \Sigma$.

$$[\Sigma] = \{S{:}S \text{ is a commutative semigroup and } S \models \Sigma\}$$

and $\Sigma \vdash e$ means $[\Sigma] \models e$. For a set of identities Σ, we let

$$\Sigma^r = \{v = w{:}(w = v) \in \Sigma\}.$$

Definition 2.1. Let Σ be a set of identities and let $v = w$ be an identity. We write $\Sigma \rightarrow v = w$ if there exists a homomorphism

$$f{:}X^\sigma \rightarrow X^\sigma$$

and an identity

$$(p = q) \in \Sigma \cup \Sigma^r$$

such that the word w is obtained from v by replacing an occurrence of a subword $f(p)$ by $f(q)$.

The completeness theorem for commutative semigroups may be phrased as follows.

LEMMA 2.2. *$\Sigma \vdash v = w$ if and only if either $v = w$ in X^σ or there exist words $u_0, u_1, \ldots, u_m$ in X^σ such that $v = u_0$ and $u_m = w$ in X^σ and*

$$\Sigma \rightarrow u_i = u_{i+1} \quad \textit{for } i = 0, \ldots, m - 1.$$

Weakening in certain directions the requirement of closure under products for varieties, one obtains classes of commutative semigroups that can still be described in terms of varieties.

THEOREM 2.3. ([**2**]). *The following statements are equivalent for a class $\mathscr{W}$ of semigroups.*

a) *$\mathscr{W}$ is closed under the formation of homomorphic images, subsemigroups, powers and finite products.*

b) *$\mathscr{W}$ is the union of some (upper) directed family of varieties.*

Definition 2.4. ([**2**]). A class $\mathscr{W}$ of semigroups is a *generalized variety* if it satisfies the equivalent conditions of Theorem 2.3.

The collection of all generalized varieties of semigroups forms a complete lattice. The meet operation is intersection and the join of a family $\{\mathscr{W}_i : i \in I\}$ of generalized varieties is the union of all finite joins of subvarieties of the $\mathscr{W}_i$.

In general we will use the notation of [**1**] with the main difference that all semigroups considered here are commutative. The following varieties and generalized varieties play an important role:

Com = class of all commutative semigroups
$\mathscr{A}$ = class of all abelian groups of bounded exponent
$\mathscr{A}p_0 = [x = y]$
$\mathscr{A}p_n = [x^n = x^{n+1}]$ for $0 < n < \infty$
$\mathscr{A}p = \cup_{0 \leqq n < \infty} \mathscr{A}p_n$ = class of all commutative bounded-aperiodic semigroups
$\mathrm{Nil}_n = [x^n = x^n y]$
$\mathrm{Nil} = \cup_n \mathrm{Nil}_n$ = class of all commutative bounded-nil semigroups
$\mathscr{N}_n = [x_1 \ldots x_n = y_1 \ldots y_n]$
$\mathscr{N} = \cup_n \mathscr{N}_n$ = class of all commutative nilpotent semigroups.

We observe that $\mathscr{A}$, $\mathscr{A}p$, Nil and $\mathscr{N}$ are not varieties. The names "bounded-aperiodic" and "bounded-nil" were introduced by Rhodes. A semigroup S is *aperiodic* (respectively *nil*) if, for all $s \in S$, there exists an integer n (which may depend on s) such that $s^n = s^{n+1}$ (resp. $s^n = 0$). See [**14**] for connections between these and other concepts. Since an aperiodic (resp. nil) commutative semigroup which is not bounded-aperiodic (resp. bounded-nil) generates Com as a generalized variety, a generalized variety of aperiodic (resp. nil) commutative semigroups is necessarily a generalized variety of bounded-aperiodic (resp. bounded-nil) semigroups. We prefer to refer to the former rather than the latter. Also, if a commutative aperiodic semigroup S satisfies an identity $v = w$ with $c(v) \neq c(w)$, then S has a zero and, as a semigroup with zero it satisfies the identities $v = 0$ and $w = 0$. An identity $v = w$ which satisfies $c(v) = c(w)$ is called *regular*; otherwise, it is called *irregular*. Thus, the commutative semigroups with zero that satisfy some nontrivial identity of the form $u = 0$ are precisely the commutative aperiodic semigroups that satisfy some irregular identity. We will deal interchangeably with the two interpretations. For example, we may write

$$\mathrm{Nil}_n = [x^n = 0]$$

$$\mathscr{N}_n = [x_1 \ldots x_n = 0].$$

An aperiodic variety $\mathscr{V} = [\Sigma]$ defined by a set Σ of irregular identities is said to be *purely nil*. The collection of all purely nil varieties is denoted by $\mathscr{L}_0$.

For a class of semigroups $\mathscr{K}$, $\mathscr{L}(\mathscr{K})$ (resp $\mathscr{G}(\mathscr{K})$) denotes the set of all varieties (resp. generalized varieties) contained in $\mathscr{K}$. $\langle \mathscr{K} \rangle$ denotes the variety of semigroups generated by $\mathscr{K}$.

The following four results are taken from [**1**].

LEMMA 2.5. *Every* $\mathscr{W} \in \mathscr{G}(\mathrm{Com})$ *is of the form* $\mathscr{W} = \cup_n \mathscr{W}_n$ *for some chain of varieties* $\mathscr{W}_1 \leqq \mathscr{W}_2 \leqq \ldots$.

THEOREM 2.6. *If* $\mathscr{W} \in \mathscr{G}(\mathscr{N}) \backslash \{\mathscr{N}\}$, *then* $\langle \mathscr{W} \rangle \in \mathscr{L}(\mathrm{Nil})$ *and* $\mathscr{W} = \langle \mathscr{W} \rangle \cap \mathscr{N}$.

COROLLARY 2.7. $\mathscr{G}(\mathscr{N})$ *is countable and has no infinite descending chains.*

THEOREM 2.8. $\mathscr{G}(\mathscr{A})$ *is uncountable and has infinite descending chains.*

It is the contrast between the two previous results that prompted the motivation for this work.

For the remainder of this section, we recall some results of [**12**] in the notation of the present work. We start with some of Nelson's notation.

Let

$$e: x_1^{\alpha_1} \ldots x_n^{\alpha_n} = x_1^{\beta_1} \ldots x_n^{\beta_n}$$

be a nontrivial identity in the commuting variables $x_i (i = 1, \ldots, n)$. Let

$$D(e) = \text{greatest common divisor } \{ |\alpha_i - \beta_i| : \alpha_i \neq \beta_i, \quad i = 1, \ldots, n \}$$

$$V(e) = \min\{\alpha_i, \beta_i : \alpha_i \neq \beta_i, i = 1, \ldots, n\}$$

$$U(e) = \begin{cases} \min\{\Sigma\alpha_i, \Sigma\beta_i\} & \text{if } \Sigma\alpha_i \neq \Sigma\beta_i \\ \Sigma\alpha_i + V(e) & \text{otherwise.} \end{cases}$$

For a variety $\mathscr{V}$ of commutative semigroups, we let

$$V(\mathscr{V}) = \min\{V(e) : e \text{ is a nontrivial identity and } \mathscr{V} \models e\}$$

where we adopt the convention $\min \emptyset = \infty$. We extend this definition for an arbitrary class $\mathscr{K}$ of commutative semigroups as follows:

$$V(\mathscr{K}) = \sup\{V(\mathscr{V}) : \mathscr{V} \in \mathscr{L}(\mathscr{K})\}.$$

Let $C_{n,k}$ denote the cyclic semigroup of index n and period k, that is, $C_{n,k}$ is given by the semigroup presentation

$$C_{n,k} = \langle a; a^n = a^{n+k} \rangle.$$

Also, let $C^I_{n,k}$ denote $C_{n,k}$ with an identity adjoined, even if $C_{n,k}$ already has an identity.

LEMMA 2.9. ([**12**]). *Let e be a nontrivial identity. Then*
a) $C_{n,k} \models e$ *if and only if* $U(e) \geqq n$ *and* $k | D(e)$;
b) $C^I_{n,k} \models e$ *if and only if* $V(e) \geqq n$ *and* $k | D(e)$.

The following results are derived from [**12**].

THEOREM 2.10. *If* $\mathscr{W} \in \mathscr{G}(\mathrm{Com}) \backslash \{\mathrm{Com}\}$, *then*

a) $\mathscr{W} = (\mathscr{W} \cap \mathscr{A}) \vee (\mathscr{W} \cap \mathscr{A}\!\!\!/\!\!\!/\mathscr{h})$;
b) $\mathscr{W} = (\mathscr{W} \cap \mathscr{A}) \vee (\mathscr{W} \cap \mathrm{Nil}) \vee \mathscr{A}\!\!\!/\!\!\!/\mathscr{h}_{V(\mathscr{W})}$.

Let $\mathbf{N}_0$ denote the lattice of nonnegative integers under its usual order.

THEOREM 2.11. *The mapping*

$$\mathscr{L}(\mathrm{Com}) \backslash \{\mathrm{Com}\} \to \mathscr{L}(\mathrm{Nil}) \times \mathscr{L}(\mathscr{A}) \times \mathbf{N}_0$$

$$\mathscr{V} \mapsto (\mathscr{V} \cap \mathrm{Nil}, \mathscr{V} \cap \mathscr{A}, V(\mathscr{V}))$$

is an embedding of meet semilattices.

COROLLARY 2.12. *The mapping*

$$\mathscr{G}(\mathrm{Com}) \to \mathscr{G}(\mathrm{Nil}) \times \mathscr{G}(\mathscr{A}) \times (\mathbf{N}_0 \cup \{\infty\})$$

$$\mathscr{W} \mapsto (\mathscr{W} \cap \mathrm{Nil}, \mathscr{W} \cap \mathscr{A}, V(\mathscr{W}))$$

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is an embedding of meet semilattices.

We note that the above components $\mathscr{L}(\mathscr{A})$ and $\mathscr{G}(\mathscr{A})$ are easily described (cf. [**1**]). Thus, we will deal mainly with the lattice $\mathscr{G}(\mathrm{Nil})$.

Definition 2.13. $\mathscr{W} \in \mathscr{G}(\mathrm{Com})$ is said to be *cycle-bounded* if $\mathscr{W} \cap \mathscr{A} \in \mathscr{L}(\mathscr{A})$, i.e., if there is a bound to the size of the cyclic groups in $\mathscr{W}$. We denote by $\mathscr{G}_{\mathrm{cb}}$ the lattice of all cycle-bounded generalized varieties of commutative semigroups.

As a particular case of Corollary 2.12, we have the following.

PROPOSITION 2.14. *The mapping*

$$\mathscr{G}_{\mathrm{cb}} \to \mathscr{G}(\mathrm{Nil}) \times \mathscr{L}(\mathscr{A}) \times (\mathbf{N}_0 \cup \{\infty\})$$

$$\mathscr{W} \mapsto (\mathscr{W} \cap \mathrm{Nil}, \mathscr{W} \cap \mathscr{A}, V(\mathscr{W}))$$

is an embedding of meet semilattices.

The last result in this section is contained in a more general result first proved by Schwabauer [**16**]; Nelson [**12**] gave a different proof. In Section 4 we will give yet another proof of this special case.

PROPOSITION 2.15. *$\mathscr{L}_0$ is a distributive lattice.*

3. Preliminaries on quasi-ordering. A reflexive and transitive relation $\leqq$ on a set Q is called a *quasi-order* (qo). We write $p \equiv q$ to mean $p \leqq q \leqq p$, and $p < q$ to mean $p \leqq q \not\leqq p$, in which case we say that p *strictly precedes* q.

A *chain* is a qo set for which $\equiv$ is the identity relation and any two elements are comparable. An *anti-chain* is a qo set in which no two distinct elements are comparable.

An *initial segment* (resp. *terminal segment*) of a qo set Q is a subset $P \subseteq Q$ such that $p \in P$ and $q \leqq p$ (resp. $p \leqq q$) imply $q \in P$. The set of all initial segments of Q forms a complete sublattice of the power set $\mathscr{P}(Q)$ which we denote by $I(Q)$. On the power set $\mathscr{P}(Q)$, we consider a qo related to the qo of Q:

$$X \leqq_m Y \text{ if there exists } f:X \to Y \text{ with } q \leqq f(q) \text{ for all } q \in X.$$

We observe that if $\equiv_m$ denotes the equivalence relation on $\mathscr{P}(Q)$ associated with $\leqq_m$, then $\mathscr{P}(Q)/\equiv_m$ is isomorphic to $I(Q)$ via the mapping

$$P/\equiv_m \mapsto \{q:q \leqq p \text{ for some } p \in P\}.$$

By

$$X^{<\omega} = \cup_{n<\omega} X^n$$

we denote the set of all finite sequences of elements of the set X. If Q is a qo set, we qo $Q^{<\omega}$ as follows: $J \leqq K$ if there exists an injective increasing function

$$f:\text{dom } J \rightarrow \text{dom } K$$

such that

$$J(\alpha) \leqq K(f(\alpha)) \quad \text{for all } \alpha \in \text{dom } J.$$

THEOREM 3.1. ([7]). *For a* qo *set Q, the following are equivalent*:

i) *every nonempty subset of Q has at least one and only finitely many minimal elements*;

ii) *$I(Q)$ has no infinite descending chains*;

iii) *Q has no infinite strictly descending chains or antichains*;

iv) *every sequence of elements of Q has an ascending subsequence*;

v) *if $q_1, q_2, \ldots \in Q$, then $q_i \leqq q_j$ for some $i < j$*;

vi) *every terminal segment of Q is of the form*

$$\{q \in Q:p \leqq q \text{ for some } p \in F\}$$

for some finite set F.

Definition 3.2. A qo set Q satisfying the conditions of Theorem 3.1 is said to be *well-quasi-ordered* (wqo).

It has been observed by many authors that the notion of wqo has certain nice transfer properties while it fails to have other desirable properties (see **[9]** for a survey of the theory of wqo, including many references). For instance, a subset, a homomorphic image, a finite union or a finite product of wqo sets is also wqo. If Q is wqo under $\leqq$, then it is also wqo under any qo which extends $\leqq$. If Q is wqo, then $Q^{<\omega}$ is also wqo. On the other hand, Q may be wqo while $I(Q)$ is not wqo.

Many examples of wqo sets are really built up from well-ordered sets. Nash-Williams found a condition on qo sets between wqo and well-ordering which has much nicer transfer properties. We follow **[10]** for the introduction of this condition.

If $s, t \subseteq \mathbf{N}_0$, then $s \leqq t (s < t)$ means that s is a (proper) initial segment of t; $s \lhd t$ means that

$$s = \{i_0, i_1, \ldots, i_m\}, \quad t = \{i_1, \ldots, i_n\}$$

for some $0 \leqq m < n$ and $i_0 < i_1 < \ldots < i_n$. Let $X \subseteq \mathbf{N}_0$ be an infinite subset. A *barrier* on X is a set B of finite subsets of X such that $\emptyset \notin B$ and the following two conditions hold:

a) for every infinite subset Y of X, there exists $s \in B$ such that $s < Y$;

b) if $s, t \in B$ and $s \neq t$, then $s \nsubseteq t$.

Definition 3.3. (Nash-Williams). Let Q be a qo set and let B be a barrier. A function $f:B \to Q$ is *good* if there exist $s, t \in B$ such that $s \lhd t$ and $f(s) \leqq f(t)$. Q is said to be *better-quasi-ordered* (bqo) if for every barrier B and function $f:B \to Q$, f is good.

In Section 1 of [**10**] the reader will find a quick presentation of some of the results of Nash-Williams along with some improvements. It is easy to show that if Q is bqo then so is every subset and every homomorphic image, and Q is also bqo under any extension of its order. We summarize some less trivial results in the following.

THEOREM 3.4. ([**11**]).
i) *Every well-ordered set is* bqo. *Every* bqo *set is* wqo.
ii) *If* $Q_1, \dots, Q_n$ *are* bqo, *then so are*

$$\bigcup_{i=1}^{n} Q_i \quad \text{and} \quad Q_1 \times \dots \times Q_n.$$

iii) *If* Q *is* bqo, *then* $\mathscr{P}(Q)$ *is also* bqo *under* $\leqq_m$.
iv) *If* Q *is* bqo, *then so is* $Q^{<\omega}$.

To complete this section we introduce a construction on a qo set which generalizes the passage from varieties to generalized varieties.

Let Q be a set with a qo $\leqq$. A *directed set* in Q is a subset $X \subseteq Q$ such that $x, y \in X$ implies $x, y \leqq z$ for some $z \in X$. Let $D(Q)$ denote the set of all directed subsets of Q. Thus, $D(Q) \subseteq \mathscr{P}(Q)$ and we qo it by $\leqq_m$. We let $G(Q)$ denote the quotient $D(Q)/\equiv_m$. From the above, we conclude the following.

PROPOSITION 3.5. *If* Q *is* bqo, *then* $G(Q)$ *is* bqo.

For a set X, $|X|$ denotes its cardinality.

The following general result will be useful in the next section. A shorter proof based on Theorem 3.1 (vi) was communicated to me by Prof. M. Pouzet (see also [**3**]).

THEOREM 3.6. *If* Q *is a* wqo *set, then* $|G(Q)| \leqq |Q|$.

Proof. First, suppose that Q is finite. If $X \in D(Q)$, let $z \in X$ be such that $x \leqq z$ for all $x \in X$. Then $X \equiv_m \{z\}$, and so

$$|G(Q)| \leqq |Q|.$$

Next, consider the case when Q is infinite and suppose that $|G(Q)| = \kappa > |Q|$. Let

$$P_1 = \{X/\equiv_m \in G(Q):\{q\} \nleqq_m X \text{ for some } q \in Q\}.$$

Note that P_1 excludes only the element $Q/\equiv_m$ of $G(Q)$. Hence $|P_1| = \kappa$.

For $q \in Q$, let

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$$\text{Excl}(q) = \{X/\equiv_m \in G(Q):\{q\} \not\leqq_m X\},$$

so that

$$P_1 = \cup\{\text{Excl}(q):q \in Q\}.$$

Since $\lambda^2 = \lambda$ for any infinite cardinal λ, there exists $q_1 \in Q$ with

$$|\text{Excl}(q_1)| = \kappa.$$

Inductively, assume that the elements $q_1, \dots, q_n \in Q$ have been obtained so as to satisfy the following conditions:

i) $|\text{Excl}(q_1) \cap \dots \cap \text{Excl}(q_n)| = \kappa$;

ii) $\{q_{r+1}\}/\equiv_m \in \text{Excl}(q_1) \cap \dots \cap \text{Excl}(q_r)$ for $r = 1, \dots, n - 1$.

We show how to obtain q_{n+1} so that the extended conditions (i) and (ii) hold. Let

$$E_n = \text{Excl}(q_1) \cap \dots \cap \text{Excl}(q_n),$$

and let

$$P_{n+1} = \{X/\equiv_m \in E_n:\text{for some } q \in Q,$$
$$\{q\} \not\leqq_m X \text{ and } \{q\}/\equiv_m \in E_n\}.$$

Then $|E_n \setminus P_{n+1}| \leqq 1$ and so, by (i) we have $|P_{n+1}| = \kappa$. Further,

$$P_{n+1} = \cup\{E_n \cap \text{Excl}(q):q \in Q, \{q\}/\equiv_m \in E_n\},$$

and $|Q| < \kappa$ give

$$|E_n \cap \text{Excl}(q_{n+1})| = \kappa$$

for some $q_{n+1} \in Q$ such that

$$\{q_{n+1}\}/\equiv_m \in E_n.$$

This completes the induction step.

Hence, there is a sequence $q_1, q_2, \dots \in Q$ which satisfies (ii) for all n. But then $q_i \not\leqq q_j$ for all $i < j$, contradicting the assumption that Q is wqo. Therefore, we must have $|G(Q)| \leqq |Q|$.

Finally, we indicate how G is related to $\mathscr{G}$. By definition, a generalized variety $\mathscr{W}$ of algebras from a class $\mathscr{K}$ is obtained as the union of a directed family in $\mathscr{L}(\mathscr{K})$. Thus, we have an onto mapping

$$\varphi:D(\mathscr{L}(\mathscr{K})) \to \mathscr{G}(\mathscr{K})$$

defined by $\varphi(X) = \cup X$. φ is obviously order-preserving, and further it factors through $G(\mathscr{L}(\mathscr{K}))$, for

$$\Psi:X/\equiv_m \mapsto \cup X$$

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is well-defined. In fact, Ψ is an isomorphism from $G(\mathscr{L}(\mathscr{K}))$ onto $\mathscr{G}(\mathscr{K})$. In particular, from Proposition 3.5 we obtain the following.

PROPOSITION 3.7. *If $\mathscr{K}$ is a class of algebras and $\mathscr{L}(\mathscr{K})$ is* bqo, *then $\mathscr{G}(\mathscr{K})$ is also* bqo.

4. Some wqo and bqo sublattices. We start by introducing some notation that will be useful in this and the next sections. Let w be a word in commuting variables from X. For $x \in X$, let $|w|_x$ denote the number of occurrences of x in w. For an integer α, let

$$\nu^{\alpha}(w) = |\{x \in X: |w|_x = \alpha\}|.$$

Let $\mathbf{N}$ denote the lattice of positive integers under the usual order. For a positive integer r, let $M_r = \mathbf{N}_0^r$ under the following order: $(a_1, \ldots, a_r) \leqq (b_1, \ldots, b_r)$ if, for $i = 1, \ldots, r$, $a_i \leqq b_i$ and $b_i = 0$ whenever $a_i = 0$. By Theorem 3.4, M_r is bqo.

The basic ideas in the following result are due to Perkins [**13**].

THEOREM 4.1. *Let $\mathscr{V} \in \mathscr{L}(\mathrm{Com})\backslash\{\mathrm{Com}\}$. Then $\mathscr{L}(\mathscr{V})$ is* bqo.

Proof. It is easy to show that $\mathscr{V} \models x^n = x^{n+k}$ for some positive integers n, k. In the presence of this identity, all identities for members of $\mathscr{V}$ can be reduced to identities of the form

$$x_1^{\alpha_1} \ldots x_r^{\alpha_r} = x_1^{\beta_1} \ldots x_r^{\beta_r} \quad \text{with } 0 \leqq \alpha_i, \beta_i < n + k.$$

Let $J(X)$ denote the set of all identities of this form on a countable set of variables $X = \{x_1, x_2, \ldots\}$. For $e \in J(X)$ with $e:v = w$, let

$$e_{ij} = |\{x \in X: |v|_x = i, |w|_x = j\}|.$$

This defines an integer vector $(e_{10}, e_{01}, e_{11}, e_{20}, e_{02}, \ldots) \in M_r$ where

$$r = (n + k)^2 - 1.$$

Conversely, for any such vector $\vec{a} \neq (0, 0, \ldots, 0)$, there is some identity $e(\vec{a}) \in J(X)$ from which $\vec{a}$ is obtained in the preceding manner. Further, we observe that

$$\vec{a} \leqq \vec{b} \text{ in } M_r \text{ implies } [e(\vec{a})] \cap \mathscr{V} \models e(\vec{b}).$$

Since M_r is bqo, it follows that $J(X)$ is bqo under the relation

$$e_1 \leqq e_2 \quad \text{if } [e_1] \cap \mathscr{V} \models e_2.$$

By Theorem 3.4, $I(J(X))$ is also bqo.

Let Σ be a closed set of identities on the set X (i.e., for any identity e on X, $\Sigma \vdash e$ implies $e \in \Sigma$) such that

$$\Sigma \vdash x^n = x^{n+k}.$$

Let

$$\Sigma' = \Sigma \cap J(X).$$

Then $J(X)\backslash\Sigma'$ is an initial segment of $J(X)$ and Σ is determined by Σ', namely

$$\Sigma = \{e{:}\Sigma' \cup \{x^n = x^{n+k}\} \vdash e\}.$$

Further, if Σ_1, Σ_2 are closed sets of identities such that

$$\Sigma_i \vdash x^n = x^{n+k} \quad (i = 1, 2),$$

then $\Sigma_1' \subseteq \Sigma_2'$ implies $\Sigma_1 \subseteq \Sigma_2$. By Birkhoff's anti-isomorphic correspondence between the lattice of varieties and the lattice of closed sets of identities, it follows that $\mathscr{L}(\mathscr{V})$ is isomorphic (as an ordered set) to a subset of $I(J(X))$. Hence, $\mathscr{L}(\mathscr{V})$ is bqo by Theorem 3.4.

COROLLARY 4.2. (**[13]**). *$\mathscr{L}$(Com) has no infinite descending chains. Every $\mathscr{V} \in \mathscr{L}$(Com) is of the form $\mathscr{V} = [\Sigma]$ for some finite set Σ of identities. $\mathscr{L}$(Com) is countable.*

Using Proposition 3.7 and Theorem 3.6, we also deduce the following.

COROLLARY 4.3. *Let $\mathscr{V} \in \mathscr{L}(\mathrm{Com})\backslash\{\mathrm{Com}\}$. Then $\mathscr{G}(\mathscr{V})$ is* bqo *and countable.*

In the next result we improve Proposition 2.15.

THEOREM 4.4. *$\mathscr{L}_0 \cup \{\mathrm{Nil}\}$ is a* bqo *complete distributive sublattice of $\mathscr{L}(\mathrm{Nil}) \cup \{\mathrm{Nil}\}$.*

Proof. For $X = \{x_1, x_2, \dots\}$ define the following qo on X^σ:

$$v \leqq w \quad \text{if } v = 0 \vdash w = 0$$

where the relation of consequence is interpreted here for commutative semigroups with zero. Then, the mapping

$$\varphi{:}\mathbf{N}^{<\omega} \to X^\sigma$$

$$(\alpha_1, \dots, \alpha_r) \mapsto x_1^{\alpha_1} \dots x_r^{\alpha_r}$$

preserves order, where the order on $\mathbf{N}^{<\omega}$ is the one defined in Section 3. Further, if $\equiv$ denotes the equivalence relation associated with $\leqq$, then every $v \in X^\sigma$ is $\equiv$-equivalent to some element in the image of φ.

By Theorem 3.4, $\leqq$ is a bqo on X^σ.

We observe that

$$\Delta = \{v_1 = 0, v_2 = 0, \dots\} \vdash w = 0$$

implies

$$v_i = 0 \vdash w = 0 \quad \text{for some } i.$$

For, by the analogue of Lemma 2.2 for commutative semigroups with zero,

$\vdash w = 0$ if and only if w has some subword z which can be obtained from v_i by substitution (i.e., $z = f(v_i)$ for some homomorphism $f:X^\sigma \to X^\sigma$).

Let

$$Z = \{v = 0 : v \in X^\sigma\}$$

and let C denote the collection of all subsets Σ of Z such that $\Sigma \vdash w = 0$ implies $(w = 0) \in \Sigma$. If $\Sigma \in C$, then

$$\Sigma' = \{v \in X^\sigma : (v = 0) \in \Sigma\}$$

is a terminal segment of X^σ. Conversely, every terminal segment P of X^σ is of the form $P = \Sigma'$, namely with

$$\Sigma = \{v = 0 : v \in P\} \in C.$$

Also, if $\Sigma \subseteq Z$ and $w_1 = w_2$ is a nontrivial identity, then $\Sigma \vdash w_1 = w_2$ if and only if

$$\Sigma \vdash w_1 = 0 = w_2.$$

Hence, the closure of $\Sigma \in C$ under $\vdash$ is obtained by taking

$$\Sigma \cup \{v = w : v, w \in \Sigma'\} \cup \{\text{all trivial identities}\}.$$

Let D be the set of all closures under $\vdash$ of sets $\Sigma \in C$. Then, by Birkhoff's theorem the set D, ordered by inclusion, is anti-isomorphic to the set $\mathscr{L}_0$. Further, D is isomorphic to C under the mapping

$$\Sigma \mapsto \Sigma \cap Z.$$

Finally, by the above C is anti-isomorphic to $I(X^\sigma)\backslash\{X^\sigma\}$ via the mapping

$$\Sigma \mapsto X^\sigma \backslash \Sigma'.$$

Hence

$$\mathscr{L}_0 \cup \{\text{Nil}\} \simeq I(X^\sigma)$$

as ordered sets; in particular, $\mathscr{L}_0$ is bqo by Theorem 3.4. Since $I(X^\sigma)$ is a complete sublattice of $\mathscr{P}(X^\sigma)$, it follows that $\mathscr{L}_0 \cup \{\text{Nil}\}$ is also a complete distributive lattice.

We denote by $\mathscr{G}_0$ the set of all generalized varieties $\mathscr{W} = \cup_n \mathscr{W}_n$ with $\mathscr{W}_n \in \mathscr{L}_0$ for all n. These are called *purely nil* generalized varieties. With this terminology, it is now an easy matter to establish the following.

COROLLARY 4.5. *$\mathscr{G}_0$ is a countable* bqo *complete distributive sublattice of* $\mathscr{G}$(Com).

For $\mathscr{L}(\mathscr{N})$ we are not yet able to establish bqo. Meanwhile, we can prove wqo.

THEOREM 4.6. *$\mathscr{L}(\mathscr{N})$ is* wqo.

Proof. Suppose $\mathscr{V}_1, \mathscr{V}_2, \dots$ are varieties in $\mathscr{L}(\mathscr{N})$ such that $i < j$ implies $\mathscr{V}_i \not\subseteq \mathscr{V}_j$. Then, for each $i > 1$ there is some identity $u_i = v_i$ such that $\mathscr{V}_1 \not\models u_i = v_i$ but $\mathscr{V}_i \models u_i = v_i$.

It is easy to show that $\mathscr{V} \in \mathscr{L}(\mathscr{N})$ implies $\mathscr{V} \in \mathscr{L}(\mathscr{N}_n)$ for some n. So, let

$$\mathscr{V}_1 \models x_1 \dots x_n = 0.$$

Since $\mathscr{V}_1 \not\models u_i = v_i$, $x_1 \dots x_n = 0 \nvdash u_i = v_i$ and so we may assume that $|u_i| < n$. We may then assume that

$$c(u_i) \subseteq \{x_1, \dots, x_{n-1}\} \quad \text{for all } i.$$

Since there are only finitely many words of length less than n in the commuting variables $x_1, \dots, x_{n-1}$, it follows that $u_i = u$ for infinitely many values of i, for some fixed word u. By considering a subsequence, we may actually assume that $u_i = u$ for all i.

We claim that a subsequence of $\mathscr{V}_2, \mathscr{V}_3, \dots$ consists of varieties satisfying some nontrivial identity. We have

$$\mathscr{V}_i \models u = v_i \quad \text{for } i > 1.$$

Since $\mathscr{V}_i \subseteq \mathscr{N}$, if $c(u) \neq c(v_i)$, then

$$\mathscr{V}_i \models u = 0.$$

Thus, we may assume that

$$c(u) = c(v_i) \quad \text{for all } i.$$

If $\{ |v_i| : i > 1\}$ is bounded, an argument similar to the one used to obtain u establishes the claim. Otherwise, we may assume that $|v_i| > |u|$. Then, since $|u| < n$, via substitution of x for all variables in $u = v_i$, we obtain

$$\mathscr{V}_i \models x^{|u|} = x^{|u|+k_i} \quad \text{for some } k_i > 0.$$

Since $\mathscr{V}_i \subset \mathscr{N}$, it follows that

$$\mathscr{V}_i \models x^{|u|} = 0 \quad \text{for all } i.$$

This proves the claim.

Finally, by the claim we may use Theorem 4.1, contradicting the initial assumption that $\mathscr{V}_i \not\subseteq \mathscr{V}_j$ for all $i < j$. Hence $\mathscr{L}(\mathscr{N})$ is wqo.

A different proof of this result was communicated to me by Prof. R. Laver. Unfortunately, his argument also fails to give bqo of $\mathscr{L}(\mathscr{N})$.

5. A description of the generalized varieties of commutative nil semigroups. In this section we obtain a more manageable finitary form for the elements of $\mathscr{G}$(Nil) than just the union of an ascending chain of nil varieties.

We start by introducing some notation. For positive integers n, α, let

$$\mathscr{N}_n^{(\alpha)} = [\,(x_1 \ldots x_n)^\alpha = 0]$$

and let $\mathscr{N}^{(\alpha)} = \cup_n \mathscr{N}_n^{(\alpha)}$. Also, let

$$\mathscr{N}^{(0)} = [x = y] \quad \text{and} \quad \mathscr{N}^{(\infty)} = \text{Nil}.$$

For a set B of identities and positive integers k, α, we let

$$B_{k,\alpha} = \{w_1(y_1 \ldots y_k)^\alpha = w_2(y_1 \ldots y_k)^\alpha : (w_1 = w_2) \in B\},$$

where $y_1, \ldots, y_k$ are variables not occurring in any identity in B.

PROPOSITION 5.1. *If B is a finite set of nontrivial identities and α is a positive integer, then*

$$(\,[B] \cap \text{Nil}) \vee \mathscr{N}^{(\alpha)} = \cup_k [B_{k,\alpha}] \cap \text{Nil}.$$

We postpone the technical proof of this result until Section 6.

THEOREM 5.2. *The mapping*

$$\varphi_\alpha : \mathscr{L}(\text{Com}) \to \mathscr{G}(\text{Nil})$$

$$\mathscr{V} \mapsto (\mathscr{V} \cap \text{Nil}) \vee \mathscr{N}^{(\alpha)}$$

is a homomorphism of complete meet semilattices for all $\alpha \in \{0, 1, \ldots, \infty\}$.

Proof. If $\alpha = 0$ or $\alpha = \infty$, the result is clear. So suppose that α is a positive integer.

We first claim that if $\mathscr{V}_1, \mathscr{V}_2 \in \mathscr{L}(\text{Com})$, then

$$\varphi(\mathscr{V}_1 \cap \mathscr{V}_2) = \varphi(\mathscr{V}_1) \cap \varphi(\mathscr{V}_2).$$

If either $\mathscr{V}_1 = \text{Com}$ or $\mathscr{V}_2 = \text{Com}$, this is obvious. Otherwise, let B^i be a finite set of nontrivial identities such that $\mathscr{V}_i = [B^i]$ (cf. Corollary 4.2). Then, by Proposition 5.1,

$$\varphi(\mathscr{V}_i) = \cup_k [B^i_{k,\alpha}] \cap \text{Nil} \quad (i = 1, 2).$$

Hence,

$$\begin{aligned}
\varphi(\mathscr{V}_1) \cap \varphi(\mathscr{V}_2) &= \cup_{k,l}(\,[B^1_{k,\alpha}] \cap [B^2_{l,\alpha}]\,) \cap \text{Nil} \\
&= \cup_{k,l}[B^1_{k,\alpha} \cup B^2_{l,\alpha}] \cap \text{Nil} \\
&= \cup_k [B^1_{k,\alpha} \cup B^2_{k,\alpha}] \cap \text{Nil} \\
&= (\,[B^1 \cup B^2] \cap \text{Nil}) \vee \mathscr{N}^{(\alpha)} \text{ by Proposition 5.1} \\
&= \varphi(\mathscr{V}_1 \cap \mathscr{V}_2).
\end{aligned}$$

To establish the theorem, since $\mathscr{L}(\text{Com})$ is countable, it is enough to show that if $\mathscr{V}_1, \mathscr{V}_2, \ldots \in \mathscr{L}(\text{Com})$, then

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$$\varphi\left(\bigcap_{i=1}^{\infty} \mathscr{V}_i\right) = \bigcap_{i=1}^{\infty} \varphi(\mathscr{V}_i).$$

Since $\mathscr{L}$(Com) has no infinite descending chains by Corollary 4.2, there exists i_0 such that

$$\bigcap_{j=1}^{i_0} \mathscr{V}_j = \bigcap_{j=1}^{i_0+1} \mathscr{V}_j = \dots .$$

Then

$$\bigcap_{i=1}^{\infty} \varphi(\mathscr{V}_i) = \bigcap_{i=1}^{\infty} \left(\bigcap_{j=1}^{i} \varphi(\mathscr{V}_j)\right)$$

$$= \bigcap_{i=1}^{\infty} \varphi\left(\bigcap_{j=1}^{i} \mathscr{V}_j\right) \quad \text{by the first part of the proof}$$

$$= \varphi\left(\bigcap_{j=1}^{i_0} \mathscr{V}_j\right)$$

$$= \varphi\left(\bigcap_{j=1}^{\infty} \mathscr{V}_j\right).$$

This completes the proof of the theorem.

By restriction of φ_α to $\mathscr{L}$(Nil), we obtain a lattice homomorphism, as stated in the following.

THEOREM 5.3. *The mapping*

$$\Psi_\alpha : \mathscr{L}(\text{Nil}) \cup \{\text{Nil}\} \to \mathscr{G}(\text{Nil})$$

$$\mathscr{V} \mapsto \mathscr{V} \vee \mathscr{N}^{(\alpha)}$$

is a homomorphism of complete lattices, for all $\alpha \in \{0, 1, \dots, \infty\}$.

The generalized varieties $\mathscr{V} \vee \mathscr{N}^{(\alpha)}$ turn out to be important building blocks in $\mathscr{G}$(Nil).

PROPOSITION 5.4. *Let* $\mathscr{W} \in \mathscr{G}$(Nil) *and let* $S \in$ Nil$\setminus\mathscr{W}$. *Then, there exist a variety* $\mathscr{V} \in \mathscr{L}$(Nil) *and an integer* α *such that*

$$S \notin \mathscr{V} \vee \mathscr{N}^{(\alpha)} \supseteq \mathscr{W}.$$

A proof of Proposition 5.4 will be given in Section 7. Meanwhile, we deduce the announced form for the elements of $\mathscr{G}$(Nil).

THEOREM 5.5. *Every* $\mathscr{W} \in \mathscr{G}$(Nil) *is of the form*

$$\mathscr{W} = \bigcap_{i=1}^{n} (\mathscr{V}_i \vee \mathscr{N}^{(\alpha_i)})$$

for some integer n, some $\mathscr{V}_1, \dots, \mathscr{V}_n \in \mathscr{L}(\mathrm{Nil})$ *and some* $\alpha_1, \dots, \alpha_n \in \{0, 1, \dots, \infty\}$ *with* $\alpha_1 < \dots < \alpha_n$.

Proof. If $\mathscr{W} = \mathrm{Nil}$, the result is obvious. So, suppose that $\mathscr{W} \neq \mathrm{Nil}$. Then, by Proposition 5.4 $\mathscr{W}$ is an intersection of generalized varieties of the form $\mathscr{V} \vee \mathscr{N}^{(\alpha)}$ with

$$\mathscr{V} \in \mathscr{L}(\mathrm{Nil}) \quad \text{and} \quad \alpha \in \{0, 1, \dots\}.$$

Since $\mathscr{L}(\mathrm{Nil})$ is countable and using Theorem 5.3, there actually exist varieties $\mathscr{V}_1, \mathscr{V}_2, \dots \in \mathscr{L}(\mathrm{Nil})$ and integers $\alpha_1 < \alpha_2 < \dots$ such that

$$\mathscr{W} = \bigcap_{i=1}^{\infty} (\mathscr{V}_i \vee \mathscr{N}^{(\alpha_i)}).$$

But $\mathscr{V}_1 \subset \mathscr{N}^{(\alpha_{n+1})}$ for some n. Hence,

$$\mathscr{W} = \bigcap_{i=1}^{n} (\mathscr{V}_i \vee \mathscr{N}^{(\alpha_i)}),$$

as desired.

We conclude this section with a number of applications of Theorem 5.5.

COROLLARY 5.6. *$\mathscr{G}(\mathrm{Nil})$ is countable.*

COROLLARY 5.7. *$\mathscr{G}(\mathrm{Nil})$ has no infinite descending chains.*

Proof. Let $\mathscr{W}_1 \supseteq \mathscr{W}_2 \supseteq \dots$ be a descending chain in $\mathscr{G}(\mathrm{Nil})$. By Theorem 5.5 we may assume that

$$\mathscr{W}_n = \bigcap_{i=1}^{n} (\mathscr{V}_i \vee \mathscr{N}^{(\alpha_i)})$$

for some $\mathscr{V}_1, \mathscr{V}_2, \dots \in \mathscr{L}(\mathrm{Nil})$, and some $\alpha_1, \alpha_2, \dots \in \{0, 1, \dots\}$, with $\alpha_1 \leqq \alpha_2 \leqq \dots$. Consider

$$\mathscr{W} = \bigcap_{i=1}^{\infty} (\mathscr{V}_i \vee \mathscr{N}^{(\alpha_i)}).$$

Using Theorem 5.3 and the fact that $\mathscr{L}(\mathrm{Nil})$ has no infinite descending chains, we may actually assume that $\alpha_1 < \alpha_2 < \dots$. Then, the same argument as in the proof of Theorem 5.5 shows that $\mathscr{W} = \mathscr{W}_n$ for some n.

From Proposition 2.14, we deduce the following.

THEOREM 5.8. *$\mathscr{G}_{\mathrm{cb}}$ is a countable lattice with no infinite descending chains. In particular, the same is true of $\mathscr{G}(\mathscr{A}p)$.*

As a final consequence of Theorem 5.5, we have the following.

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THEOREM 5.9. *If* $\mathscr{W} \in \mathscr{G}_{\text{cb}}$ *is such that* $\langle \mathscr{W} \rangle = \text{Com}$, *then* $\mathscr{W}$ *contains* $\mathscr{N}$.

Proof. By Theorem 2.10 (b), it is enough to establish the result for $\mathscr{W} \in \mathscr{G}(\text{Nil})$. Write $\mathscr{W} \in \mathscr{G}(\text{Nil})$ in the form given by Theorem 5.5. If $\alpha_1 = 0$, then

$$\langle \mathscr{W} \rangle \subseteq \mathscr{V}_1 \neq \text{Com}.$$

Thus, $\alpha_1 > 0$ giving

$$\mathscr{N} \subseteq \mathscr{N}^{(\alpha_i)} \quad (i = 1, \dots, n).$$

Hence, $\mathscr{N} \subseteq \mathscr{W}$.

6. Proof of proposition 5.1. We start with a few simple observations.

LEMMA 6.1. *Let* $S \in \text{Nil}$ *be such that*

$$S \models u = v{:}x_1^{\alpha_1} \dots x_r^{\alpha_r} = x_1^{\beta_1} \dots x_r^{\beta_r}$$

with $\alpha_i, \beta_i \geqq 0$ *and* $\alpha_1 < \beta_1$. *Then*

$$S \models uy^{\alpha_1} = 0 = vy^{\alpha_1}.$$

Hence, if y *is a variable not occurring in* uv *and* $s = V(u = v)$, *then*

$$[u = v] \cap \text{Nil} \models uy^s = 0 = vy^s.$$

Proof. Substituting yx_1 for x_1, we get

$$S \models y^{\alpha_1}x_1^{\alpha_1} \dots x_r^{\alpha_r} = y^{\beta_1}x_1^{\beta_1} \dots x_r^{\beta_r} = y^{\beta_1}x_1^{\alpha_1} \dots x_r^{\alpha_r}$$

and so

$$\begin{aligned} S \models y^{\alpha_1}v = y^{\alpha_1}u &= y^{\alpha_1 + k(\beta_1 - \alpha_1)}u \quad \text{for all } k \geqq 0 \\ &= 0 \text{ since } S \in \text{Nil}. \end{aligned}$$

The next result is intended to serve as motivation for what will appear later.

LEMMA 6.2. a) *If* $\text{Nil}_r \nsubseteq \mathscr{W} \in \mathscr{G}(\text{Nil})$, *then* $\mathscr{W} \subseteq \mathscr{N}^{(r-1)}$.
b) $\mathscr{G}(\text{Nil}) = \{\text{Nil}\} \cup \bigcup_{0<k<\infty} \mathscr{G}(\mathscr{N}^{(k)})$.

Proof. a) Write $\mathscr{W} = \cup_n \mathscr{W}_n$ where $(\mathscr{W}_n)_n$ is an ascending chain of varieties. Then, for each n there exists a nontrivial identity $u_n = v_n$ such that

$$\mathscr{W}_n \models u_n = v_n \quad \text{but} \quad x^r = 0 \not\models u_n = 0.$$

Since $\mathscr{W}_n \in \mathscr{L}(\text{Nil})$, it follows from Lemma 6.1 that

$$\mathscr{W}_n \models u_n y^k = 0 \quad \text{for some } k < r,$$

and so

$$\mathscr{W}_n \subseteq \mathscr{N}_{m_n}^{(r-1)} \quad \text{for big enough } m_n.$$

b) This is an easy consequence of (a).

LEMMA 6.3. *Let B be a set of identities such that*

$$[B] \cap \mathscr{A}\!\!p \vDash x^r = x^{r+1}.$$

Then

$$[B_{k,\alpha}] \cap \mathscr{A}\!\!p \vDash x^r(y_1 \dots y_k)^\alpha = x^{r+1}(y_1 \dots y_k)^\alpha.$$

Proof. Since $C_{r+1,1} \in \mathscr{A}\!\!p$ but $C_{r+1,1} \nvDash x^r = x^{r+1}$, we have

$$C_{r+1,1} \notin [B].$$

Therefore, $V(w_1 = w_2) \leqq r$ for some identity $(w_1 = w_2) \in B$ by Lemma 2.9 (a). Thus, we can obtain $x^r = x^{r+s}$ for some $s > 0$ from $w_1 = w_2$ by substituting some power of x for each variable in $w_1 = w_2$ and multiplying both sides of the resulting identity by an appropriate power of x. Then, the same operations show that

$$\begin{aligned} w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha \vdash x^r(y_1 \dots y_k)^\alpha \\ = x^{r+s}(y_1 \dots y_k)^\alpha \end{aligned}$$

and this implies the lemma.

LEMMA 6.4. *Let $w \in \{x_1, x_2, \dots\}^\sigma$ and let α be a positive integer. Let $m \geqq |w|/\alpha$ and, for an integer k, let $n = n(k) = m + k$. Then*

$$w = 0 \vdash u = v \quad \textit{and} \quad (y_1 \dots y_n)^\alpha = 0 \vdash u = v$$

imply

$$w(y_1 \dots y_k)^\alpha = 0 \vdash u = v.$$

Proof. We may assume that $u = v$ is nontrivial. Then

$$w = 0 \vdash u = 0 = v \quad \text{and}$$

$$(y_1 \dots y_n)^\alpha = 0 \vdash u = 0 = v,$$

so that each of u and v admits a subword of the form $w(y_1 \dots y_k)^\alpha$, with appropriate substitutions for the variables in this word. Hence

$$w(y_1 \dots y_k)^\alpha = 0 \vdash u = 0 = v.$$

LEMMA 6.5. *Let $w_1 = w_2$ be an identity in the commuting variables $\{x_1, x_2, \dots\}$, $r = V(w_1 = w_2)$, α a positive integer and*

$$m \geqq \frac{1}{\alpha}(|w| + r).$$

For an integer k, let $n = m + k$. *Let* $u = v$ *be an identity of the form* $w_1'z = w_2'z$ *where the identity* $w_1' = w_2'$ *is obtained from* $w_1 = w_2$ *by replacing every occurrence of each variable* x_i *by a term* t_i, *and*

$$z \in \{x_1, x_2, \dots\}^\sigma.$$

Suppose further that

$$(y_1 \dots y_n)^\alpha = 0 \vdash u = v.$$

a) *If* $|t_{i_0}| > 1$ *for some* i_0 *such that* $|w_1|_{x_{i_0}} \neq |w_2|_{x_{i_0}}$, *then*

$$[w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha] \cap \mathrm{Nil} \vDash u = 0 = v.$$

b) *If* $|t_i| = 1$ *for all i such that* $|w_1|_{x_i} \neq |w_2|_{x_i}$, *then*

$$w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha \vdash u = v.$$

Proof. a) Assuming the type of substitution indicated,

$$w_1y^r = 0 = w_2y^r \vdash u = 0 = v.$$

Thus, the result follows from Lemmas 6.1 and 6.4, namely

$$[w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha] \cap \mathrm{Nil} \vDash w_1(y_1 \dots y_k)^\alpha y^r$$
$$= 0 = w_2(y_1 \dots y_k)^\alpha y^r \vdash u = 0 = v.$$

b) In this case, we can write $u = v$, in the form $w_1''z' = w_2''z'$ where $w_1'' = w_2''$ is obtained from $w_1 = w_2$ by a variable for variable substitution and

$$z' \in \{x_1, x_2, \dots\}^\sigma.$$

In particular,

$$(y_1 \dots y_k)^\alpha = 0 \vdash z' = 0.$$

Combining a substitution which gives a subword of z' from $(y_1 \dots y_k)^\alpha$ and a substitution which yields $w_1'' = w_2''$ from $w_1 = w_2$, we deduce that

$$w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha \vdash w_1''z' = w_2''z'.$$

Notation for Lemma 6.6 *and Proposition* 6.7. Let B be a finite set of nontrivial identities. Let

$$r = \max\{V(e): e \in B\},$$

$$l = \max\{\,|w|:(w = v) \in B \cup B^r \text{ for some } v\},$$

and let

$$[B] \cap \mathcal{A}p \vDash x^q = x^{q+1}.$$

For $D \subseteq B$, let

$$m(D) = \sum_{e \in D} \ulcorner \frac{1}{\alpha} \chi(e) V(e) \urcorner + \sum_{(w_1 = w_2) \in D} \ulcorner \frac{1}{\alpha} (|w_1| + |w_2|) \urcorner + \ulcorner \frac{1}{\alpha} (l + r) \urcorner$$

where $\ulcorner a \urcorner$ denotes the least integer not less than a and $\chi(e)$ denotes the number of variables appearing in the identity e. We fix positive integers k and α, and let $n = m(B) + k$.

LEMMA 6.6. *Let* $u_0, u_1, \dots, u_p \in \{x_1, x_2, \dots\}^\sigma$ *and let* $e_0, \dots, e_{p-1} \in B$ *be such that*

$$e_i \rightarrow u_i = u_{i+1}$$

but

$$w_{1i} y^{s_i} = 0 = w_{2i} y^{s_i} \nvdash u_i = u_{i+1}$$

where $e_i : w_{1i} = w_{2i}$ *and* $s_i = V(e_i)$ $(i = 0, \dots, p - 1)$. *If*

$$(y_1 \dots y_n)^\alpha = 0 \vdash u_0 = 0,$$

then

$$(y_1 \dots y_{n'})^\alpha = 0 \vdash u_i = 0 \quad (i = 0, \dots, p),$$

where

$$n' = \ulcorner \frac{1}{\alpha} (l + r) \urcorner + k.$$

Proof. First, we observe that the hypotheses imply that each identity e_i is regular, i.e., $s_i > 0$. Write

$$u_i = w'_{1i} z'_i z_i, \quad u_{i+1} = w'_{2i} z'_i z_i$$

where $w'_{1i} = w'_{2i}$ is obtained from $w_{1i} = w_{2i}$ by a variable for variable substitution,

$$c(z'_i) \subseteq c(w'_{1i}), \quad c(z_i) \cap c(w'_{1i}) = \emptyset \quad \text{and}$$

$$y^{s_i} = 0 \nvdash z'_i z_i = 0 \quad (i = 0, \dots, p - 1).$$

Let

$$n_j = m(B \backslash \{e_{i_0}, e_{i_1}, \dots, e_{i_j}\}) + k$$

where $i_0 = 0$ and i_j is defined below.

Then

$$(y_1 \dots y_{n_0})^\alpha = 0 \vdash z_0 = 0,$$

since $|z'_0| < \chi(e_0) s_0$. Further, no identity e with $V(e) \geqq s_0$ has the capacity to change the exponents in z_0 since

$$y^{s_0} = 0 \nvdash z_0 = 0.$$

Hence, if i_1 is the least index for which $s_{i_1} < s_0$ but $s_j \geqq s_0$ for $0 < j < i_1$, then

$$(y_1 \dots y_{n_0})^\alpha = 0 \vdash z_j = 0 \quad \text{for } 0 \leqq j < i_1.$$

More generally, let $0 = i_0, i_1, \dots, i_h$ be indices such that i_{j+1} is the least index for which $s_{i_{j+1}} < s_{i_j}$ but $s_\rho \geqq s_{i_j}$ for $i_j \leqq \rho < i_{j+1}$, and $s_\rho \geqq s_{i_h}$ for $i_h \leqq \rho \leqq p - 1$. Let $i_{h+1} = p$.

Inductively, we assume that

$$(y_1 \dots y_{n_{j-1}})^\alpha = 0 \vdash z_\rho = 0 \quad \text{for } \rho < i_j,$$

where $j > 0$. Then

$$(y_1 \dots y_{n_j})^\alpha = 0 \vdash z_{i_j} = 0$$

since

$$(y_1 \dots y_{n_{j-1}})^\alpha = 0 \vdash u_{j-1} = 0 \text{ and } |z'_{i_j}| < \chi(e_{i_j})s_{i_j}.$$

Also, no e_ρ with $i_j \leqq \rho < i_{j+1}$ has the capacity to change the exponents in z_{ij}. Hence

$$(y_1 \dots y_{n_j})^\infty = 0 \vdash z_\rho = 0 \quad \text{for } \rho < i_{j+1}.$$

By induction, and since $n' \leqq n_j$ for all j, we obtain

$$(y_1 \dots y_{n'})^\alpha = 0 \vdash u_i = 0 \quad (i = 0, \dots, p),$$

as desired.

PROPOSITION 6.7. *If*

$$B \cup \{x^q = 0\} \vdash u = v \quad \textit{and}$$

$$(y_1 \dots y_n)^\alpha = 0 \vdash u = v,$$

then

$$[B_{k,\alpha}] \cap \text{Nil} \vDash u = v.$$

Proof. We assume that $u = v$ is a nontrivial identity, so that

$$(y_1 \dots y_n)^\alpha = 0 \vdash u = 0 = v$$

and there exist words $u_0, u_1, \dots, u_p$ and identities $e_0, \dots, e_{p-1} \in B \cup \{x^q = 0\}$ such that, as words, $u = u_0$ and $v = u_p$, and $e_i \rightarrow u_i = u_{i+1}$ (cf. Lemma 2.2). Again, let $s_i = V(e_i)$ and $e_i{:}w_{1i} = w_{2i}$. If for all i,

$$w_{1i}y^{s_i} = 0 \nvdash u_i = 0 \quad \text{or} \quad w_{2i}y^{s_i} = 0 \nvdash u_{i+1} = 0,$$

then the result follows from Lemmas 6.6 and 6.5 (b). Otherwise, let i_1 and i_2 be respectively the first and last steps for which the previous hypothesis fails. Then, by Lemma 6.6,

(*) $(y_1 \dots y_{n'})^\alpha = 0 \vdash u_j = 0$ for $j = 0, \dots, i_1$ and $j = i_2, \dots, p$,

so that, by Lemma 6.5 (b), one step $u_i = u_{i+1}$ at a time, we obtain

$$B_{k,\alpha} \vdash u_0 = u_1 = \dots = u_{i_1}, \quad u_{i_2} = \dots = u_p.$$

Now,

$$w_{1i_1}y^{s_{i_1}} = 0 \vdash u_{i_1} = 0, \quad w_{2i_2}y^{s_{i_2}} = 0 \vdash u_{i_2} = 0$$

(or $x^q = 0 \vdash u_{i_j} = 0$ in case e_{i_j} is $x^q = 0$) and (*). Hence, by Lemmas 6.1, 6.3 and 6.4,

$$[B_{k,\alpha}] \cap \mathrm{Nil} \vDash w_{1i_1}(y_1 \dots y_k)^\alpha y^{s_{i_1}} = 0$$
$$= w_{2i_2}(y_1 \dots y_k)^\alpha y^{s_{i_2}} \vdash u_{i_1} = 0 = u_{i_2}.$$

Hence

$$[B_{k,\alpha}] \cap \mathrm{Nil} \vDash u_0 = u_1 = \dots = u_{i_1} = u_{i_2} = \dots = u_p,$$

and so

$$[B_{k,\alpha}] \cap \mathrm{Nil} \vDash u = v,$$

as claimed.

From Proposition 6.7 it is easy to deduce Proposition 5.1. Here, we prove a special case.

COROLLARY 6.8.

a) $[w = 0] \vee \mathcal{N}^{(\alpha)} = \bigcup_n [w(y_1 \dots y_n)^\alpha = 0]$

b) $[w_1 = w_2] \vee \mathcal{N}^{(\alpha)} = \bigcup_n [w_1(y_1 \dots y_n)^\alpha$

$$= w_2(y_1 \dots y_n)^\alpha] \cap \mathrm{Nil}.$$

Proof. (a) is a particular case of (b). For (b), we first observe that the left side is clearly contained in the right side. For the reverse inclusion, we use Proposition 6.7 with $B = \{w_1 = w_2\}$:

$$[w(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha] \cap \mathrm{Nil}$$
$$\subseteq ([w_1 = w_2] \cap \mathrm{Nil}) \vee \mathcal{N}^{(\alpha)}_{n(k)}$$

where $n(k) = n$ is given by Proposition 6.7.

7. Proof of proposition 5.4. Let $\mathcal{W} \in \mathcal{G}(\mathrm{Nil})$ and $S \in \mathrm{Nil} \setminus \mathcal{W}$.

Let $\mathcal{W} = \cup_n \mathcal{W}_n$, the union of an ascending chain of varieties (cf. Lemma 2.5). Then, for each n there exists an identity $u_n = v_n$ such that

$$\mathcal{W}_n \vDash u_n = v_n \quad \text{but} \quad S \nvDash u_n = v_n.$$

Since $S \in \text{Nil}$, there exists m such that $S \models x^m = 0$, and we may assume that

$$x^m = 0 \nvdash u_n = 0 \quad \text{for all } n.$$

Note that for every $n_1 < n_2 < \dots$,

$$S \notin \bigcup_i [u_{n_i} = v_{n_i}] \cap \text{Nil} \supseteq \mathscr{W}.$$

We then proceed to find an appropriate subsequence of $(u_n = v_n)_n$. First, we may assume that all identities in this sequence are of the form $u_n = 0$ or they are all regular.

a) Suppose that $S \notin \bigcup_n [u_n = 0] \supseteq \mathscr{W}$. Since $x^m = 0 \nvdash u_n = 0$ for all n, we can represent each word u_n by a vector $e_n \in M_{m-1}$ whose k^{th} coordinate is $\nu^\alpha(u_n)$ (cf. Section 4). Then $e_n \leqq e_l$ implies $u_n = 0 \vdash u_l = 0$. Hence, we may assume that for all n, $u_n = 0 \vdash u_{n+1} = 0$, for this can be achieved by considering a subsequence (since M_{m-1} is wqo).

Let α be the maximum exponent for which

$$\{\nu^\alpha(u_n): n = 1, 2, \dots\}$$

is unbounded (if $\{\nu^\alpha(u_n): n = 1, 2, \dots\}$ is bounded for all α, then

$$S \notin [u = 0] \supseteq \mathscr{W} \quad \text{for some word } u,$$

and we are done). We may assume that $\nu^\alpha(u_n) < \nu^\alpha(u_{n+1})$ for all n and that $\lambda_\rho = \nu^\rho(u_n)$ is independent of n for each $\rho > \alpha$. Let

$$w = (x_{11} \dots x_{1\lambda_{\alpha+1}})^{\alpha+1} \dots (x_{m-\alpha-1,1} \dots x_{m-\alpha-1,\lambda_{m-1}})^{m-1}.$$

Then

$$\bigcup_n [u_n = 0] = \bigcup_n [w(y_1 \dots y_n)^\alpha = 0] = [w = 0] \vee \mathscr{N}^{(\alpha)}$$

by Corollary 6.8 (a). Thus, we are done in case the identities $u_n = v_n$ are all of the form $u_n = 0$.

b) Suppose now that the identities $u_n = v_n$ are all regular. Since

$$x^m = 0 \nvdash u_n = 0,$$

we have $V(u_n = v_n) < m$ and so we may assume that

$$V(u_n = v_n) = r \quad \text{for all } n.$$

With the notation used in (a), we may also assume that $e_n \leqq e_{n+1}$ for all n.

Suppose first that the set $\{\text{maximum exponent in } v_n : n = 1, 2, \dots\}$ is unbounded. In this case, suppose that n is such that for all $i > 0$,

$$u_n y^r = 0 \nvdash u_{n+i} = 0.$$

Then $\nu^\rho(u_{n+j})$ is independent of $j \geqq 0$, for all $\rho \geqq r$. Also, any variable which appears in u_{n+j} with exponent less than r, must appear in v_{n+j} with the same exponent. Hence, for big enough i,

$$u_{n+i} = 0 \vdash v_{n+i} = 0$$

and so,

$$[u_{n+i} = v_{n+i}] \cap \mathrm{Nil} \vDash u_{n+i} = 0.$$

Thus, we may replace the identities $u_{n+i} = v_{n+i}$ by $u_{n+i} = 0$ and use (a). Therefore, we may assume that

$$u_n y^r = 0 \vdash u_{n+i} = 0 \quad \text{for all } n \text{ and } i > 0.$$

Then, for each n,

$$u_n y^r = 0 \vdash u_{n+i} = 0 = v_{n+i}$$

for some big enough i. It follows that

$$\bigcup_i [u_{n_i} = v_{n_i}] \cap \mathrm{Nil} = \bigcup_i [u_{n_i} y^r = 0] \quad \text{for some } n_1 < n_2 < \dots$$

and this again reduces the proof to case (a).

Hence we may assume that, for all n, there is no variable in $u_n = v_n$ with exponent greater than N. Now, represent each identity $u_n = v_n$ by a vector $f_n \in M_{(N+1)^2-1}$ in which the (i, j)th coordinate is

$$|\{x : |u_n|_x = i \quad \text{and} \quad |v_n|_x = j\}|.$$

Then, $f_n \leqq f_l$ implies $u_n = v_n \vdash u_l = v_l$. Also, we may assume that $f_n \leqq f_{n+1}$ for all n.

Let α (resp. β) be the maximum exponent for which

$$\{\nu^\alpha(u_n) : n = 1, 2, \dots\}$$

(resp. $\{\nu^\beta(v_n) : n = 1, 2, \dots\}$) is unbounded. Note that, since the identities $u_n = v_n$ are regular, if one of these numbers α, β exists, then so does the other. Also, if neither α nor β exist, then

$$S \notin [u = v] \cap \mathrm{Nil} \supseteq \mathscr{W}$$

for some regular identity $u = v$, and we are done. Thus, we consider the remaining case in which both α and β exist. We observe that either $r > \alpha, \beta$ or $r \leqq \alpha, \beta$, e.g., $\alpha < r < \beta$ is ruled out by the definition of r and the assumption that the identities $u_n = v_n$ are regular.

Let w_1 (resp. w_2) be the subword of u_n (resp. v_n) formed by the factors x_i^ρ where x_i is a variable appearing in u_n (resp. v_n) with exponent $\rho > \alpha$ (resp. $\rho > \beta$). We may assume that these words w_1 and w_2 are independent of n. We separate two subcases.

b.1) $r > \alpha, \beta$. Then $\alpha = \beta$. Also

$$u_n = v_n \vdash w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha$$

for big enough k and

$$w_1(y_1 \dots y_k)^\alpha = w_2(y_1 \dots y_k)^\alpha \vdash u_n = v_n$$

for big enough n.

Hence

$$\bigcup_n [u_n = v_n] \cap \text{Nil} = \bigcup_k [w_1(y_1 \dots y_k)^\alpha$$
$$= w_2(y_1 \dots y_k)^\alpha] \cap \text{Nil}$$
$$= ([w_1 = w_2] \cap \text{Nil}) \vee \mathcal{N}^{(\alpha)}$$

by Corollary 6.8 (b).

b.2) $\alpha, \beta \geqq r$. As in (b.1), we obtain

$$\bigcup_n [u_n = v_n] \cap \text{Nil} = \bigcup_k [w_1(y_1 \dots y_k)^\alpha = 0 = w_2(y_1 \dots y_k)^\beta]$$
$$= \bigcup_k [w_1(y_1 \dots y_k)^\alpha = 0]) \cap (\bigcup_k [w_2(y_1 \dots y_k)^\beta = 0])$$
$$= ([w_1 = 0] \vee \mathcal{N}^{(\alpha)}) \cap ([w_2 = 0] \vee \mathcal{N}^{(\beta)})$$

where the last line uses Corollary 6.8 (a).

This completes the proof of Proposition 5.4.

To help to understand the previous proof, we indicate the following.

Examples.

1) $$\bigcup_n [x^4y^3(z_1 \dots z_n)^2(t_1 \dots t_n)$$
$$= x^3y^4(z_1 \dots z_n)^2(t_1 \dots t_n)] \cap \text{Nil}$$
$$= ([x^4y^3 = x^3y^4] \cap \text{Nil}) \vee \mathcal{N}^{(2)}.$$

2) $$\bigcup_n [x^4(y_1 \dots y_n)^3(z_1 \dots z_n)^2t$$
$$= x^5(y_1 \dots y_n)(z_1 \dots z_n)^3t^2] \cap \text{Nil}$$
$$= [x^4 = 0] \vee \mathcal{N}^{(3)}.$$

3) $$\bigcup_n [x^4(y_1 \dots y_n)^3z^2(t_1 \dots t_n)$$
$$= x^3z^6(y_1 \dots y_n)^2(t_1 \dots t_n)] \cap \text{Nil}$$
$$= ([x^4 = 0] \vee \mathcal{N}^{(3)}) \cap ([x^6y^3 = 0] \vee \mathcal{N}^{(2)}).$$

8. Pseudovarieties of commutative semigroups.

Definition 8.1. (**[6]**). A class $\mathcal{V}$ of finite semigroups is a *pseudovariety* if it is closed under the formation of homomorphic images, subsemigroups and finite products.

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The basic link between pseudovarieties and varieties is described in the following.

THEOREM 8.2. ([**2**]). *A class $\mathscr{V}$ of finite semigroups is a pseudovariety if and only if it consists of the finite members of some generalized variety.*

This section is dedicated to the translation of some of the results on $\mathscr{G}$(Com) to finite commutative semigroups via the previous theorem. The simplification that appears when we restrict our attention to finite semigroups is due mainly to the following.

PROPOSITION 8.3. ([**5**], Proposition III.9.2). *Let S be a finite semigroup with n elements and let E be the set of all idempotents in S. Then $S^n = SES$. In particular, every finite nil semigroup is nilpotent.*

For a class $\mathscr{K}$ of semigroups, we denote by $\mathscr{K}^F$ the class of all finite semigroups in $\mathscr{K}$. By Proposition 8.3, we have $\text{Nil}^F = \mathscr{N}^F$. $\mathscr{P}\!s(\mathscr{K})$ denotes the collection of all pseudovarieties of semigroups from $\mathscr{K}$. If $\mathscr{K} \in \mathscr{G}(\text{Com})$, using Theorem 8.2 it is easy to show that the mapping $\mathscr{W} \mapsto \mathscr{W}^F$ defines a lattice homomorphism from $\mathscr{G}(\mathscr{K})$ onto $\mathscr{P}\!s(\mathscr{K})$. Essentially because of Lemma 2.9 (b), the assignment $V_F(\mathscr{W}^F) = V(\mathscr{W})$ defines a function on $\mathscr{P}\!s$(Com).

LEMMA 8.4. *If $\mathscr{V} \in \mathscr{L}$(Nil), then $\mathscr{V} = \langle \mathscr{V}^F \rangle$.*

Proof. Suppose that $\mathscr{V}^F$ satisfies an identity e in the variables $x_1, \dots, x_m$. Let $S \in \mathscr{V}$ and $s_1, \dots, s_m \in S$. The subsemigroup T of S generated by $s_1, \dots, s_m$ is nil and so it is finite. Thus $T \in \mathscr{V}^F$ and $T \models e$. It follows that the substitution of s_i for x_i ($i = 1, \dots, m$) in e gives a valid equality in T. Hence $S \models e$ and so $\mathscr{V} \models e$. This proves the lemma.

PROPOSITION 8.5. *The mapping*

$$\varphi : \mathscr{L}(\text{Nil}) \to \mathscr{P}\!s(\mathscr{N}) \setminus \{\mathscr{N}^F\}$$

$$\mathscr{V} \mapsto \mathscr{V}^F$$

is a lattice isomorphism.

Proof. φ is injective by the lemma. φ clearly preserves the meet operation. By Theorem 8.2, for each $Z \in \mathscr{P}\!s(\mathscr{N}) \setminus \{\mathscr{N}^F\}$ there is some $\mathscr{W} \in \mathscr{G}(\mathscr{N}) \setminus \{\mathscr{N}\}$ such that $\mathscr{W}^F = Z$. By Theorem 2.6, $\langle \mathscr{W} \rangle \in \mathscr{L}$(Nil) and $\mathscr{W} = \langle \mathscr{W} \rangle \cap \mathscr{N}$. By Proposition 8.3 $\langle \mathscr{W} \rangle^F \subset \mathscr{N}$. Hence, $\langle \mathscr{W} \rangle^F = Z$, proving that φ is surjective. So, φ is a semilattice isomorphism. It follows that φ is a lattice isomorphism.

Remark. It is not hard to give a direct proof that if $Z \in \mathscr{P}\!s(\mathscr{N}) \setminus \{\mathscr{N}^F\}$ then $\langle Z \rangle^F = Z$ using the observations that a finitely generated commutative nil semigroup is finite and $\langle Z \rangle \in \mathscr{L}$(Nil).

THEOREM 8.6. *The lattices* $\mathscr{L}(\mathrm{Nil}) \cup \{\mathrm{Nil}\}$, $\mathscr{G}(\mathscr{N})$ *and* $\mathscr{P}\!s(\mathscr{N})$ *are all isomorphic.*

Proof. Let

$$\psi: \mathscr{L}(\mathrm{Nil}) \cup \{\mathrm{Nil}\} \to \mathscr{G}(\mathscr{N})$$
$$\mathscr{W} \mapsto \mathscr{W} \cap \mathscr{N}$$

and

$$\theta: \mathscr{G}(\mathscr{N}) \to \mathscr{P}\!s(\mathscr{N})$$
$$\mathscr{W} \mapsto \mathscr{W}^F.$$

Then $\theta \circ \psi$ is a lattice isomorphism by Proposition 8.5. Since ψ is onto by Theorem 2.6, it follows that both ψ and θ are bijections. Obviously ψ and θ preserve order. Hence ψ and θ are lattice isomorphisms.

We now state a version of Theorem 2.10 and Corollary 2.12 for pseudovarieties.

THEOREM 8.7. *The mapping*

$$\mathscr{P}\!s(\mathrm{Com}) \to (\mathscr{L}(\mathrm{Nil}) \cup \{\mathrm{Nil}\}) \times \mathscr{P}\!s(\mathscr{A}) \times (\mathbf{N}_0 \cup \{\infty\})$$
$$Z \mapsto (\langle Z \cap \mathrm{Nil}\rangle, Z \cap \mathscr{A}, V_F(Z))$$

is an embedding of meet semilattices. Further,

$$Z = (Z \cap \mathscr{A}) \vee (Z \cap \mathrm{Nil}) \vee \mathscr{A}\!p^F_{V_F}(Z).$$

We also observe that $\mathscr{G}(\mathscr{A}) \simeq \mathscr{P}\!s(\mathscr{A})$ via the mapping $\mathscr{W} \mapsto \mathscr{W}^F$, so that the "unknown" part in the product of Theorem 8.7 is only $\mathscr{L}(\mathrm{Nil}) \cup \{\mathrm{Nil}\}$.

A pseudovariety Z is said to be *proper* if $\langle Z\rangle^F \neq Z$. From Theorem 8.7 one easily deduces the following.

COROLLARY 8.8. *A pseudovariety* $Z \in \mathscr{P}\!s(\mathrm{Com})$ *is proper if and only if either* Z *contains* $\mathscr{N}^F$ *or* Z *contains a proper pseudovariety of abelian groups.*

Remark. Although no $Z \in \mathscr{P}\!s(\mathscr{N})\setminus\{\mathscr{N}^F\}$ is proper, $\langle Z\rangle$ is not necessarily nilpotent. In fact, by a result of **[1]**, $\langle Z\rangle \subseteq \mathscr{N}$ if and only if $[x^2 = 0]^F \cap \mathscr{N} \not\subseteq Z$.

9. Final remarks and questions. None of the mappings considered in Theorem 2.11, Corollary 2.12, Proposition 2.14 and Theorem 8.7 are lattice homomorphisms. For instance,

$$([x^2y = xy^2, x^4 = 0] \vee [x^2 = x^3]) \cap \text{Nil} \not\subseteq$$
$$([x^2y = xy^2, x^4 = 0] \cap \text{Nil}) \vee ([x^2 = x^3] \cap \text{Nil})$$

for the left side is $[x^2yz = 0]$ while the right side is $[x^2y = xy^2, x^4 = 0]$. Nevertheless, since $\mathscr{L}_0$ is distributive and the second and third components of those mappings are well-behaved in general, the restrictions to $\mathscr{L}_0$, $\mathscr{G}_0$, $\mathscr{G}_{cb} \cap \mathscr{G}_0$ and $\mathscr{P}\!s_0 = \{\mathscr{W}^F : \mathscr{W} \in \mathscr{G}_0\}$ are lattice homomorphisms.

Since $\mathscr{L}(\mathscr{A})$ is isomorphic to the lattice of positive integers under division, it follows from Theorem 2.11, Corollary 2.12 and Theorem 8.7 that the unknown order components of $\mathscr{L}$(Com), $\mathscr{G}$(Com) and $\mathscr{P}\!s$(Com) are respectively $\mathscr{L}$(Nil), $\mathscr{G}$(Nil) and $\mathscr{L}$(Nil). By Theorem 8.6, as long as the operation $Q \mapsto G(Q)$ of Section 3 is sufficiently understood, it is enough to study the order properties of the lattice $\mathscr{L}(\mathscr{N})$. For example, if $\mathscr{L}(\mathscr{N})$ is bqo, then so are $\mathscr{L}$(Nil) and $\mathscr{G}$(Nil).

Thus, the lattice $\mathscr{L}(\mathscr{N})$ plays a central role in $\mathscr{L}$(Com). Burris and Nelson [**4**] showed that $\mathscr{L}(\mathscr{N})$ does not satisfy any special lattice identities (see also [**8**]). On the other hand, $\mathscr{L}(\mathscr{N}) = \cup_n \mathscr{L}(\mathscr{N}_n)$ is the union of an ascending chain of finite lattices. Also, we showed in Theorem 4.6 that $\mathscr{L}(\mathscr{N})$ is wqo. According to Nash-Williams, every wqo that "occurs in nature" should be bqo. Corollaries 4.3, 4.5, 5.6 and 5.7 are all consistent with $\mathscr{L}(\mathscr{N})$ bqo. Thus, we conjecture that $\mathscr{L}(\mathscr{N})$ is bqo. If this turns out to be the case, we conclude that, "modulo groups", $\mathscr{L}$(Com), $\mathscr{G}$(Com) and $\mathscr{P}\!s$(Com) are also bqo.

We wish to further call the attention of algebraists to the existence of the theory of wqo. With this in mind, we give a simple application of our results. Many others of the same type could be indicated.

Following [**5**], we say that a semigroup S *divides* a semigroup T and write $S \prec T$ if S is a homomorphic image of a subsemigroup of T.

THEOREM 9.1. *Let $\mathscr{S}$ be an infinite set of finite commutative semigroups, all of which satisfy some nontrivial identity in commuting variables. Then, there exist distinct $S, T \in \mathscr{S}$ and an integer n such that $S \prec T^n$.*

Proof. Since $\mathscr{S}$ satisfies some nontrivial identity, $\langle \mathscr{S} \rangle \neq$ Com. By Corollary 4.3 and Theorem 8.2, it follows that $\mathscr{P}\!s(\langle \mathscr{S} \rangle)$ is bqo. To conclude the proof, apply this observation to the pseudovarieties generated by each $S \in \mathscr{S}$.

Results such as Theorem 9.1 lead to numerous questions concerning certain classes of commutative semigroups under various qo. For $\mathscr{K} \subseteq$ Com, let $H\mathscr{K}$, $S\mathscr{K}$ and $P_f\mathscr{K}$ denote respectively the classes of all homomorphic images, subsemigroups and finite products of members of $\mathscr{K}$. Let $\mathscr{S}$ be a pseudovariety such that $\langle \mathscr{S} \rangle \subsetneq$ Com. Then, Theorem 9.1 says that $\mathscr{S}$ is bqo under the qo $R \leqq_V T$ if $R \in \text{HSP}_f\{T\}$. What can be said for the qo $R \leqq_D T$ if $R \in \text{HS}\{T\}$ (i.e., $R \prec T$)? What about the even weaker orderings $R \leqq_Q T$ if $R \in H\{T\}$, or $R \leqq_E T$ if up to isomorphism

$R \in S\{T\}$? The search for an affirmative answer for a special class of semilattices (trees) under the embeddability order $\leqq_E$ is in part responsible for the work that led to the development of the theory of wqo. For related results see, e.g., [**10**].

Note. In a recent communication to the conference Graphs and Order (Banff, Canada, May 1984) we showed that most of the indicated relations are not wqo on even very restrictive classes of commutative semigroups. For instance, finite semilattices are not wqo by $\leqq_D$. Again in contrast, positive results are obtained fairly trivially when attention is restricted to classes of abelian groups.

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