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A COUNTEREXAMPLE IN THE THEORY OF HERMITIAN LIFTINGS

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1. Introduction

In [3], [8], and [2], it was shown that if T is an essentially Hermitian operator on l_p, $1 \leqq p < \infty$, or on $L_p[0,1]$, $1 < p < \infty$, then T is a compact perturbation of a Hermitian operator. In [1], this result was established for operators on Orlicz sequence space l_M, where $2 \notin [\alpha_M, \beta_M]$ (the associated interval for M). In that same paper, it was conjectured that this result does not in general hold if $2 \in [\alpha_M, \beta_M]$. In this paper, we show that this conjecture is correct by exhibiting an Orlicz sequence space l_M and an essentially Hermitian operator on l_M which is not a compact perturbation of a Hermitian operator.

2. Orlicz sequence spaces

We refer the reader to [1] and [9] for detailed information concerning the following facts on Orlicz sequence spaces, and to [4] and [5] for material on numerical ranges.

If M is an Orlicz function, then l_M is the corresponding Orlicz sequence space with the norm

$$\|\bar{a}\|_M = \|\{a_n\}\|_M = \inf\left\{k: \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{k}\right) \leqq 1\right\}$$

and $l_{(M)}$ is the same space with the norm

$$\|\bar{a}\|_{(M)} = \|\{a_n\}\|_{(M)} = \sup\left\{\left|\sum_{n=1}^{\infty} a_n b_n\right|: \sum_{n=1}^{\infty} N(|b_n|) \leqq 1\right\},$$

where N is the complementary Orlicz function to M. If M and N both satisfy the Δ_2-condition, then l_M^* is isometrically isomorphic to $l_{(N)}$. For each unit vector $\bar{a} = \{a_n\}$ in l_M, let $\bar{a}' = \{a_n'\}$, where $a_n' = \alpha M'(|a_n|) \operatorname{sgn} a_n$ and $\alpha = \|\{M'(|a_n|)\}\|_{(N)}^{-1}$. Then $\bar{a}'$ is the unique unit vector in $l_{(N)}$ satisfying $\langle \bar{a}, \bar{a}' \rangle = 1$. Furthermore, there is a $K > 0$ such that $1 \leqq \alpha \leqq K$ for all unit vectors $\bar{a} \in l_M$.

The spatial numerical range of an operator T on l_M is the set $V(T) = \{\langle T\bar{a}, \bar{a}' \rangle: \|\bar{a}\|_M = 1\}$ and the essential numerical range is $V_{\text{ess}}(T) = \cap\{V(T+K): K \text{ is a compact operator on } l_M\}$. T is Hermitian if $V(T) \subseteq R$ and essentially Hermitian if $V_{\text{ess}}(T) \subseteq R$. Tam has shown in [10] that unless l_M is isometrically isomorphic to Hilbert space, the only

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Hermitian operators on l_M are represented by real diagonal matrices with respect to the natural basis vectors on l_M.

3. The example

Let $M(t)$ be an Orlicz function which agrees with t^2 on $[0,\sqrt{3}/2]$, but does not agree with t^2 on $[\sqrt{3}/2,\infty]$.

Lemma 1. *Let T_n be the operator on l_M defined by the matrix*

$$\begin{pmatrix} 0 & A_n & 0 \\ A_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where A_n is an $n\times n$ block of elements, all equal to $1/n$. Then there are constants C_1 and C_2 such that $1\leqq \|T_n\|\leqq C_1$ and

$$r_i(T_n)=\sup\{|\mathrm{Im}\, z|: z\in V(T_n)\}\leqq C_2/\sqrt{n}$$

Proof. Let $\bar{a}_0=\{a_k\}$, where $a_k=1$ for $k=1,\ldots,2n$, and $a_k=0$ for $k>2n$. Then $\|\bar{a}_0\|_M\neq 0$ and $T\,\bar{a}_0=\bar{a}_0$, hence $\|T_n\|\geqq 1$. Since l_M is isomorphic to l_2, we can think of T_n as an operator on l_2, in which case its norm is 1. It follows that there is a constant C_1 such that $\|T_n\|\leqq C_1$ when considered as an operator on l_M.

To establish the second inequality, let $\bar{a}=\{a_k\}$ be an arbitrary unit vector. Then since $\sum_{k=1}^{\infty} M(|a_k|)=1$, at most one a_k satisfies $|a_k|\geqq\sqrt{3}/2$. Now if all a_k satisfy $|a_k|<\sqrt{3}/2$, then $M'(|a_k|)=2|a_k|$, and it is easy to see that $\langle T_n\bar{a},\bar{a}'\rangle=0$. Hence we assume, without loss of generality, that $|a_1|\geqq\sqrt{3}/2$, but $|a_k|<\sqrt{3}/2$ for $k>1$. We can also assume that $a_k=0$ for $k>2n$. Then we have

$$|\mathrm{Im}\langle T_n\bar{a},\bar{a}'\rangle|=$$

$$\left|\mathrm{Im}\left[\left(\frac{a_{n+1}+\cdots+a_{2n}}{n}\right)(\alpha M'(|a_1|)\,\mathrm{sgn}\,\bar{a}_1+\cdots+\alpha M'(|a_n|)\,\mathrm{sgn}\,\bar{a}_n)\right.\right.$$
$$\left.\left.+\left(\frac{a_1+\cdots+a_n}{n}\right)(\alpha M'(|a_{n+1}|)\,\mathrm{sgn}\,\bar{a}_{n+1}+\cdots+\alpha M'(|a_{2n}|)\,\mathrm{sgn}\,\bar{a}_{2n})\right]\right|$$

$$=\alpha\left|\mathrm{Im}\left[\left(\frac{a_{n+1}+\cdots+a_{2n}}{n}\right)M'(|a_1|)\,\mathrm{sgn}\,\bar{a}_1+\frac{a_1}{n}(2|a_{n+1}|\,\mathrm{sgn}\,\bar{a}_{n+1}+\cdots+2|a_{2n}|\,\mathrm{sgn}\,\bar{a}_{2n})\right]\right|$$

$$\leqq\alpha\left(\frac{|a_{n+1}|+\cdots+|a_{2n}|}{n}\right)(M'(|a_1|)+2|a_1|)$$

Now it is easy to show that the maximum value of $(x_1+\cdots+x_n)/n$ subject to $x_1^2+\cdots$

$+x_n^2 \leqq 1$ is $1/\sqrt{n}$. Also,

$$\alpha(M'(|a_1|)+2|a_1|) \leqq K[M'(M^{-1}(1))+2M^{-1}(1)] \equiv C_2.$$

Hence the above calculation shows that $|\mathrm{Im}\langle T_n\bar{a}, \bar{a}'\rangle| \leqq C_2/\sqrt{n}$ for all unit vectors $\bar{a}$. It follows that $r_i(T_n) \leqq C_2/\sqrt{n}$.

Theorem 2. *Let T be the operator on l_M defined by the block matrix*

$$\begin{pmatrix} T_{n_1} & & \\ & T_{n_2} & \\ & & \ddots \end{pmatrix}$$

where T_{n_k} is defined as in Lemma 1 and $n_k = 4^k$. Then T is an essentially Hermitian operator on l_M which is not a compact perturbation of a Hermitian operator.

Proof.

$$\text{Let } K_j = \begin{pmatrix} T_{n_1} & & & & \\ & \ddots & & & \\ & & T_{n_j} & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Then clearly K_j is compact, and

$$r_i(T-K_j) \leqq C_2 \sum_{k=j+1}^{\infty} 2^{-k} \to 0 \quad \text{as} \quad j \to \infty.$$

Hence T is essentially Hermitian. However, since l_M is not Hilbert space, the only Hermitian operators on l_M are diagonal. Clearly T is not a compact perturbation of a diagonal operator.

Remarks. The techniques of (1) and (2) can clearly be adapted to prove the following theorem:

Theorem 3. *Let $L_M[0,1]$ be the Orlicz function space on $[0,1]$ associated with the Orlicz function M. Assume M and its complementary function N both satisfy the Δ_2-condition. Define*

$$\alpha_M = \sup\left\{p\colon \sup_{0<\lambda,t} \frac{M(\lambda t)}{M(\lambda)t^p} < \infty\right\}$$

$$\beta_M = \inf\left\{p\colon \inf_{0<\lambda,t} \frac{M(\lambda t)}{M(\lambda)t^p} > 0\right\}$$

If $2 \notin [\alpha_M, \beta_M]$, *then any essentially Hermitian operator on* $L_M[0,1]$ *can be written as* $D + K$, *where* D *is a real multiplication operator* ($Df = hf$ *for some real valued* $h \in L_\infty[0,1]$) *and* K *is a compact operator.*

A good reference for Orlicz function spaces is [7].

Hence the Hermitian lifting problem has been settled in the affirmative for operators on $\bar{X}$, where $\bar{X} = l_p$, $1 \leqq p < \infty$; $\bar{X} = lM$, $2 \notin [\alpha_M, \beta_M]$; $\bar{X} = L_p[0,1]$, $1 < p < \infty$; and $\bar{X} = L_M[0,1]$, $2 \notin [\alpha_M, \beta_M]$. Counterexamples have been found in $\bar{X} = A(D)$ (see [6, Example 4.1]) and $\bar{X} = l_M$ where M is some Orlicz function with $2 \in [\alpha_M, \beta_M]$.

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