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FACTOR REPRESENTATIONS AND FACTOR STATES ON A C^*-ALGEBRA

JAMES A. SCHOEN

Let A be a C^*-algebra and H a Hilbert space of large enough (infinite at least) dimension so that every π_f, where f is a factor state on A, can be unitarily represented on H. Let Fac (A, H) denote the set of all factor representations of A on H. If π is in Fac (A, H) we call its essential subspace the smallest, closed, vector subspace K of H such that $\pi(A)$ is null on $H \ominus K$. We define $\mathrm{Fac}_\infty(A, H)$ to be the set of elements in Fac (A, H) whose essential subspace is H. Equip $\mathrm{Fac}_\infty(A, H)$ with the topology of strong pointwise convergence, i.e., $\pi_\nu \rightarrow \pi$ if $||\pi_\nu(x)\,\alpha - \pi(x)\alpha|| \rightarrow 0$ for all x in A and α in H. Bichteler [**1**, p. 90] shows that this topology is the same as that of weak convergence. What we show in this paper is that there is a continuous open surjection of $\mathrm{Fac}_\infty(A, H)$ with this strong topology onto the set of factor states, $F(A)$, of A with the weak topology. It then follows from [**3**, Proposition 11] that the map $\pi \rightarrow [\pi]$ is a continuous open surjection of $\mathrm{Fac}_\infty(A, H)$ onto the quasi-dual of A.

Let α be a unit vector in H and $F(A)$ denote the set of factor states on A. Define the map w_α from $\mathrm{Fac}_\infty(A, H)$ to $F(A)$ by

$$(w_\alpha(\pi))(x) = (\pi(x)\alpha, \alpha).$$

We note that for each π in $\mathrm{Fac}_\infty(A, H)$, $w_\alpha(\pi)$ is a state and the representation ρ induced by $w_\alpha(\pi)$ is unitarily equivalent to π restricted to cl $\{\pi(x)\alpha | x \in A\}$. Hence $w_\alpha(\pi)$ is in $F(A)$ and $\rho \in [\pi]$, where $[\pi]$ is the quasi-equivalence class of π.

Let X be a subset of $\mathrm{Fac}_\infty(A, H)$. Then $X^\sim$ will denote

$$\{\rho \in \mathrm{Fac}_\infty(A, H) | \rho \in [\pi], \pi \in X\}.$$

Let Y be a subset of $F(A)$. Then $Y^\sim$ will denote $\{g \in F(A) | \pi_g \in [\pi_f], f \in Y\}$. The following lemma is clear.

LEMMA 1. *Let B be a subset of* $\mathrm{Fac}_\infty(A, H)$ *and α and β unit vectors in H.*
(a) $w_\alpha(B)^\sim = w_\alpha(B^\sim)$.
(b) *If* $B = B^\sim$, $w_\alpha(B) = w_\beta(B)$.

Let π be in $\mathrm{Fac}_\infty(A, H)$ and $h_1 = \alpha$, where α is fixed unit vector. We define $\pi_1 = \pi | H_1$, where $H_1 = \mathrm{cl}\,\{\pi(x)h_1 | x \in A\}$. Assume that for all ordinal numbers $\nu < \nu'$ we have defined h_ν such that $h_\nu \in H \ominus \mathrm{cl} \bigcup H_\mu$, $\mu < \nu$, where $H_\mu =$

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cl $\{\pi(x)h_\mu | x \in A\}$; $H_\nu = \text{cl}\,\{\pi(x)h_\nu | x \in A\}$; and $\pi_\nu = \pi | H_\nu$. If cl $\bigcup H_\nu$, $\nu < \nu'$, is not H, pick $h_{\nu'}$ in $H \ominus \text{cl} \bigcup H_\nu$, $\nu < \nu'$. Let $\pi_{\nu'} = \pi | H_{\nu'}$, where $H_{\nu'} = \text{cl}\,\{\pi(x)h_{\nu'} | x \in A\}$. Then, by transfinite induction, we may write $\pi = \Sigma \oplus \pi_\nu$, $H = \text{cl}\,(\Sigma \oplus H_\nu)$, and for each ν, $H_\nu = \text{cl}\,\{\pi(x)h_\nu | x \in A\}$.

LEMMA 2. *Using the notation above, sets of the form*

$$\bigcap_i \bigcap_j \{\rho | \, ||\rho(x_{ij})h_{\nu_i} - \pi(x_{ij})h_{\nu_i}|| < \epsilon\}$$

is a neighborhood system for π.

Proof. It is sufficient to show that contained in a set of the form

$$\{\rho | \, ||\rho(x)h - \pi(x)h|| < \epsilon\}$$

is a set of the form $\bigcap_i \{\rho | \, ||\rho(x_i)h_{\nu_i} - \pi(x_i)h_{\nu_i}|| < \delta\}$. We may assume $x \neq 0$ and we can find a vector $\beta = \sum_i \pi(x_i)h_{\nu_i}$ such that $||h - \beta|| < \epsilon/(3\, ||x||)$. Then, by the triangle inequality, we have that

$$||\rho(x)h - \pi(x)h|| < 2\epsilon/3 + ||\rho(x)\beta - \pi(x)\beta||.$$

Since H_{ν_i} is orthogonal to H_{ν_j}, for $i \neq j$, we can find a $\delta > 0$ such that

$$t \in \bigcap_i \{\rho | \, ||\rho(x_i)h_{\nu_i} - \pi(x_i)h_{\nu_i}|| < \delta\} \cap \bigcap_i \{\rho | \, ||\rho(x\, x_i)h_{\nu_i} - \pi(x\, x_i)h_{\nu_i}|| < \delta\}$$

implies that $||t(x)\alpha - \pi(x)\alpha|| < \epsilon$.

LEMMA 3. *Let* π *be in* $\text{Fac}_\infty(A, H)$ *and*

$$O = \bigcap_i \bigcap_j \{\rho | \, ||\rho(x_{ij})h_{\nu_i} - \pi(x_{ij})h_{\nu_i}|| < \epsilon\}$$

where the h_{ν_i}*'s satisfy the conditions of* 2. *Then*

$$w_\alpha(O) = \bigcap_i w_\alpha \left(\bigcap_j \{\rho | \, ||\rho(x_{ij})h_{\nu_i} - \pi(x_{ij})h_{\nu_i}|| < \epsilon\} \right).$$

Proof. The left side is obviously contained in the right. Let

$$f \in \bigcap_i w_\alpha \left(\bigcap_j \{\rho | \, ||\rho(x_{ij})h_{\nu_i} - \pi(x_{ij})h_{\nu_i}|| < \epsilon\} \right).$$

This means that for each $i = 1, 2, \ldots, N$, there is a ρ_i such that

$$||\rho_i(x_{ij})h_{\nu_i} - \pi(x_{ij})h_{\nu_i}|| < \epsilon$$

for $j = 1, 2, \ldots, M$ and $w_\alpha(\rho_i) = f$. We may assume $h_{\nu_1} = \mu$. Let K_i be the finite dimensional Hilbert space generated by

$$\{\pi(x_{ij})h_{\nu_i} | j = 1, 2, \ldots, M\} \cup \{h_{\nu_i}\}.$$

Then the K_i's are mutually orthogonal. Let H_N be the direct sum of N copies of H. The subspace

$$H_N \ominus (K_1 \oplus K_2 \oplus \ldots \oplus K_N)$$

has dimension equal to that of H_N and so there is an isometric isomorphism of $H_N \ominus (K_1 \oplus K_2 \oplus \ldots \oplus K_N)$ onto $H \ominus (K_1 \oplus K_2 \oplus \ldots \oplus K_N)$. Thus there is an isometric isomorphism U of H_N onto H such that

$$U(\xi_1, \xi_2, \ldots, \xi_N) = \xi_1 + \xi_2 + \ldots + \xi_N \text{ for } (\xi_1, \xi_2, \ldots, \xi_N) \text{ in } \Sigma \oplus K_i.$$

Let ρ be the representation $U(\Sigma \oplus \rho_i)U^{-1}$ on H. Then ρ is in O and $w_\alpha(\rho) = f$.

The next lemma is a result that was pointed out to me by Herbert Halpern.

LEMMA 4. *Let H be a Hilbert space, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be vectors in H, and let $\epsilon > 0$ be given. There is a $\delta > 0$ such that for any vectors $\beta_1, \beta_2, \ldots, \beta_n$ in H with*

$$||\alpha_1|| = ||\beta_1|| \text{ and } |(\beta_i, \beta_j) - (\alpha_i, \alpha_j)| < \delta,$$

there is a unitary operator U on H with

$$U\beta_1 = \alpha_1 \text{ and } ||U\beta_i - \alpha_i|| < \epsilon.$$

Proof. For $n = 1$, there is a unitary U with $U\beta_1 = \alpha_1$. Now suppose that for any set $\{\beta_{1,\delta}, \beta_{2,\delta}, \ldots, \beta_{n,\delta}\}$ of vectors in H, with

$$||\beta_{1,\delta}|| = ||\alpha_1|| = 1 \text{ and } (\beta_{i,\delta}, \beta_{j,\delta}) \rightarrow (\alpha_i, \alpha_j) \text{ as } \delta \rightarrow 0,$$

the relation

$$\lim_{\delta \rightarrow 0} \inf_{U(\beta_{1,\delta}, \alpha_1)} (||U\beta_{1,\delta} - \alpha_1|| + \ldots + ||U\beta_{n,\delta} - \alpha_n||) = 0,$$

where $U(\beta_{1,\delta}, \alpha_1)$ is the set of unitary operators on H with $U\beta_{1,\delta} = \alpha_1$, holds.

Let $\{\beta_{1,\delta}, \beta_{2,\delta}, \ldots, \beta_{n+1,\delta}\}$ be vectors in H with

$$||\beta_{1,\delta}|| = ||\alpha_1|| = 1 \text{ and } (\beta_{i,\delta}, \beta_{j,\delta}) \rightarrow (\alpha_i, \alpha_j) \text{ for } 1 \leqq i \leqq n + 1.$$

We may find U_δ in $U(\beta_{1,\delta}, \alpha_1)$ such that

$$||U_\delta\beta_{1,\delta} - \alpha_1|| + \ldots + ||U_\delta\beta_{n,\delta} - \alpha_n|| \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Let H' be the space generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$, let $H'' = H \ominus H'$, let P' be the projection onto H', and P'' the projection onto H''. For $1 \leqq i \leqq n$, we have that

$$|(P'\alpha_{n+1} - P'U_\delta\beta_{n+1,\delta}, \alpha_i)| \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

Thus, $P' U_\delta\beta_{n+1,\delta} \rightarrow P' \alpha_{n+1}$ in H' since H' is finite dimensional. Also,

$$||U_\delta\beta_{n+1,\delta}||^2 = ||\beta_{n+1,\delta}||^2 \rightarrow ||\alpha_{n+1}||^2.$$

Thus,

$$||P''U_\delta\beta_{n+1,\delta}||^2 = ||\beta_{n+1,\delta}||^2 - ||P'U_\delta\beta_{n+1,\delta}||^2$$

converges to

$$||\alpha_{n+1}||^2 - ||P'\alpha_{n+1}||^2 = ||P''\alpha_{n+1}||^2.$$

Then there is a unitary operator V_δ on H such that V_δ is the identity on H' and

$$V_\delta P'' U_\delta \beta_{n+1,\delta} \to P''\alpha_{n+1}.$$

Thus, for $1 \leqq i \leqq n$,

$$||P'' U_\delta \beta_{i,\delta}|| \to ||P''\alpha_i|| = 0$$

and so

$$||V_\delta U_\delta \beta_{i,\delta} - U_\delta \beta_{i,\delta}|| \to 0.$$

This means that $||V_\delta U_\delta \beta_{i,\delta} - \alpha_i||$ converges to zero. Also, $V_\delta U_\delta \beta_{1,\delta} = V_\delta \alpha_1 = \alpha_1$. Furthermore,

$$||V_\delta U_\delta \beta_{n+1,\delta} - \alpha_{n+1}||^2 = ||P' U_\delta \beta_{n+1,\delta} - P'\alpha_{n+1}||^2 + ||V_\delta P'' U_\delta \beta_{n+1,\delta} - P''\alpha_{n+1}||^2 \to 0 \text{ as } \delta \to 0.$$

Hence, our result follows.

We are now ready for our major result.

THEOREM 5. *Let A be a C^*-algebra, H be a Hilbert space of large enough dimension (at least infinite) so that each factor representation induced by a factor state on A can be unitarily represented on H, and α be a fixed unit vector in H. Then the map, w_α, is a continuous open surjection from* $\mathrm{Fac}_\infty(A, H)$ *onto* $F(A)$.

Proof. We first show that w_α is onto. Let f be in $F(A)$, π_f be the factor representation defined by f on the Hilbert space H_f, and h_f be in H_f such that $f(x) = (\pi_f(x)h_f, h_f)$. Let N be the cardinality of a maximal set of orthonormal vectors in H. We form the Hilbert space K by taking the direct sum of H_f with itself N times. Let $\{\beta_\nu\}_{\nu \in I}$ (resp. $\{\gamma_\nu\}_{\nu \in I}$) be a maximal set of orthonormal vectors in K (resp. H) such that $\beta_1 = h_f \oplus O \oplus O \oplus \ldots$ (resp. $\gamma_1 = \alpha$). We define an isometric isomorphism U of K onto H by $U\beta_\nu = \gamma_\nu$ for each $\nu \in I$. Let π' be the representation of A on K formed by taking $\Sigma \oplus \pi_f$. Let $\pi = U\pi' U^{-1}$. Then π is in $\mathrm{Fac}_\infty(A, H)$ and $w_\alpha(\pi) = f$. Hence w_α is onto.

Let $f_0 \in F(A)$ and $\pi_0 \in \mathrm{Fac}_\infty(A, H)$ such that $w_\alpha(\pi_0) = f_0$. Then

$$w_\alpha^{-1}(\{f \in F(A) | |f(x) - f_0(x)| < \epsilon\}) = \{\pi | |(\pi(x)\alpha, \alpha) - (\pi_0(x)\alpha, \alpha)| < \epsilon\}.$$

Hence, w_α is continuous.

Our final task is to show w_α is open. By 2 and 3, we need only show that sets of the form

$$w_\alpha(\bigcap \{\rho| ||\rho(x_i)\alpha - \pi(x_i)\alpha|| < \epsilon\})$$

and of the form

$$w_\alpha(\bigcap \{\rho| ||\rho(x_i)\beta - \pi(x_i)\beta|| < \epsilon\}),$$

where α is orthogonal to cl $\{\pi(x)\beta | x \in A\}$, are open in $F(A)$. We treat the

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latter case first. Let

$$O = \bigcap \{\rho | \, ||\rho(x_i)\beta - \pi(x_i)\beta|| < \epsilon\}, \rho \in O,$$

and $f \in F(A)$ such that π_f is quasi-equivalent to ρ. Let $\rho_0 = \pi_f \oplus \rho$ on $H_f \oplus H$. Then there is an isometric isomorphism U from $H_f \oplus H$ onto H such that $U(h_f \oplus O) = \alpha$ and $U(O \oplus \beta) = \beta$. Then $U\rho_0 U^{-1} \in O$ and $w_\alpha(U\rho_0 U^{-1}) = f$. Hence, $w_\alpha(O)$ is saturated. By 1, $w_\alpha(O) = w_\alpha(O^\sim) = w_\beta(O^\sim)$. By [**1**, Proposition 4] and [**3**, Proposition 11], it now follows that $w_\alpha(O)$ is open in $F(A)$. (We now assume

$$O = \bigcap \{\rho | \, ||\rho(x_i)\alpha - \pi(x_i)\alpha|| < \epsilon\}$$

and observe that it is sufficient to replace O by an open set O' containing π, such that $w_\alpha(O') = w_\alpha(O' \cap O)$. We construct O'. By 4, there is a $\delta > 0$ and a unitary operator U on H such that $U\alpha = \alpha$ and if t is in

$$O' = \left(\bigcap_i \{\rho | \, |(\rho(x_i)\alpha, \alpha) - (\pi(x_i)\alpha, \alpha)| < \delta\}\right) \cap \left(\bigcap_i \bigcap_j \{\rho | \, |(\rho(x_j^* x_i)\alpha, \alpha) - (\pi(x_j^* x_i)\alpha, \alpha)| < \delta\}\right),$$

then UtU^{-1} is in O. Thus, O may be replaced by O' and $w_\alpha(O')$ is open in $F(A)$. Our result then follows.

COROLLARY 6. *Let A be a C^*-algebra. Then the map $\pi \to [\pi]$ of* $\text{Fac}_\infty(A, H)$ *onto the quasi-dual of A is continuous and open.*

Proof. This map is the composition of the maps defined in 5 and [**3**, Proposition 11].

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University of Wisconsin,
LaCrosse, Wisconsin