Canad. Math. Bull. Vol. 29 (3), 1986

INJECTIVE APPROXIMATIONS

BY
K. VARADARAJAN*

ABSTRACT. Let C be a cochain complex satisfying the condition that $\overline{\lim}$ inj.dim. $H^q(C) < \infty$. Then we prove that C admits an injective approximation.

Introduction. Let R be an associative ring with $1 \neq 0$ and R-mod the category of unitary left R-modules. We will be considering cochain complexes in R-mod. If C is a cochain complex, we will denote the coboundary maps in C by $\delta^q_C: C^q \to C^{q+1}$. When there is no possibility of confusion we just write δ^q instead of δ^q_C. A cochain complex I in which each I^q is injective will be referred to as an injective complex. A cochain complex C is said to be bounded below if there exists some $k \in Z$ with $C^q = 0$ for $q < k$. By an injective approximation to a cochain complex C we mean an injective cochain complex I together with a monomorphism $\tau: C \to I$ of cochain complexes satisfying $H^*(\tau): H^*(C) \simeq H^*(I)$. A cochain complex C is said to be positive if $C^q = 0$ for $q < 0$. It is well-known that if C is a cochain complex which is bounded below (resp. positive) then there exists an injective approximation $\tau: C \to I$ to C with I bounded below (resp. positive) [4, page 42]. It is also known that if R is of finite global dimension and C an arbitrary cochain complex over R then C admits an injective approximation [3].

For any $M \in R$-mod, let i.d.M denote the injective dimension of M. Given a cochain complex C over R, let $e^q(C) = \text{i.d.}\, H^q(C)$. Let $\mathscr{C}$ denote the class of cochain complexes C satisfying the condition that $\overline{\lim}_{q \to -\infty} e^q(C) < \infty$. (Here $\overline{\lim}$ denotes lim sup). For any $C \in \mathscr{C}$, let $e(C) = \overline{\lim}_{q \to -\infty} e^q(C)$. Then $e(C)$ is an integer ≥ -1. The object of the present paper is to prove the following.

THEOREM. *Any cochain complex C in $\mathscr{C}$ admits an injective approximation.*

1. **Preliminary results**. In this section we present some preliminary results needed to prove the above Theorem. A cochain map $\tau: C \to D$ with $H^*(\tau): H^*(C) \simeq H^*(D)$ will be referred to as a quasi-isomorphism.

Received by the editors September 20, 1985.

* Research done while the author was spending part of his sabbatical leave at the Forschungsinstitut für Mathematik, ETH, Zurich. The author was partially supported by NSERC grant A 8225 while this research was being carried out.

Key words: Cochain Complexes, injective approximations.
1980 AMS Classification: 18G35, 55U15.

Let $f: C \to D$ be a cochain map. By lowering the indices and changing signs we could regard C and D as chain complexes and f as a chain map. Let C_f be the mapping cone of f. We could convert C_f into a cochain complex by raising the indices and changing signs. The cochain complex C_f thus got will be referred to as the mapping cone of the cochain map f. More explicitly $C_f^q = D^q \oplus C^{q+1}$ for each q and $\delta_f^q(a, x) = (f(x) + \delta_D^q(a), -\delta_C^{q+1}(x))$ is the coboundary map in C_f. There is an associated exact sequence $0 \to D \xrightarrow{i} C_f \xrightarrow{\eta} \Gamma C \to 0$ of cochain complexes, where ΓC is the cochain complex defined by $(\Gamma C)^q = C^{q+1}$ for each q and $\delta_{\Gamma C}^q = -\delta_C^{q+1}$. The associated cohomology exact sequence is of the form

$$\begin{array}{ccccccccc} \to H^q(D) \to & H^q(C_f) & \to & H^q(\Gamma C) & \xrightarrow{\delta} & H^{q+1}(D) & \to & H^{q+1}(C_f) \to \\ & & & \| & \nearrow & H^{q+1}(f) & & \\ & & & H^{q+1}(C) & & & & \end{array}$$

For later use, we define Γ_C^{-1} to be the cochain complex with $(\Gamma_C^{-1})^q = C^{q-1}$ for each q and $\delta_{\Gamma_C^{-1}}^q = -\delta_C^{q-1}$.

PROPOSITION 1.1. *Let C be any cochain complex. Then there exists a monomorphism $f{:}C \to I$ of cochain complexes with I injective $H^*(f){:}H^*(C) \to H^*(I)$ a split monomorphism and $e^q(C_f) \leq$ Max$(0, e^{q-1}(C) - 2)$.*

PROOF. Let $0 \to C^q/B^q(C) \xrightarrow{\alpha^q} J^q$ be exact with J^q injective. Let $K^q = \alpha^q(Z^q(C)/B^q(C))$. Then $\alpha^q|H^q(C) = Z^q(C)/B^q(C)$ is an isomorphism of $H^q(C)$ onto K^q. Moreover α^q induces a monomorphism $\bar{\alpha}^q{:}C^q/Z^q(C) \to J^q/K^q$. Let $0 \to J^q/K^q \xrightarrow{\beta^q} T^{q+1}$ be exact with T^{q+1} injective. Write $\gamma^q{:}J^q \to T^{q+1}$ for the map $\gamma^q(a) = \beta^q(a + K^q)$ for any $a \in J^q$. Since T^{q+1} is injective, there exists a map $g^{q+1}{:}C^{q+1} \to T^{q+1}$ with

$$\begin{array}{ccc} 0 \to C^q/Z^q(c) & \xrightarrow{\delta_C^{-q}} & C^{q+1} \\ \Big\downarrow \beta^q\alpha^{-q} & \swarrow g^{q+1} & \\ T^{q+1} & & \end{array}$$

Diagram 1

commutative.

We define a cochain complex I as follows: $I^q = J^q \oplus T^q$ for every $q \in Z$. The map $\delta_I^q: I^q \to I^{q+1}$ is given by $\delta_I^q(x, y) = (0, \gamma^q(x))$ for every $x \in J^q$, $y \in T^q$. It is clear that (I, δ_I) is an injective cochain complex. Define f^q: $C^q \to I^q$ by $f^q(c) = (\alpha^q(c + B^q(C)), g^q(c))$ for any $c \in C^q$. Then $\delta_I^q f^q(c) = (0, \gamma^q\alpha^q(c + B^q(C))) = (0, \beta^q\alpha^{-q}(c + Z^q(C))) = (0, g^{q+1}\delta^{-q}(c + Z^q(C))) = (0, g^{q+1}\delta_C^q(c))$ and $f^{q+1}\delta_C^q(c) = (\alpha^{q+1}(\delta_C^q(c) + B^{q+1}(C)), g^{q+1}(\delta_C^q(c))) = (0, g^{q+1}\delta_C^q(c))$ since $\delta_C^q(c) + B^{q+1}(C) = 0$ in $C^{q+1}/B^{q+1}(C)$. Thus the maps f^q: $C^q \to I^q$ defined above yield a cochain map f: $C \to I$. Also, $f^q(c) = 0 \Rightarrow \alpha^q(c + B^q(C)) = 0$ and $g^q(c) = 0$. Since $\alpha^q{:}C^q/B^q(C) \to J^q$

https://doi.org/10.4153/CMB-1986-059-6 Published online by Cambridge University Press

is monic, we get $c \in B^q(C)$. Let $c = \delta_C^{q-1}y$ with $y \in C^{q-1}$. Then $0 = g^q(c) = g^q(\delta_C^{q-1}y) = \beta^{q-1}\alpha^{q-1}(y + Z^{q-1}(C))$ from diagram 1. Since β^{q-1} is monic, we get $\alpha^{-q-1}(y + Z^{q-1}(C)) = 0$. This means $\alpha^{q-1}(y + \beta^{q-1}(C)) \in K^{q-1}$. This in turn implies $y \in Z^{q-1}(C)$. Hence $c = \delta_C^{q-1}y = 0$. Thus $f^q(c) = 0 \Rightarrow c = 0$. This proves that $f: C \to I$ is a monomorphism.

From the definition of δ_I^q we see immediately that $Z^q(I) = \text{Ker}\,\gamma^q \oplus T^q = K^q \oplus T^q$ and that $B^q(I) = 0 \oplus \text{Im}\,\gamma^{q-1}$. Hence $H^q(I) = K^q \oplus (T^q/\text{Im}\,\gamma^{q-1})$. If $p^q:H^q(I) \to K^q$ denotes the projection onto K^q, it is clear that $p^q \circ H^q(f):H^q(C) \to K^q$ is the same as the isomorphism $\alpha^q:H^q(C) \simeq K^q$. This proves that $H^*(f): H^*(C) \to H^*(I)$ is a split injection, with coker $H^q(f) \simeq T^q/\text{Im}\,\gamma^{q-1} = T^q/\text{Im}\,\beta^{q-1}$ for each q. From the exactness of $0 \to K^{q-1} \to J^{q-1} \xrightarrow{\gamma^{q-1}} T^q \to T^q/\text{Im}\,\gamma^{q-1} \to 0$ with J^{q+1} and T^q injective, we get inj dimn Coker $H^q(f) = \text{inj dim}\, T^q/\text{Im}\,\gamma^{q-1} \leq \text{Max}\,(0, \text{inj dim}\, K^{q-1} - 2)$. But $H^{q-1}(C) \simeq K^{q-1}$. Hence inj dim Coker $H^q(f) \leq \text{Max}\,(0, \text{inj dim}\, H^{q-1}(C) - 2) = \text{Max}\,(0, e^{q-1}(C) - 2)$. From the exactness of

$$
\begin{array}{ccccccccc}
\to H^{q-1}(I) \to H^{q-1}(C_f) \to & H^{q-1}(\Gamma C) & \xrightarrow{\delta} & H^q(I) \to H^q(C_f) \to \\
& \| & \nearrow H^q(f) & \\
& H^q(C) & &
\end{array}
$$

using the fact that all the $H^j(f)$ are monomorphisms, we see that $H^q(C_f) \simeq \text{Coker}\, H^q(f)$ for every q. Hence inj dim $H^q(C_f) \leq \text{Max}\,(0, e^{q-1}(C) - 2)$ or $e^q(C_f) \leq \text{Max}\,(0, e^{q-1}(C) - 2)$. This completes the proof of proposition 1.1 □

PROPOSITION 1.2. *Let C be a cochain complex satisfying the condition that $e^q(C) \leq 1$ for all q. Then C admits an injective approximation.*

PROOF. If $e^q(C) \leq 1$ for all q, from $H^q(C) \simeq K^q$ and J^q injective (in the proof of proposition 1.1) we see that J^q/K^q is injective for each q. Hence we can choose $T^{q+1} = J^q/K^q$ and $\beta^q = \text{Id}\, J^q/K^q$ for each q. For the map $f: C \to I$ constructed as in proposition 1.1, we have $H^*(f):H^*(C) \to H^*(I)$ a split mono with coker zero. Thus $f:C \to I$ is an injective approximation to C. □

PROPOSITION 1.3. *Let $\varphi: C \to I$ be a cochain map with I injective. Suppose C_φ admits an injective approximation. Then C also admits an injective approximation.*

PROOF. Let $\tau:C_\varphi \to J$ be an injective approximation to C_φ. Let $i:I \to C_\varphi$ denote the inclusion and $\eta:C_\varphi \to \Gamma C$ the quotient map. Let $\mu = \tau \circ i: I \to J$. Then μ is an inclusion of injective cochain complexes. Let $L = \text{coker}\,\mu$ and $\epsilon:J \to L$ the projection. We know that $0 \to I \xrightarrow{i} C_\varphi \xrightarrow{\eta} \Gamma C \to 0$ is exact and that

$$
\begin{array}{ccccc}
0 \to & I & \xrightarrow{i} & C_\varphi \\
& \| & & \downarrow \\
0 \to & I & \xrightarrow[\mu]{} & J
\end{array}
$$

is commutative. Hence τ yields a cochain map $\bar{\tau}: \Gamma C \to L$ making

$$
\begin{array}{ccccccc}
0 \to I & \xrightarrow{i} & C_\varphi & \xrightarrow{\eta} & \Gamma C & \to 0 \\
& & \downarrow \tau & & \downarrow \bar{\tau} & \\
0 \to I & \xrightarrow[\mu]{} & J & \xrightarrow[\epsilon]{} & L & \to 0
\end{array}
$$

Diagram 2

https://doi.org/10.4153/CMB-1986-059-6 Published online by Cambridge University Press

commutative. Since I^q is injective for each q, the lower exact sequence splits for each q. Since J^q is injective for q, it follows that L^q is injective for each q. From diagram 2, using the fact that τ is monic, it is easily checked that $\bar{\tau}$ is monic. Five lemma and exact cohomology sequences show that $H^*(\bar{\tau}):H^*(\Gamma C) \to H^*(L)$ is an isomorphism. Hence $C \overset{\Gamma^{-1}\bar{\tau}}{\to} \Gamma^{-1}L$ is an injective approximation to C. □

Proposition 1.4. *The following are equivalent for a cochain complex. (1) C admits an injective approximation (2) There exists a quasi-isomorphism $\varphi:C \to I$ with I injective.*

Proof. The implication (1) $\to$ (2) is trivial. (2) $\to$ (1): Assume (2). Then C_φ is a cochain complex satisfying $H^*(C_\varphi) = 0$. From proposition 1.2 we see that C_φ admits an injective approximation. Now, applying proposition 1.3 we see that C itself admits an injective approximation. □

Definition 1.5. *A cochain complex C is said to be cohomologically bounded below if there exists a $k \in Z$ with $H^q(C) = 0$ for $q < k$.*

Lemma 1.6. *Let C be a cochain complex which is cohomologically bounded below. Then there exists a quasi-isomorphism $\theta:C \to D$ with D bounded below.*

Proof. Let $H^q(C) = 0$ for $q < k$. The map $\delta^{k-1}:C^{k-1} \to C^k$ induces a monomorphism $\bar{\delta}^{k-1}:C^{k-1}/Z^{k-1}(C) \to C^k$. Let $\epsilon^{k-1}:C^{k-1} \to C^{k-1}/Z^{k-1}(C)$ denote the canonical quotient map. Let $\square(C)$ be the cochain complex defined as follows. $\square(C)^q = C^q$ for $q \geq k$, $\square(C)^{k-1} = C^{k-1}/Z^{k-1}(C)$ and $\square(C)^q = 0$ for $q < k - 1$. Let $\delta^q_{\square(C)} = \delta^q_C$ for $q \geq k$, $\delta^{k-1}_{\square(C)} = \bar{\delta}^{k-1}$. Let $\theta:C \to \square(C)$ be defined by, $\theta^q = \mathrm{Id}_{C^q}$ for $q \geq k$, $\theta^{k-1} = \epsilon^{k-1}$. Then $\theta:C \to \square(C)$ is a quasi-isomorphism. Clearly $\square(C)$ is a cochain complex which is bounded below. □

Corollary 1.7. *Any cochain complex C which is cohomologically bounded below admits an injective approximation.*

Proof. Immediate consequence of 1.4, 1.6 and the known fact [4] that any bounded cochain complex admits an injective approximation. □

2. **Proof of the main theorem**.

Lemma 2.1. *Let C be a cochain complex belonging to the class $\mathscr{C}$. Then there exists a monomorphism $f:C \to I$ of cochain complexes with I injective, $C_f \in \mathscr{C}$ and $e(C_f) \leq$* Max(0, *$e(C) - 2$*).

Proof. This is an immediate consequence of proposition 1.1. □

Proposition 2.2. *Let C be a cochain complex belonging to $\mathscr{C}$. Suppose $e(C) \leq 1$. Then there exists a monomorphism f: $C \to I$ with I injective and C_f cohomologically bounded below.*

Proof. Since $\lim_{q \to -\infty} e^q(C) \leq 1$, there exists an integer $k \in Z$ with $e^q(C) \leq 1$ for all $q < k$. In the proof of proposition 1.1, we have $H^q(C) \simeq K^q$ and J^q injective with

https://doi.org/10.4153/CMB-1986-059-6 Published online by Cambridge University Press

$J^q \supset K^q$. From $e^q(C) \leq 1$ we see that J^q/K^q is injective for $q < k$. Hence we can choose $T^{q+1} = J^q/K^q$ and $\beta^q = \mathrm{Id}_{J^q/K^q}$ for $q < k$. Then for C_f with $f:C \to I$ constructed in the proof of proposition 1.1, we have $H^q(C_f) \simeq \mathrm{Coker}\, H^q(f) \simeq T^q/\mathrm{Im}\, \beta^{q-1} = 0$ for $q < k + 1$. Thus C_f is cohomologically bounded below. □

COROLLARY 2.3. *Let $C \in \mathscr{C}$. Suppose $e(C) \leq 1$. Then C has an injective approximation.*

PROOF. By proposition 2.2 there exists a cochain map $f:C \to I$ with I injective and C_f cohomologically bounded below. By Corollary 1.7, C_f has an injective approximation. From proposition 1.3 it follows that C itself has an injective approximation. □

THEOREM 2.4. *Any cochain complex C in $\mathscr{C}$ admits an injective approximation.*

PROOF. By induction on $e(C)$. If $e(C) \leq 1$, corollary 2.3 implies that C has an injective approximation. Suppose $e(C) > 1$. Then from lemma 2.1 we get a cochain map $f:C \to I$ with I injective, $C_f \in \mathscr{C}$ and $e(C_f) \leq e(C) - 2$. By the inductive assumption C_f has an injective approximation. Now proposition 1.3 implies that C itself has an injective approximation. □

REMARK 2.5. In [7] we have shown that any chain complex over any ring whatsoever admits a free approximation. The proof combines the techniques used in [5] and [6] with a result of *I*. Berstein [1] on the projective dimension of countable direct limits. Also that homology commutes with direct limits plays a role in the proof given in [7]. There are no dual results dealing with injective dimension of countable inverse limits. We do not know whether every cochain complex over any ring R whatsoever admits an injective approximation.

Finally I wish to thank Professors Eckmann and Mislin for giving me an opportunity to spend part of my sabbatical leave at the Forschungsinstitut für Mathematik, ETH.

REFERENCES

1. I. Berstein, *On the dimension of modules and algebras IX:* Nagoya Math. J. **13** (1958) pp. 83–84.
2. H. Cartan and S. Eilenberg, *Homological Algebra;* Princeton University Press, 1956.
3. A. Dold, *Zur Homotopietheorie der Kettencomplexe;* Math. Annalen **140** (1960) pp. 278–298.
4. R. Hartshorne, *Residues and Duality:* Springer Lecture Notes **20** (1966).
5. K. Varadarajan, *Projective approximation;* Canadian Journal of Math. **36** (1984) pp. 178–192.
6. K. Varadarajan, *Projective and free approximations;* Contemporary Mathematics, **37** (1985) pp. 153–161.
7. K. Varadarajan, *Free approximations:* To appear in "Questiones Mathematicae".

THE UNIVERSITY OF CALGARY
CALGARY, ALBERTA, CANADA T2N 1N4

FORSCHUNGSINSTITUT FÜR MATHEMATIK
ETH, ZURICH.