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METRISATION OF MOORE SPACES AND ABSTRACT TOPOLOGICAL MANIFOLDS

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The problem of metrising abstract topological spaces constitutes one of the major themes of topology. Since, for each new significant class of topological spaces this question arises, the problem is always current. One of the famous metrisation problems is the Normal Moore Space Conjecture. It is known from relatively recent work that one must add special conditions in order to be able to get affirmative results for this problem. In this paper we establish such special conditions. Since these conditions are characterised by local simplicity and global coherence they are referred to in this paper generically as "abstract topological manifolds." In particular we establish a generalisation of a classical development of Bing, giving a proof which is complete in itself, not depending on the result or arguments of Bing. In addition we show that the spaces recently developed by Collins designated as "$\mathcal{W}$ satisfying open $G(N)$" are metrisable if they are locally separable and locally connected and regular. Finally, we establish a new necessary and sufficient condition for spaces to be metrisable.

1. INTRODUCTION

Two important themes of topology are the interplay of local and global properties of a topological space, and the metrisation of abstract topological spaces. In this paper we develop results concerning both of these themes.

First we give definitions of terms that will be used in this development.

A topological space X is said to have a *point-countable basis* if X has a basis $\mathcal{B}$ such that each point of X is an element of at most countably many elements of $\mathcal{B}$.

A topological space X is said to be *screenable* if for each open covering $\mathcal{G}$ of X there exists a countable sequence $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots$ of collections of open sets such that

(a) the members of $\mathcal{H}_i$ are mutually exclusive, $i = 1, 2, 3\ldots$,

(b) the union $\mathcal{H}$ of the collection of sets $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots$ is a covering of X,

and

(c) each member of $\mathcal{H}$ is a subset of at least one member of $\mathcal{G}$.

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A topological space X is said to have the property $\mathcal{W}$ *satisfying open* $G(N)$ if there exists a collection of open sets $\mathcal{W}$ having the following properties:

(a) for each point p of X there is an associated countable collection $W(1, p)$, $W(2, p)$, $W(3, p)$, $\dots$ of members of $\mathcal{W}$ such that each $W(n, p)$ contains p, $n = 1, 2, 3, \dots$,

(b) if U is an arbitrary open set and x is any point of U then there is an open set V such that $x \in V \subset U$ and furthermore if y is any point of V then there exists an integer k such that $x \in W(k, y) \subset U$.

This last concept, "$\mathcal{W}$ satisfying open $G(N)$," was developed recently by P.J. Collins and associates at Oxford, and others in general topology.

A topological space X is *developable* if there exists a sequence of open coverings $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \dots$ of X such that for each point $x \in X$ and each open subset U of X such that $x \in U$ there is some positive integer n such that if $G \in \mathcal{G}_n$ and $x \in G$ then $G \subset U$.

A topological space X is a *Moore space* if it is regular and developable.

A topological space X is *paracompact* if for every open covering $\mathcal{G}$ of X there is a locally finite open refinement $\mathcal{H}$ of $\mathcal{G}$ such that $\mathcal{H}$ also covers X.

We shall use the notation $St(p, \mathcal{G})$ to denote the union of all members of $\mathcal{G}$ that contain the point p. In words, $St(p, \mathcal{G})$ is referred to as "the star of the point p in the collection of sets $\mathcal{G}$."

2. Metrisation of normal Moore spaces

One of the famous questions of topology is the question of whether or not every normal Moore space can be metrised. It is known that in order to obtain affirmative results concerning the question of under what conditions a normal Moore space can be metrised, further conditions, in addition to the Zermelo-Frankel axioms with the Axiom of Choice, must be part of the hypothesis. In this section we establish theorems involving such conditions.

The first theorem, which follows, establishes a result whose proof generalises a classical development of Bing [**1**, Theorem 5].

Theorem 2.1. *Let X be a Moore space which has the additional properties of local separability and (global) screenability. Then X is a metrisable normal Moore space.*

Proof: The proof uses the Axiom of Choice in the form of well-ordering, and transfinite induction. Let X be well ordered.

Since X is locally separable there exists an open covering $\mathcal{G}$ of X such that each member of $\mathcal{G}$ is separable. Now X is screenable. Hence there exists a sequence

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$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \ldots$ of collections of mutually exclusive open sets such that their union $\mathcal{H}$ covers X and is a refinement of $\mathcal{G}$. Thus, since every subset of $\mathcal{H}$ is an open subset of a member of $\mathcal{G}$, then every member of $\mathcal{H}$ is a separable subset of X.

Let p_0 be the first point of X in the well-ordering of the points of X. Since the members of $\mathcal{H}_n$ are mutually exclusive collections of subsets of X, for each n; $n = 1, 2, 3, \ldots$, the point p_0 is a point of at most one member of each $\mathcal{H}_n$. Hence, p_0 is a member of at most countably many members of $\mathcal{H}$. Since each member of $\mathcal{H}$ is separable, it follows that the star of p_0 in $\mathcal{H}$ is also separable.

Let $D_0(p_0)$ be a countable dense subset of $St(p_0, \mathcal{H})$. We define a related set $O_1(p_0)$ as follows: $O_1(p_0) = \bigcup\{St(q, \mathcal{H}) \mid q \in D_0(p_0)\}$. Since each of the sets $St(q, \mathcal{H})$ is separable and $D_0(p_0)$ is countable then $O_1(p_0)$ is also separable. Inductively we define $O_n(p_0) = \bigcup\{St(q, \mathcal{H}) \mid q \in D_{n-1}(p_0)\}$ where $D_n(p_0)$ is a countable dense subset of $O_n(p_0)$ and $D_{n-1}(p_0) \subset D_n(p_0)$, $n = 1, 2, 3, \ldots$. Thus $O_1(p_0), O_2(p_0), O_3(p_0), \ldots$ is an expanding sequence of separable open sets, each of which contains p_0, with respective countable sense sets $D_1(p_0), D_2(P_0), D_3(p_0), \ldots$ which also form an expanding sequence. Let V_0 denote the union of the sets $O_1(p_0), O_2(p_0), O_3(p_0), \ldots$ and E_0 denote the union of the sets $D_1(p_0), D_2(p_0), D_3(p_0), \ldots$. We proceed by transfinite induction to define a possibly uncountable family of sets $\{V_\alpha\}$ as follows:

Assume V_α has been defined for every subscript α less than some ordinal subscript $\beta > 0$, and let p_β be the first element (if any) of the set $X - \bigcup\{V_\alpha \mid \alpha < \beta\}$ relative to the well-ordering that was assigned originally to X. We construct sets $O_1(p_\beta), O_2(p_\beta), O_3(p_\beta), \ldots$ relative to p_β in the same manner as $O_1(p_0), O_2(p_0), O_3(p_0), \ldots$ were constructed relative to p_0. Then we define $V_\beta = \bigcup\{O_i(p_\beta) \mid i = 1, 2, 3, \ldots\}$. The resulting collection $\{V_\alpha\}$ is a collection of sets which covers X. Moreover, we shall show that the collection $\{V_\alpha\}$ is discrete in the sense that no member of this collection intersects the closure of the union of the other members of this collection.

To prove that the collection $\{V_\alpha\}$ is discrete we show first that each member of this collection is closed. Let V_β, as defined above, be a representative of the collection $\{V_\alpha\}$, and denote by z a limit point of V_β. Let $H(z)$ be any member of the covering $\mathcal{H}$ of X such that $z \in H(z)$. Since $H(z)$ is open and z is a limit point of V_β then $H(z)$ must intersect one of the sets $O_\beta(p_\beta)$ of the collection $\{O_1(p_\beta), O_2(p_\beta), O_3(p_\beta), \ldots\}$ whose union is V_β. But $O_n(p_\beta)$ is also open. Thus $H(z) \cap O_n(p_\beta)$ is a non-empty open set which therefore most contain a point q of the associated countable dense subset $D_n(p_\beta)$ of $O_n(p_\beta)$. Hence, $z \in St(q, \mathcal{H}) \subset O_{n+1}(p_\beta) \subset V_\beta$, and consequently V_β is closed. Next we show that no point of V_β is a point or a limit point of the union of the other members of the collection $\{V_\alpha\}$. The sets of the collection $\{V_\alpha\}$ are both open and closed, each being a union of open sets and each having been shown to be closed.

Hence, it is sufficient to show that the members of the collection $\{V_\alpha\}$ are disjoint. Suppose there is a point of V_β which is also an element of V_α for some $\alpha < \beta$. The point p_β we chose as the first point in our construction of V_β is by definition not in V_α. We may choose $O_n(p_\beta)$ such that n is the first integer for which $O_n(p_\beta)$ intersects V_α. Note that since $O_{n-1}(p_\beta)$ (or p_β if $n = 0$) does not intersect V_α, $D_{n-1}(p_\beta)$ does not intersect V_α (if $n = 0$, then let $D_{n-1}(p_\beta)$ denote p_β). Then let $w \in O_n(p_\beta) \cap V_\alpha$. Then w is an element of some open set $H(w) \in \mathcal{H}$ such that $H(w)$ contains some point q in $D_{n-1}(p_\beta)$ by the construction of $O_n(p_\beta)$. Since $H(w)$ is open and intersects V_α, $H(w)$ contains a point of $D_m(p_\alpha)$ for some integer m. By construction, $H(w)$ is then contained in V_α. But this contradicts the fact that $D_{n-1}(p_\beta)$ does not intersect V_α because $q \in H(w)$.

Now we use the fact that a Moore space is developable in showing that each member V_α of the collection $\{V_\alpha\}$ has a countable basis. To do this note that if $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$ is a development of X and if $\mathcal{G}_n \cap V_\alpha$ denotes the open covering of V_α which consists of the intersections of $\mathcal{G}_n$ with V_α then $\{\mathcal{G}_n \cap V_\alpha \mid n = 1, 2, 3, \ldots\}$ is a development of V_α. Likewise, using similar notation $\{\mathcal{H}_n \cap V_\alpha \mid n = 1, 2, 3, \ldots\}$ is a screening of V_α. Since V_α is separable there cannot exist uncountably many disjoint open sets in V_α. Hence, for each positive integer n, $\{\mathcal{H}_n \cap V_\alpha\}$ is a countable collection. Consequently, $\bigcup\{\mathcal{H}_n \cap V_\alpha \mid n = 1, 2, 3, \ldots\}$ is a countable open cover of V_α. Hence, since this is true for any screening of any open cover of V_α, for each covering $\mathcal{G}_n \cap V_\alpha$ of V_α we may choose a countable open cover of V_α which refines $\mathcal{G}_n \cap V_\alpha$. The union of all these countable open refining coverings is a countable basis for V_α. Therefore, V_α is completely separable. Also, V_α inherits regularity from X. We conclude, by the Urysohn metrisation theorem, that V_α has a metric d_α which is consistent with the topology that V_α inherits from X. Also, without loss of generality, we may choose d_α such that $d_\alpha(x, y) < 1$ for all $x, y \in V_\alpha$. Finally, since the collection $\{V_\alpha\}$ is discrete, we can create a metric for the whole of X by defining $d(x, y) = d_\alpha(x, y)$ if x and y are elements of the same V_α and $d(x, y) = 1$ otherwise. We conclude that X is metrisable. ∎

The next theorem provides a metrisation result whose proof does not use the Axiom of Choice. Instead the result uses the structuring mechanism known as "$\mathcal{W}$ satisfying open $G(N)$" developed by Collins and associates [**3**]. The theorem we give does not require the monotonicity restrictions of Collins and Roscoe [**2**]. For convenience of formulation we introduce first the definition of an additional term.

DEFINITION: A *Moore pseudo-manifold* is a Moore space which is locally separable and (globally) connected.

THEOREM 2.2. *A Moore pseudo-manifold is metrisable if it has the property of $\mathcal{W}$ satisfying open $G(N)$.*

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PROOF: Let X be a normal Moore pseudo-manifold having the property $\mathcal{W}$ satisfying open $G(N)$. For each point x of X let $\{W(n, x)\}$ be a countable collection of open sets containing x having property (b) of the definition of $\mathcal{W}$ satisfying open $G(N)$. Since X is locally separable there is, for each x in X, a non-empty open set $U(x) = \bigcup\{W(n, x) \mid W(n, x) \text{ is separable}\}$. Furthermore, since there exist only countably many $W(n, x)$ associated with x, the set $U(x)$ is separable.

Choose a point p of X and let $E(p)$ be a countable set which is (everywhere) dense in $U(p)$. We define a set $C_1(p) = \bigcup\{U(x) \mid x \in E(p)\}$ and note that also $C_1(p)$ is separable. Thus $C_1(0)$ contains a countable dense subset $E_1(p)$. We define $C_2(p) = \bigcup\{U(x) \mid x \in E_1(p)\}$. We now repeat this process to obtain a sequence $C_1(p), C_2(p), C_3(p), \ldots$ with respective countable dense sets $E_1(p), E_2(p), E_3(p), \ldots$. Next we make stronger use of the property $\mathcal{W}$ satisfying open $G(N)$ in order to show that $\{C_n(p)\}$ covers X. Suppose on the contrary that $H = C_1(p) \cup C_2(p) \cup C_3(p) \cup \ldots$ and $K = X - H$ are non-empty sets. Then, since X is connected and H is open, there must be a point q of K which is a limit point of H. Using the local separability of X again, let U be an open set containing q such that U is separable. Moreover, since X has the property of $\mathcal{W}$ satisfying open $G(N)$ there is an open set V containing q such that for every point x of V there exists a $W(n, x)$ such that $q \in W(n, x) \subset U$. But, since q is a limit point of H and $H = \bigcup\{C_n(p)\}$, x can be chosen to be a point of a set $E_k(p)$ which is dense in a set $C_k(p)$, for some positive integer k. But $W(n, x)$ is separable and hence $W(n, x) \subset E_{k+1}(p)$. It follows that q is contained in H, which involves a contradiction. Thus $H = X$ and therefore X is separable, and $E = E_1(p) \cup E_2(p) \cup E_3(p) \cup \ldots$ is a countable dense subset of X. Furthermore, X has a countable basis consisting of $\{W(n, x) \mid x \in E\}$ because if U is any open set and p is any point of U there exists an open set V such that $p \in V \subset U$ and each point z of $V \cap E$ has a neighbourhood $W(k, z)$ which contains p and lies in U. Therefore, since X has the regularity property of a Moore space, X is metrisable. ∎

In the third theorem on conditions under which normal Moore spaces are metrisable we are able to give more emphasis yet to local properties.

THEOREM 2.3. *Let X be a Moore space having the property of $\mathcal{W}$ satisfying open $G(N)$. Then X is metrisable if X is locally separable and locally connected.*

PROOF: Since X is locally connected the components $\{C_\alpha\}$ of X are both open and closed. Since no two components have a point in common, and the union of any subcollection of $\{C_\alpha\}$ is closed, the collection $\{C_\alpha\}$ is discrete. Hence, by the last section of the proof of Theorem 2.1 it is sufficient to establish the desired results for each individual component under the subspace topology. But each such component satisfies the hypothesis of Theorem 2.2. Therefore X is a metrisable Moore space. ∎

3. Necessary and sufficient conditions for metrisability

It is desirable to find necessary and sufficient conditions for a result to hold. Notable examples of metrisability characterisations are those of Bing [**1**] and of Smirnov [**4**]. Generally each such characterisation is applicable to a particular family of abstract topological spaces. Now, in this paper, we establish necessary and sufficient conditions for a topological space to be metrisable which applies to spaces that are locally "nice" and globally "coherent" fitting the general criteria of what we have called an abstract topological manifold.

Theorem 3.1. *A necessary and sufficient condition for a regular and locally separable space X to be metrisable is that X have a (global) basis which is point-countable.*

Proof of sufficiency: Let $\mathcal{B}$ be a point-countable basis of X, and let X be well-ordered. Since X is locally separable there is, for every point x of X a non-empty separable open set $U(x)$ which equals the (countable) union of all separable members of $\mathcal{B}$ that contain x. Let p_1 be the first element of X with repsect to the well-ordering of X and let $E(p_1)$ be a countable set that is dense in $U(p_1)$. We define $C_1(p) = \bigcup\{U(x) \mid x \in E(p_1)\}$. This countable union of separable sets is separable and hence has a countable dense set $E_1(p_1)$. In general then we define $C_1(p_1) = \bigcup\{U(x) \mid x \in E_{n-1}(p_1)\}$, and define V_1 to be equal to the union of the collection $\{C_1(p_1), C_2(p_1), C_3(p_1), \ldots\}$, and note that V_1 is separable.

We use transfinite induction to define a possibly uncountable family of sets such as V_1. If V_α has been defined for all α less than β then choose p to be the first member of well-ordered X that is not contained in $\bigcup\{V_\alpha \mid \alpha < \beta\}$ and define V_β to be the union of sets $C_1(p)$, $C_2(p)$, $C_3(p)$, $\ldots$ which are defined relative to p in the same way that the sets $C_1(p_1)$, $C_2(p_1)$, $C_3(p_1)$, $\ldots$ were defined relative to p_1.

Then $\{V_\alpha\}$ is a discrete collection by the argument given in Theorem 2.1 and each member V_α of $\{V_\alpha\}$ is separable as well as regular. We need to show that there is a countable basis for each member V_β of the collection $\{V_\alpha\}$, assigning to V_β the subspace topology induced by the topology of X, in order to complete the proof that X is metrisable.

The required countable basis for X is obtained as follows. Since V_β is separable there is a countable subset $W_\beta = \{q_i\}$ of V_β such that W_β is dense in V_β. For each point q_n of W_β we choose $\mathcal{B}_n$ to be the subcollection of $\mathcal{B}$ such that each member of $\mathcal{B}_n$ contains q_n. Since $\mathcal{B}$ is point countable, each collection $\mathcal{B}_n$ is countable. Hence, $\mathcal{B}_\beta = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \ldots$ is a countable collection of open sets covering the dense set W_β of V_β. To show that moreover $\{B \cap V_\beta \mid B \in \mathcal{B}_\beta\}$ gives a basis for all of V_β, let z be an arbitrary point of V_β and let G be an open set of X which contains z. There is a member B of $\mathcal{B}$ such that $z \in B \subset G$. But B must contain an element q_k

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of W_β since B intersects V_β and W_β is dense in V_β. Hence, B is a member of $\mathcal{B}_k$ which is contained in $\mathcal{B}_\beta$. Therefore the intersections of members of $\mathcal{B}_\beta$ with V_β form a countable basis for V_β and the proof of sufficiency is complete.

PROOF OF NECESSITY: Let X be metric. Then for each positive integer n, the set $\mathcal{C}_n = \{N(x, 1/n) \mid x \in X\}$ of $1/n$ neighbourhoods about points of X is an open cover of X. Since a metric space is paracompact by [**5**], then for each $\mathcal{C}_n$ cover we may choose a refinement $\mathcal{V}_n$ which also covers X. Each point is in at most countably many elements of the union of $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots$ which is a basis for X. □

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