6

Bulk properties of strongly coupled plasma

Up to this point in this book, we have laid the groundwork needed for what is to come in two halves. In Chapters 2 and 3 we have introduced the theoretical, phenomenological and experimental challenges posed by the study of the deconfined phase of QCD and in Chapters 4 and 5 we have motivated and described gauge/string duality, providing the reader with most of the conceptual and computational machinery necessary to perform many calculations. Although we have foreshadowed their interplay at various points, these two long introductions have to a large degree been separately self-contained. In the next four chapters, we weave these strands together. In these chapters, we shall describe applications of gauge/gravity duality to the study of the strongly coupled plasma of $\mathcal{N} = 4$ SYM theory at nonzero temperature, focusing on the ways in which these calculations can guide us toward the resolution of the challenges described in Chapters 2 and 3.

The study of the zero temperature vacuum of strongly coupled $\mathcal{N} = 4$ SYM theory is a rich subject with numerous physical insights into the dynamics of gauge theories. Given our goal of gaining insights into the deconfined phase of QCD, we will largely concentrate on the description of strongly coupled $\mathcal{N} = 4$ SYM theory at nonzero temperature, where it describes a strongly coupled non-Abelian plasma with $\mathcal{O}(N_c^2)$ degrees of freedom. The vacua of QCD and $\mathcal{N} = 4$ SYM theory have very different properties. However, when we compare $\mathcal{N} = 4$ SYM at $T \neq 0$ with QCD at a temperature above the temperature T_c of the crossover from a hadron gas to quark–gluon plasma, many of the qualitative distinctions disappear or become unimportant. In particular, we have the following.

(1) QCD confines, while $\mathcal{N} = 4$ SYM does not. This is a profound difference in vacuum. But, above its T_c QCD is no longer confining. The fact that its $T = 0$ quasiparticles are hadrons within which quarks are confined is not particularly relevant at temperatures above T_c.

(2) In QCD, chiral symmetry is broken by a chiral condensate which sets a scale that is certainly not present in $\mathcal{N} = 4$ SYM theory. However, in QCD above its T_c the chiral condensate melts away and this distinction between the vacua of the two theories also ceases to be relevant.

(3) $\mathcal{N} = 4$ SYM is a scale-invariant theory while in QCD scale invariance is broken by the confinement scale, the chiral condensate, and the running of the coupling constant. Above T_c, we have already dispensed with the first two scales. Also, as we have described in Chapter 3, QCD thermodynamics is significantly nonconformal just above $T_c \sim 170$ MeV, but at higher temperatures the quark–gluon plasma becomes more and more scale invariant, at least in its thermodynamics. (Thermodynamic quantities converge to their values in the noninteracting limit, due to the running of the coupling towards zero, only at vastly higher temperatures which are far from the reach of any collider experiment.) So, here again, QCD above (but not asymptotically far above) its T_c is much more similar to $\mathcal{N} = 4$ SYM theory at $T \neq 0$ than the vacua of the two theories are.

(4) $\mathcal{N} = 4$ SYM theory is supersymmetric. However, supersymmetry is explicitly broken at nonzero temperature. In a thermodynamic context, this can be seen by noting that fermions have antiperiodic boundary conditions along the Euclidean time circle while bosons are periodic. For this reason, supersymmetry does not play a major role in the characterization of properties of the $\mathcal{N} = 4$ SYM plasma at nonzero temperatures.

(5) QCD is an asymptotically free theory and, thus, high energy processes are weakly coupled. However, as we have described in Chapter 2, in the regime of temperatures above T_c that are accessible to heavy ion collision experiments the QCD plasma is strongly coupled, which opens a window of applicability for strong coupling techniques.

For these and other reasons, the strongly coupled plasma of $\mathcal{N} = 4$ SYM theory has been studied by many authors with the aim of gaining insights into the dynamics of deconfined QCD plasma.

In fairness, we should also mention the significant differences between the two theories that remain at nonzero temperature.

(1) $\mathcal{N} = 4$ SYM theory with $N_c = 3$ has more degrees of freedom than QCD with $N_c = 3$. To seek guidance for QCD from results in $\mathcal{N} = 4$ SYM, the challenge is to evaluate how an observable of interest depends on the number of degrees of freedom, as we do at several points in Chapter 8. The best case scenario is that there is no such dependence. For example, the ratio η/s between the shear viscosity and the entropy density that we introduced in Chapter 2 and that we shall discuss in Section 6.2 is such a case.

(2) Most of the calculations that we shall report are done in the strong coupling ($\lambda \to \infty$) limit. This is of course a feature not a bug. The ability to do these calculations in the strong coupling regime is a key part of the motivation for all this work. But, although in the temperature regime of interest $g^2(T)N_c = 4\pi N_c \alpha_s(T)$ is large, it is not infinite. This motivates the calculation of corrections to various results that we shall discuss that are proportional to powers of $1/\lambda$, for the purpose of testing the robustness of conclusions drawn from calculations done with $\lambda \to \infty$.

(3) QCD has $N_c = 3$ colors, while all the calculations that we shall report are done in the $N_c \to \infty$ limit. Although the large-N_c approximation is familiar in QCD, the standard way of judging whether it is reliable in a particular context is to compute corrections suppressed by powers of $1/N_c$. And, determining the $1/N_c^2$ corrections to the calculations done via the gauge/string duality that we review remains an outstanding challenge.

(4) Although we have argued above that the distinction between bosons and fermions is not important at nonzero temperature, the distinction between degrees of freedom in the adjoint or fundamental representation of $SU(N_c)$ is important. QCD has $N_f = 3$ flavors in the fundamental representation, namely $N_f = N_c$. These fundamental degrees of freedom contribute significantly to its thermodynamics at temperatures above T_c. And, the calculations that we shall report are either done with $N_f = 0$ or with $0 < N_f \ll N_c$. Extending methods based upon gauge/string duality to the regime in which $N_f \sim N_c$ remains an outstanding challenge.

The plasmas of QCD and strongly coupled $\mathcal{N} = 4$ SYM theory certainly differ. At the least, using one to gain insight into the other follows in the long tradition of modelling, in which a theoretical physicist employs the simplest instance of a theory that captures the essence of a suite of phenomena that are of interest in order to gain insights. The gravitational description of $\mathcal{N} = 4$ SYM makes it clear that it is in fact the simplest, most symmetric, strongly coupled non-Abelian plasma. The question then becomes whether there are quantities or phenomena that are universal across many different strongly coupled plasmas. The qualitative, and in some instances even semi-quantitative, successes that we shall review that have been achieved in comparing results or insights obtained in $\mathcal{N} = 4$ SYM theory to those in QCD suggest a positive answer to this question, but no precise definition of this new kind of universality has yet been conjectured. In the absence of a precise understanding of such a universality, we can hope for reliable insights into QCD but not for controlled calculations.

We begin our description of the $\mathcal{N} = 4$ SYM strongly coupled plasma in this section by characterizing its macroscopic properties, i.e. those that involve temporal and spatial scales much larger than the microscopic scale $1/T$. In Section 6.1 we briefly review the determination of the thermodynamics of $\mathcal{N} = 4$ SYM theory. The quantities that we calculate are accessible in QCD, via lattice calculations as we have described in Chapter 3, meaning that in Section 6.1 we will be able to compare calculations done in $\mathcal{N} = 4$ SYM theory via gauge/string duality to reliable information about QCD. In Section 6.2 we turn to transport coefficients like the shear viscosity η, which govern the relaxation of small deviations away from thermodynamic equilibrium. Lattice calculations of such quantities remain challenging for reasons that we have described in Section 3.2 but, as we have seen in Section 2.2, phenomenological analyses of collective effects in heavy ion collisions in comparison to relativistic, viscous, hydrodynamic calculations are yielding information about η/s in QCD. Section 6.3 will be devoted to illustrating one of the most important qualitative differences between the strongly coupled $\mathcal{N} = 4$ SYM plasma and any weakly coupled plasma: the absence of quasiparticles. As we will argue in this section, this is a generic feature of strong coupling which, at least at a qualitative level, provides a strong motivation in the context of the physics of QCD above T_c for performing studies within the framework of gauge/string duality. Finally, in Section 6.4 we shall see how long-lived collective hydrodynamic excitations of the plasma, as well as a plethora of excitations of the plasma with lifetimes that are short compared to the inverse of their energies, emerge from the gravitational point of view where they correspond to perturbations of the metric.

6.1 Thermodynamic properties

6.1.1 Entropy, energy and free energy

As discussed in Section 5.2.1, $\mathcal{N} = 4$ SYM theory in equilibrium at nonzero temperature is described in the gravity theory by introducing black branes which change the AdS_5 metric to the black brane metric (5.34) with an event horizon at position z_0. As in standard black hole physics, the presence of the horizon allows us to compute the entropy in the gravity description, which is given by the Bekenstein–Hawking formula

$$S_{\lambda=\infty} = S_{\rm BH} = \frac{A_3}{4G_5}\,, \tag{6.1}$$

where A_3 is the three-dimensional area of the event horizon of the non-compact part of the metric and G_5 is the five-dimensional Newton constant. This entropy is to be identified as the entropy of the gauge theory plasma in the strong coupling limit [391]. The area A_3 is determined from a spatial section of the horizon metric, obtained by setting $t = \text{const}$, $z = z_0$ in Eq. (5.34), i.e.

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$$ds^2_{\rm Hor} = \frac{R^2}{z_0^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right) . \tag{6.2}$$

The total horizon area is then

$$A_3 = \frac{R^3}{z_0^3}\int dx_1 dx_2 dx_3 \, , \tag{6.3}$$

where $\int dx_1 dx_2 dx_3$ is the volume in the gauge theory. While the total entropy is infinite, the entropy density per unit gauge theory volume is finite and is given by

$$s_{\lambda=\infty} = \frac{S_{\rm BH}}{\int dx_1 dx_2 dx_3} = \frac{R^3}{4G_5 z_0^3} = \frac{\pi^2}{2} N_c^2 T^3 \, , \tag{6.4}$$

where in the last equality we have used Eqs. (5.12) and (5.36) to translate the gravity parameters z_0, R and G_5 into the gauge theory parameters T and N_c. Note that we would have obtained the same result if we had used the full ten-dimensional geometry, which includes the S^5. In this case the horizon would have been nine-dimensional, with a spatial area of the form $A_8 = A_3 \times S^5$, and the entropy would have taken the form

$$S_{\rm BH} = \frac{A_8}{4G} = \frac{A_3 V_{S^5}}{4G} \, , \tag{6.5}$$

which equals (6.1) by virtue of the relation (5.12) between the ten- and the five-dimensional Newton constants.

Once the entropy density is known, the rest of the thermodynamic potentials are obtained through standard thermodynamic relations. In particular, the pressure P obeys $s = \partial P/\partial T$, and the energy density is given by $\varepsilon = -P + Ts$. Thus we find:

$$\varepsilon_{\lambda=\infty} = \frac{3\pi^2}{8} N_c^2 T^4, \qquad P_{\lambda=\infty} = \frac{\pi^2}{8} N_c^2 T^4 \, . \tag{6.6}$$

The N_c and temperature dependence of these results could have been anticipated. The former follows from the fact that the number of degrees of freedom in an $SU(N_c)$ gauge theory in its deconfined phase grows as N_c^2, whereas the latter follows from dimensional analysis, since the temperature is the only scale in the $\mathcal{N} = 4$ SYM theory. What is remarkable about these results is that they show that the prefactors in front of the N_c and temperature dependence in these thermodynamic quantities attain finite values in the limit of infinite coupling, $\lambda \to \infty$, which is the limit in which the gravity description becomes strictly applicable.

It is instructive to compare the above expressions at infinite coupling with those for the free $\mathcal{N} = 4$ SYM theory, i.e. at $\lambda = 0$. Since $\mathcal{N} = 4$ SYM has eight bosonic and eight fermonic adjoint degrees of freedom and since the contribution of each boson to the entropy is $2\pi^2 T^3/45$ whereas the contribution of each fermion is $7/8$ of that of a boson, the zero coupling entropy is given by

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$$s_{\lambda=0} = \left(8 + 8 \times \frac{7}{8}\right) \frac{2\pi^2}{45} (N_c^2 - 1) T^3 \simeq \frac{2\pi^2}{3} N_c^2 T^3 \,, \tag{6.7}$$

where in the last equality we have used the fact that $N_c \gg 1$. As before, the N_c and T dependences are set by general arguments. The only difference between the infinite and zero coupling entropies is an overall numerical factor: comparing Eqs. (6.4) and (6.7) we find [391]

$$\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{P_{\lambda=\infty}}{P_{\lambda=0}} = \frac{\varepsilon_{\lambda=\infty}}{\varepsilon_{\lambda=0}} = \frac{3}{4} \,. \tag{6.8}$$

This is a very interesting result: while the coupling of $\mathcal{N} = 4$ SYM changes radically between the two limits, the thermodynamic potentials vary very mildly. This observation is, in fact, not unique to the special case of $\mathcal{N} = 4$ SYM theory, but seems to be a generic phenomenon for field theories with a gravity dual. In fact, in Ref. [653] it was found that for several different classes of theories, each encompassing infinitely many instances, the change in entropy between the infinitely strong and infinitely weak coupling limit is

$$\frac{s_{\text{strong}}}{s_{\text{free}}} = \frac{3}{4} h \,, \tag{6.9}$$

with h a factor of order one, $\frac{8}{9} \leq h \leq 1.096\,62$. These explicit calculations strongly suggest that the thermodynamic potentials of non-Abelian gauge-theory plasmas (at least for near-conformal ones) are quite insensitive to the particular value of the gauge coupling. This is particularly striking since, as we will see in Sections 6.2 and 6.3, the transport properties of these gauge theories change dramatically as a function of coupling, going from a nearly ideal gas-like plasma of quasiparticles at weak coupling to a nearly ideal liquid with no quasiparticles at strong coupling. So, we learn an important lesson from the calculations of thermodynamics at strong coupling via gauge/string duality: thermodynamic quantities are not good observables for distinguishing a weakly coupled gas of quasiparticles from a strongly coupled liquid; transport properties and the physical picture of the composition of the plasma are completely different in these two limits, but no thermodynamic quantity changes much.

Returning to the specific case of $\mathcal{N} = 4$ SYM theory, in this case the leading finite-λ correction to (6.8) has been calculated [402] as has the leading finite-N_c correction [640], yielding

$$\begin{aligned} \frac{s_{\lambda,\, N_c\to\infty}}{s_{\lambda=0,\, N_c\to\infty}} &= \frac{P_{\lambda,\, N_c\to\infty}}{P_{\lambda=0,\, N_c\to\infty}} = \frac{\varepsilon_{\lambda,\, N_c\to\infty}}{\varepsilon_{\lambda=0,\, N_c\to\infty}} \\ &= \frac{3}{4}\left(1 + \frac{15\,\zeta(3)}{8}\frac{1}{\lambda^{3/2}} + \frac{5}{128}\frac{\lambda^{1/2}}{N_c^2} + \cdots\right) \,, \end{aligned} \tag{6.10}$$

where ζ is the Riemann zeta function and $\zeta(3) \approx 1.20$. Note that equation (6.10) is obtained by taking $N_c \to \infty$ first and then taking $\lambda \to \infty$. In this limit, the last term is always much smaller than the other terms despite the $\lambda^{1/2}$ factor in the numerator. Note also that the $\mathcal{O}(1/N_c^2)$ corrections that are zeroth order in λ have not yet been computed. The expression (6.10) suggests that $s_{\lambda=\infty}/s_{\lambda=0}$ increases from 3/4 to 7/8 as λ drops from infinity down to $\lambda \sim 6$, corresponding to $\alpha_{\text{SYM}} \sim 0.5/N_c$. This reminds us that the control parameter for the strong coupling approximation is $1/\lambda$, meaning that it can be under control down to small values of α_{SYM}.

It is also interesting to compare (6.8) to what we know about QCD thermodynamics from lattice calculations like those described in Section 3.1. The ratio (6.8) has the advantage that the leading dependence on the number of degrees of freedom drops out, making it meaningful to compare directly to QCD. While theories that have been analyzed in Ref. [653] are rather different from QCD, the regularity observed in these theories compel us to evaluate the ratio of the entropy density computed in the lattice calculations to that which would be obtained for free quarks and gluons. Remarkably, Fig. 3.1 shows that, for $T = (2-3)\,T_c$, the coefficient defined in (6.9) is $h \simeq 1.07$, which is in the ballpark of what the calculations done via gauge/gravity duality have taught us to expect for a strongly coupled gauge theory. While this observation is interesting, by itself it is not strong evidence that the QCD plasma at these temperatures is strongly coupled. The central lesson is, in fact, that the ratio (6.8) is quite insensitive to the coupling. The proximity of the lattice results to the value for free quarks and gluons should never have been taken as indicating that the quark–gluon plasma at these temperatures is a weakly coupled gas of quasiparticles. And, now that experiments at RHIC and at the LHC that we described in Section 2.2 combined with calculations that we shall describe in Section 6.2 have shown us a strongly coupled QCD plasma, the even closer proximity of the lattice results for QCD thermodynamics to that expected for a strongly coupled gauge theory plasma should also not be overinterpreted.

6.1.2 Holographic susceptibilities

The previous discussion focused on a plasma at zero chemical potential μ. While gauge/gravity duality allows us to explore the phase diagram of the theory at nonzero values of μ, in order to parallel our discussion of QCD thermodynamics in Chapter 3, in our analysis of strongly coupled $\mathcal{N} = 4$ SYM theory here we will concentrate on the calculation of susceptibilities. As explained in Section 3.1.1, their study requires the introduction of $U(1)$ conserved charges. In $\mathcal{N} = 4$ SYM, there is an $SU(4)$ global symmetry, the R-symmetry, which in the dual gravity theory corresponds to rotations in the five-sphere. A chemical potential for R-charge can be introduced by studying black branes that rotate in these coordinates [720, 302, 557, 393]; these solutions demand non-vanishing values of an Abelian vector potential A_μ in the gravitational theory which, in turn, lead to a non-vanishing R-charge density n in the gauge theory proportional to the angular momentum density of the black hole. The chemical potential can be extracted from the boundary value of the temporal component of the Maxwell field as in (5.37) and is also a function of the angular momentum of the black hole. The explicit calculation performed in Ref. [748] leads to

$$n = \frac{N_c^2 T^2}{8}\mu \tag{6.11}$$

in the small chemical potential limit. Note that, unlike in QCD, the susceptibility $dn/d\mu$ inferred from Eq. (6.11) is proportional to N_c^2 instead of N_c. This is a trivial consequence of the fact that R-symmetry operates over adjoint degrees of freedom.

As in the case of the entropy, the different number of degrees of freedom can be taken into account by comparing the susceptibility at strong coupling to that in the noninteracting theory, which yields

$$\frac{\chi_{\lambda=\infty}}{\chi_{\lambda=0}} = \frac{1}{2}, \tag{6.12}$$

where $\chi_{\lambda=0} = N_c^2 T^2/4$ [777]. Similarly to the case of the entropy density, the ratio of susceptibilities between these two extreme limits saturates into an order one constant. Despite the radical change in the dynamics of the degrees of freedom in the two systems, the only variation in this observable is a 50% reduction, comparable to the 25% reduction of the energy density in the same limit. This 50% reduction can be contrasted with the results from the lattice calculations reviewed in Section 3.1.1 which show a slow rise in the quark number fluctuations above T_c, seemingly saturating at about 90% of their value in the noninteracting limit. As in the early interpretations of lattice calculations of the energy density and pressure, the proximity of the diagonal susceptibilities to their Stefan–Boltzmann values has been interpreted by some as a sign that the QCD quark–gluon plasma is not strongly coupled [540, 726, 171, 676]. However, although the susceptibilities calculated on the lattice come numerically closer to their values in the noninteracting limit than in the case of the pressure, their temperature-dependence is qualitatively quite similar. Therefore, it is not clear whether the values of the susceptibilities pose any challenge to the interpretation of the QCD plasma as a strongly coupled one, given the manifest insensitivity of thermodynamic quantities to the coupling. Furthermore, the value of the ratio of susceptibilities (6.12) is not universal: it can be different in holographic gauge theories which are closer to QCD than $\mathcal{N} = 4$ SYM. It is tempting to speculate that if it were possible to use gauge/string duality to analyze strongly coupled theories with $N_f \sim N_c$ and compute the susceptibility for a $U(1)$ charge carried by the fundamental degrees of freedom in such a theory, we may be able to find examples in which the susceptibility is as close to its weak coupling value as is the case in QCD, even when all degrees of freedom are strongly coupled. Were this speculation to prove correct, it would be an example of a result from QCD leading to insight into strongly coupled gauge theories with a gravitational description, i.e. it would be an example of insight in the opposite direction from that that throughout most of this book.

The study of off-diagonal susceptibilities as in (3.3) requires the introduction of an additional $U(1) \times SU(N_f)$ global symmetry in the plasma, with $N_f \geq 2$. (The global $SU(4)$ symmetry that is already a feature of $\mathcal{N} = 4$ SYM theory cannot be used for this purpose because the off-diagonal susceptibilities of two commuting $U(1)$ subgroups within $SU(4)$ must vanish.) As explained in Section 5.5, fundamental flavor degrees of freedom are introduced in the holographic set-up via D-branes, which, in addition to a $SU(N_f)$ global symmetry, also lead to an additional $U(1)$ charge (baryon number). Analogously to the way the diagonal susceptibilities are analyzed above, non-vanishing values of the different chemical potentials arising in off-diagonal susceptibilities (3.3) are associated with non-vanishing non-Abelian gauge fields in the brane. In the probe approximation ($N_f \ll N_c$), the study of susceptibilities corresponds to determining the reaction of the partition function of the branes to small values of these non-Abelian fields up to quadratic order. However, off-diagonal susceptibilities are suppressed by an additional power of N_c with respect to the diagonal susceptibilities, as shown in Ref. [170]. This can be inferred from the fact that there is no mixing (at quadratic order) between different gauge fields in the non-Abelian Yang-Mills Lagrangian. On the gravity side, this means that the off-diagonal susceptibilities vanish at the classical level and a one-loop calculation is required. While the complete determination of the one-loop correction to the partition function is technically very demanding, since it must include an analysis of all the gravitational fields, the contribution to the flavor correlations that is leading order in N_c can be obtained by restricting the calculation to open string fluctuations, since closed string modes cannot distinguish among different flavors. After this simplification, the analysis of the one-loop determinant in Ref. [250] yields the leading parametric dependence of the off-diagonal susceptibilities on both λ and N_c.

The main result of the analysis in Ref. [250] is that the ratio of off-diagonal to diagonal susceptibilities becomes independent of the coupling in the limit $\lambda \to \infty$, which is in marked contrast to expectations based upon extending the perturbative result (which is that the off-diagonal susceptibilities are suppressed relative to the diagonal ones by a factor of order $(\lambda^3/N_c)\log(1/g)$ with g the gauge coupling constant [170]) to strong coupling. Within the D3/D7 model for holographic flavor, the off-diagonal susceptibilities as in (3.3) can be expressed as [250]

$$\chi_{11}^{ud} = h\left(\frac{M}{T}\right)T^2\,, \tag{6.13}$$

with $h\left(\frac{M}{T}\right)$ a model dependent numerical constant that depends on the mass of the quarks, M, but has no dependence on either λ or N_c. The fact that in the large λ limit this thermodynamic quantity becomes independent of coupling and differs from its value at $\lambda = 0$ only by a modest numerical factor reflects, once again, the insensitivity of thermodynamic potentials to the underlying degrees of freedom. Furthermore, the rich structure of the D3/D7 model, which will be discussed in depth in Section 9.3 , allows us to use the opportunity to vary M/T to compare the off-diagonal susceptibilities in a plasma at large M/T in which these susceptibilities are dominated by quasiparticles (infinitely narrow bound mesons that can be thought of as analogous to quarkonia) to that in a plasma at small M/T in which there are no quasiparticles. (In this high temperature phase, the quarkonium-like mesons have dissolved and there are no quasiparticles.) As inferred from Eq. (6.13), the off-diagonal susceptibilities remain parametrically the same in the large and small M/T limits, even given the radical change in the degrees of freedom and in the nature of the plasma. From this study, we can conclude that when we see non-vanishing values of the off-diagonal susceptibilities, as in the lattice QCD calculations that we have described in Section 3.1.1, this does not imply the existence of resonances of any type, let alone bound states. The holographic analysis of susceptibilities in strongly coupled plasma demonstrates that drawing conclusions about the strength of the coupling constant or about the nature of the effective degrees of freedom in the QGP from the lattice computation of susceptibilities should be treated with just as much caution as drawing such conclusions from the values of thermodynamic quantities.

6.2 Transport properties

We now turn to the calculation of the transport coefficients of a strongly coupled plasma with a dual gravitational description, which control how such a plasma responds to small deviations from equilibrium. We shall see that in the strong coupling limit these quantities take on very different values, both parametrically and numerically, than in a noninteracting plasma. This makes them much better suited to diagnosing whether a plasma is gas-like or liquid-like, weakly coupled or strongly coupled, than the thermodynamic quantities and susceptibilities of the previous section. As we have reviewed in Section 3.2, since the relaxation of perturbations toward equilibrium is intrinsically a real time process, the lattice determination of transport coefficients is very challenging. While initial steps toward determining them in QCD have been taken, definitive results are not in hand. As a consequence, the determination of transport coefficients via

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gauge/string duality is extremely valuable since it opens up their analysis in a regime which is not tractable otherwise. A remarkable consequence of this analysis, which we describe in Section 6.2.2, is a universal relation between the shear viscosity and the entropy density for the plasmas in all strongly coupled large-N_c gauge theories with a gravity dual [554, 212, 552, 206]. This finding, together with the comparison of the universal result $\eta/s = 1/(4\pi)$ with values extracted by comparing data on azimuthally asymmetric flow in heavy ion collisions to analyses in terms of viscous hydrodynamics as we have described in Section 2.2, has been one of the most influential results obtained via the gauge/string duality.

6.2.1 A general formula for transport coefficients

The most straightforward way in which transport coefficients can be determined using the gauge/gravity correspondence is via Green–Kubo formulas, see Appendix A, which rely on the analysis of the retarded correlators in the field theory at small four-momentum. The procedure for determining these correlators using the correspondence has been outlined in Section 5.3. In this section we will try to keep our analysis as general as possible so that it can be used for the transport coefficient that describes the relaxation of any conserved current in the theory. In addition, we will not restrict ourselves to the particular form of the metric (5.34) so that our discussion can be applied to any theory with a gravity dual. Our discussion will closely follow the formalism developed in [482], which builds upon earlier analyses in Refs. [690, 554, 212, 552, 206, 725, 761, 364].

In general, if the field theory at nonzero temperature is invariant under translations and rotations, the gravitational theory will be described by a (4 + 1)-dimensional metric of the form

$$ds^2 = -g_{tt}dt^2 + g_{zz}dz^2 + g_{xx}\delta_{ij}dx^i dx^j = g_{MN}dx^M dx^N \tag{6.14}$$

with all the metric components solely dependent on z. Since a nonzero temperature is characterized in the dual theory by the presence of an event horizon, we will assume that g_{tt} has a first order zero and g_{zz} has a first order pole at a particular value $z = z_0$.

We are interested in computing the transport coefficient χ associated with some operator $\mathcal{O}$ in this theory, namely

$$\chi = -\lim_{\omega\to 0}\lim_{\vec{k}\to 0}\frac{1}{\omega}\,\mathrm{Im}\,G_R(\omega, \mathbf{k})\,. \tag{6.15}$$

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(See Appendix A for the exact definition of G_R and for a derivation of this formula.) For concreteness, we assume that the quadratic effective action for the bulk mode ϕ dual to $\mathcal{O}$ has the form of a massless scalar field[1]

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\,\frac{1}{q(z)}\,g^{MN}\partial_M\phi\partial_N\phi, \tag{6.16}$$

where $q(z)$ is a function of z and can be considered a spacetime-dependent coupling constant. As we will see below, Eqs. (6.14) and (6.16) apply to various examples of interest including the shear viscosity and the momentum broadening for the motion of a heavy quark in the plasma. Since transport coefficients are given by the Green–Kubo formula Eq. (6.15), the general expression for the retarded correlator (5.62) with $\Delta = d$ and $m = 0$ leads to

$$\chi = -\lim_{k_\mu\to 0}\lim_{z\to 0}\mathrm{Im}\left\{\frac{\Pi(z,k_\mu)}{\omega\phi(z,k_\mu)}\right\} = -\lim_{k_\mu\to 0}\lim_{z\to 0}\frac{\Pi(z,k_\mu)}{i\omega\phi(z,k_\mu)}\,, \tag{6.17}$$

where Π is the canonical momentum of the field ϕ:

$$\Pi = \frac{\delta S}{\delta\partial_z\phi} = -\frac{\sqrt{-g}}{q(z)}\,g^{zz}\partial_z\phi\,. \tag{6.18}$$

The last equality in (6.17) follows from the fact that the real part of $G_R(k)$ vanishes faster than linearly in ω as $k \to 0$, as is proven by the fact that the final result that we will obtain, Eq. (6.25), is finite and real.

In (6.17) both Π and ϕ must be solutions of the classical equations of motion which, in the Hamiltonian formalism, are given by (6.18) together with

$$\partial_z\Pi = -\frac{\sqrt{-g}}{q(z)}\,g^{\mu\nu}k_\mu k_\nu\phi\,. \tag{6.19}$$

The evaluation of χ, following Eq. (6.17), requires the determination of both $\omega\phi$ and Π in the small four momentum $k_\mu \to 0$ limit. Remarkably, in this limit the equations of motion (6.18) and (6.19) are trivial[2]

$$\partial_z\Pi = 0 + \mathcal{O}(k_\mu\omega\phi)\,, \qquad \partial_z(\omega\phi) = 0 + \mathcal{O}(\omega\Pi)\,, \tag{6.20}$$

and both quantities become independent of z, which allows their evaluation at any z. For simplicity, and since the only restriction on the general metric (6.14) is that it

[1] Note that restricting to a massless mode does not result in much loss of generality, since almost all transport coefficients calculated to date are associated with operators whose gravity duals are massless fields. The only exception is the bulk viscosity.

[2] Note from (6.18) and (6.19) that the $\mathcal{O}(k_\mu\omega\phi)$ terms neglected in the first equation in (6.20) contain a term multiplied by g^{tt} while the $\mathcal{O}(\omega\Pi)$ term neglected in the second equation in (6.20) is multiplied by g_{zz}. Since both quantities diverge at the horizon, Eqs. (6.20) are not valid there. They are valid anywhere outside the horizon for sufficiently small ω. Note, however, that the ratio in (6.17) does have a well defined limit approaching the horizon.

possesses a horizon, we will evaluate them for $z \to z_0$ where the infalling boundary condition should be imposed. Our assumptions about the metric imply that in the vicinity of the horizon $z \to z_0$

$$g_{tt} = -c_0(z_0 - z), \qquad g_{zz} = \frac{c_z}{z_0 - z}, \tag{6.21}$$

and eliminating Π from (6.18) and (6.19) we find an equation for ϕ given by

$$\sqrt{\frac{c_0}{c_z}}(z_0 - z)\partial_z \left(\sqrt{\frac{c_0}{c_z}}(z_0 - z)\partial_z \phi \right) + \omega^2 \phi = 0\,. \tag{6.22}$$

The two general solutions for this equation are

$$\phi \propto e^{-i\omega t}\,(z_0 - z)^{\pm i\omega\sqrt{c_z/c_0}}\,. \tag{6.23}$$

Imposing infalling boundary condition implies that we should take the negative sign in the exponent. Therefore, from Eq. (6.23) we find that at the horizon

$$\partial_z \phi = \sqrt{\frac{g_{zz}}{-g_{tt}}}(i\omega\phi)\,, \tag{6.24}$$

and using Eqs. (6.18) and (6.23) we obtain

$$\chi = -\lim_{k_\mu \to 0}\lim_{z \to 0} \frac{\Pi(z, k_\mu)}{i\omega\phi(z, k_\mu)} = -\lim_{k_\mu \to 0}\lim_{z \to z_0} \frac{\Pi(z, k_\mu)}{i\omega\phi(z, k_\mu)} = \frac{1}{q(z_0)}\sqrt{\frac{-g}{-g_{zz}g_{tt}}}\Bigg|_{z_0}\,. \tag{6.25}$$

Note that the last equality in (6.25) can also be written as

$$\chi = \frac{1}{q(z_0)}\frac{A}{V}\,, \tag{6.26}$$

where A is the area of the horizon and V is the spatial volume of the boundary theory. From our analysis of the thermodynamic properties of the plasma in Section 6.1, the area of the event horizon is related to the entropy density of the boundary theory via

$$s = \frac{A}{V}\frac{1}{4G_N}\,. \tag{6.27}$$

From this analysis we conclude that in theories with a gravity dual the ratio of any transport coefficient to the entropy density depends solely on the properties of the dual fields at the horizon,

$$\frac{\chi}{s} = \frac{4G_N}{q(z_0)}\,. \tag{6.28}$$

In the next section we will use this general expression to compute the shear viscosity of the AdS plasma.

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Finally, we would like to remark that the above discussion applies to more general effective actions of the form

$$S = -\frac{1}{2}\int \frac{d\omega d^{d-1}k}{(2\pi)^d} dz\sqrt{-g}\left[\frac{g^{zz}(\partial_z\phi)^2}{Q(z;\omega,k)} + P(z;\omega,k)\phi^2\right], \tag{6.29}$$

provided that the equations of motion (6.20) remain trivial in the zero-momentum limit. This implies that Q should go to a nonzero constant at zero momentum and P must be at least quadratic in momenta. For (6.29) the corresponding transport coefficient χ is given by

$$\chi = \frac{1}{Q(z_0, k_\mu = 0)}\frac{A}{V} \quad \text{and} \quad \frac{\chi}{s} = \frac{4G_N}{Q(z_0, k_\mu = 0)}. \tag{6.30}$$

6.2.2 Universality of the shear viscosity

We now apply the result of the last section to the calculation of the shear viscosity η of a strongly coupled plasma described by the metric (6.14). As in Appendix A we must compute the correlation function of the operator $\mathcal{O} = T_{xy}$, where the coordinates x and y are orthogonal to the momentum vector. The bulk field ϕ dual to $\mathcal{O}$ should have a metric perturbation h_{xy} as its boundary value. It then follows that $\phi = (\delta g)^x_y \underset{z\to 0}{\longrightarrow} h_{xy}$, where δg is the perturbation of the bulk metric. For Einstein gravity in a geometry with no off-diagonal components in the background metric, as in (6.14), a standard analysis of the Einstein equations to linear order in the perturbation, upon assuming that the momentum vector is perpendicular to the (x, y)-plane, shows that the effective action for ϕ is simply that of a minimally coupled massless scalar field, namely

$$S = -\frac{1}{16\pi G_N}\int d^{d+1}x\,\sqrt{-g}\left[\frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi\right]. \tag{6.31}$$

The prefactor $1/16\pi G_N$ comes from that of the Einstein–Hilbert action. This action has the form of Eq. (6.16) with

$$q(z) = 16\pi G_N = \text{const}, \tag{6.32}$$

which, together with Eq. (6.28), leads to the celebrated result

$$\frac{\eta_{\lambda=\infty}}{s_{\lambda=\infty}} = \frac{1}{4\pi} \tag{6.33}$$

that was first obtained in 2001 by Policastro, Son and Starinets [690]. In (6.32), we have added the subscript $\lambda = \infty$ to stress that the numerator and denominator are both computed in the strict infinite coupling limit. Remarkably, this ratio converges

https://doi.org/10.1017/9781009403504.006 Published online by Cambridge University Press

to a constant at strong coupling. And, this is not only a feature of $\mathcal{N} = 4$ SYM theory because this derivation applies to any gauge theory with a gravity dual given by Einstein gravity coupled to matter fields, since in Einstein gravity the coupling constant for gravity is always given by Eq. (6.32). In this sense, this result is universal [554, 212, 552, 206] since it applies in the strong coupling and large-N_c limits to the large class of theories with a gravity dual, regardless of whether the theories are conformal or not, confining or not, supersymmetric or not and with or without chemical potential. In particular, if large-N_c QCD has a string theory dual, there should exist temperature ranges where its η/s is well approximated by $1/(4\pi)$ up to corrections due to the finiteness of the coupling. Even if large-N_c QCD does not have a string theory dual, Eq. (6.33) may still provide a reasonable approximation in certain temperature ranges since the universality of this result may be due to generic properties of strongly coupled theories (for example the absence of quasiparticles, see Section 6.3) which may not depend on whether they are dual to a gravitational theory.

The original calculation of η/s and the original demonstration of its universality were based on the relationship between the absorption cross-section σ for a graviton incident on a black D3-brane in the limit of zero graviton energy and the shear viscosity η [690, 552]. These authors showed that $\eta = \sigma/(16\pi G)$, with G being the ten-dimensional Newton constant. General results from black hole physics include $\sigma = A$, where A is the area of the black brane horizon, and $s = A/(4G)$. So one then finds $\eta/s = 1/(4\pi)$, namely (6.33). This derivation is intuitive and geometrical in the way that it relates dissipation in the gauge theory (η) to falling into a horizon in the dual gravitational description and in the way that it relates both η and s to A, thus giving an immediate sense of the universality of the result (6.33). However, the definition of σ requires considering scattering states in the asymptotically flat region of the D3-brane that lies beyond the $AdS_5 \times S^5$ region of the D3-brane where the physics of actual interest resides. The self-contained derivation that we have presented in full above refers only to physics in $AdS_5 \times S^5$ and, as we shall see, it generalizes immediately to the calculation of other transport coefficients.

The leading finite-coupling and finite-N_c corrections to Eq. (6.33) in $\mathcal{N} = 4$ SYM theory have been computed and are given by [215, 211, 210, 640]

$$\frac{\eta_{N_c,\lambda\to\infty}}{s_{N_c,\lambda\to\infty}} = \frac{1}{4\pi}\left(1 + \frac{15\,\zeta(3)}{\lambda^{3/2}} + \frac{5}{16}\,\frac{\lambda^{1/2}}{N_c^2} + \cdots\right). \tag{6.34}$$

The above equation is obtained by taking $N_c \to \infty$ first and then $\lambda \to \infty$. In this limit, the last term is always much smaller than the other terms. While the expression (6.34) is only valid as written for $\mathcal{N} = 4$ SYM theory, if the leading finite-λ correction proportional to $1/\lambda^{3/2}$ and the finite-N_c correction proportional to $\lambda^{1/2}/N_c^2$ are re-expressed instead in terms of the parameters R and l_s in the gravity theory, in this form the expression would then apply to a larger class of theories (those dual to compactifications of type IIB supergravity on various different five-dimensional manifolds) [216]. It is also important to stress that the correction proportional to $\lambda^{1/2}/N_c^2$ is not the full correction of order $1/N_c^2$ [640]. The prefactor in front of the order $1/N_c^2$ correction can be expanded in powers of λ, and the $\lambda^{1/2}/N_c^2$ term in (6.34) is the leading term in this expansion. The higher order terms have not yet been computed. It is interesting to notice that, according to Eq. (6.34) with N_c set to 3, η/s increases to $\sim 2/(4\pi)$ once λ decreases to $\lambda \sim 7$, meaning $\alpha_{\rm SYM} \sim 0.2$. This is the same range of couplings at which the finite coupling corrections (6.10) to thermodynamic quantities become significant. These results together suggest that strongly coupled theories with gravity duals may yield insight into the quark-gluon plasma in QCD even down to apparently rather small values of α_s, at which λ is still large.

To put the result (6.33) into further context, we can compare this strong coupling result to results for the same ratio η/s at weak coupling in both $\mathcal{N} = 4$ SYM theory and QCD. These have been computed at next to leading log accuracy, and take the form

$$\frac{\eta_{N_c,\lambda\to\infty}}{s_{N_c,\lambda\to\infty}} = \frac{A}{\lambda^2 \log\left(B/\sqrt{\lambda}\right)}\,, \tag{6.35}$$

with $A = 6.174$ and $B = 2.36$ in $\mathcal{N} = 4$ SYM theory and $A = 34.8$ (46.1) and $B = 4.67$ (4.17) in QCD with $N_f = 0$ ($N_f = 3$) [78, 474], where we have defined $\lambda = g^2 N_c$ in QCD as in $\mathcal{N} = 4$ SYM theory. Quite unlike the strong coupling result (6.33), these weak coupling results show a strong dependence on λ, and in fact diverge in the weak coupling limit. The divergence reflects the fact that a weakly coupled gauge theory plasma is a gas of quasiparticles, with strong dissipative effects. In a gas, η/s is proportional to the ratio of the mean free path of the quasiparticles to their average separation. A large mean free path, and hence a large η/s, mean that momentum can easily be transported over distances that are long compared to the average spacing between particles. In the $\lambda \to 0$ limit the mean free path diverges. The strong $1/\lambda^2$ dependence of η/s can be traced to the fact that the two-particle scattering cross-section is proportional to g^4. It is natural to guess that the λ-dependence of η/s in $\mathcal{N} = 4$ SYM theory is monotonic, increasing from $1/(4\pi)$ as in (6.34) as λ decreases from ∞ and then continuing to increase until it diverges according to (6.35) as $\lambda \to 0$. The weak coupling result (6.35) also illustrates a further important point: η/s is not universal for weakly coupled gauge theory plasmas. The coefficients A and B can vary significantly from one theory to another, depending on their particle content. It is only in the strong coupling limit that universality emerges, with all large-N_c theories with a gravity dual having plasmas with $\eta/s = 1/(4\pi)$. And, we shall see in Section 6.3 that a strongly coupled gauge theory plasma does not have quasiparticles, which makes it less surprising that η/s at strong coupling is independent of the particle content of the theory at weak coupling.

One lesson from the calculations of η/s is that this quantity changes significantly with the coupling constant, going from infinite at zero coupling to $1/(4\pi)$ at strong coupling, at least for large-N_c theories with gravity duals. This is in marked contrast to the behavior of the thermodynamic quantities described in Section 6.1, which change only by 25% over the same large range of couplings. Thermodynamic observables are insensitive to the coupling, whereas η/s is a much better indicator of the strength of the coupling because it is a measure of whether the plasma is liquid-like or gaseous.

These observations prompt us to revisit the phenomenological extraction of the shear viscosity of quark–gluon plasma in QCD from measurements of azimuthally anisotropic flow in heavy ion collisions, described in Section 2.2. As we saw, the comparison between data and calculations done using relativistic viscous hydrodynamic yields the current estimate that η/s seems to lie within the range $(1-2)/(4\pi)$ in QCD, in the same ballpark as the strong coupling result (6.33) And, as we reviewed in Section 3.2, current lattice calculations of η/s in $N_f = 0$ QCD come with caveats but also indicate a value that is in the ballpark of $1/(4\pi)$, likely somewhat above it. Given the sensitivity of η/s to the coupling, these comparisons constitute one of the main lines of evidence that, in the temperature regime accessible at RHIC and at the LHC, the quark–gluon plasma is a strongly coupled fluid. If we were to attempt to extrapolate the weak coupling result (6.35) for η/s in QCD with $N_f = 3$ to the values of η/s favored by experiment, we would need $\lambda \sim (14-24)$, well beyond the regime of applicability of perturbation theory. (To make this estimate we had to set the log in (6.35) to 1 to avoid negative numbers, which reflects the fact that the perturbative result is being applied outside its regime of validity.)

A central lesson from the strong coupling calculation of η/s via gauge/string duality, arguably even more significant than the qualitative agreement between the result (6.33) and current extractions of η/s from heavy ion collision data, is simply the fact that values of $\eta/s \ll 1$ are possible in non-Abelian gauge theories, and in particular in non-Abelian gauge theories whose thermodynamic observables are not far from weak coupling expectations. These calculations, done via gauge/string duality, provided theoretical support for considering a range of small values of η/s that had not been regarded as justified previously, and inferences drawn from RHIC data have now pushed η/s into this regime. The computation of the shear viscosity that we have just described is one of the most influential results supporting the notion that the application of gauge/string duality can yield insights into the phenomenology of hot QCD matter.

It has also been conjectured [552] that the value of η/s in Eq. (6.33) is, in fact, a lower bound for all systems in nature. This conjecture is supported by the finite-coupling corrections shown in Eq. (6.34). And, all substances known in the laboratory satisfy the bound. Among conventional liquids, the lowest η/s is achieved by liquid helium, but it is about an order of magnitude above $1/(4\pi)$; water – after which hydrodynamics is named – has an η/s that is larger still, by about another order of magnitude. The best liquids known in the laboratory are the quark–gluon plasma produced in heavy ion collisions and an ultracold gas of fermionic atoms at the unitary point, at which the s-wave atom–atom scattering length has been dialed to infinity [730], both of which have η/s in the ballpark of $1/(4\pi)$ but, according to current estimates, somewhat larger.

However, in recent years the conjecture that (6.33) is a lower bound on η/s has been questioned and counter-examples have been found among theories with gravity duals. As emphasized in Chapter 5, Einstein gravity in the dual gravitational description corresponds to the large-λ and large-N_c limit of the boundary gauge theory. When higher order corrections to Einstein gravity are included, which correspond to $1/\sqrt{\lambda}$ or $1/N_c$ corrections in the boundary gauge theory, Eq. (6.33) will no longer be universal. In particular, as pointed out in Refs. [197, 523] and generalized in Refs. [203, 204, 229, 373, 662, 737, 49, 205, 234], generic higher derivative corrections to Einstein gravity can violate the proposed bound. Eq. (6.30) indicates that η/s is smaller than Eq. (6.33) if the "effective" gravitational coupling for the h^x_y polarization at the horizon is *stronger* than the universal value (6.32) for Einstein gravity. Gauss–Bonnet gravity as discussed in Refs. [197, 196] is an example in which this occurs. There, the effective action for h^y_x has the form of Eq. (6.29) with the effective coupling $Q(r)$ at the horizon satisfying [197]

$$\frac{1}{Q(r_0)} = \frac{(1-4\,\lambda_{\rm GB})}{16\pi G_N}\,, \tag{6.36}$$

leading to

$$\frac{\eta}{s} = \frac{(1-4\,\lambda_{\rm GB})}{4\pi}\,, \tag{6.37}$$

where $\lambda_{\rm GB}$ is the coupling for the Gauss–Bonnet higher derivative term. Thus, for $\lambda_{\rm GB} > 0$ the graviton in this theory is more strongly coupled than that of Einstein gravity and the value of η/s is smaller than $1/4\pi$. In Ref. [523], an explicit gauge theory has been proposed whose gravity dual corresponds to $\lambda_{\rm GB} > 0$. (See Refs. [640, 217, 639, 741] for generalizations.) It is interesting to note that in all these examples the bound-violating gauge theory includes degrees of freedom in the fundamental representation. Indeed, in all the theories that these authors have investigated that contain fundamental matter, the presence of fundamental matter pushes η/s toward values below $1/(4\pi)$.

Despite not being a lower bound, the smallness of η/s in Eq. (6.33), the qualitative agreement between Eq. (6.33) and values obtained from heavy ion collisions, and the universality of the result (6.33) which applies to any gauge theory with a gravity dual in the large-N_c and strong coupling limits, are responsible for the great impact that this calculation done via gauge/string duality has had on our understanding of the properties of deconfined QCD matter.

As we have mentioned in Section 2.2, the determination of η/s in hot QCD matter by comparing data on azimuthally asymmetric heavy ion collisions and hydrodynamic calculations is rapidly improving, as theorists begin to use very new data on the damping of higher-order-than-two harmonics of the azimuthal asymmetry sourced by fluctuations. Looking ahead a few years, we anticipate that η/s will be sufficiently well understood that effort will then be spent on tightening constraints on its temperature dependence and on the values of other transport coefficients. Although it remains to be demonstrated, it is certainly possible that in a few years string theorists could be debating what the then well-determined value of η/s for the quark–gluon plasma of QCD is telling us about quantum gravity (finite $1/N_c^2$) and stringy (finite coupling) corrections in the as yet unknown dual description of QCD itself. Although current analyses of heavy ion collision data do not support this speculation, we can also muse about what would happen if η/s were to turn out to be lower than $1/(4\pi)$ in QCD. We would be asking what features of the gravitational physics dual to QCD, and indeed in QCD itself, yield this result. We can speculate that, if this were to happen, the culprit on the QCD side could be N_f/N_c, given the presence of fundamental matter in the presently known examples where $\eta/s < 1/(4\pi)$ and given that $N_f = N_c$ in the strongly coupled plasma of QCD. It is worth noting, though, that in the models of Ref. [607] η/s is unaffected by the presence of fundamental matter at order $\lambda N_f/N_c$, which is the leading order at which such effects might have arisen. The reduction in η/s that we described in (6.37), due to Gauss–Bonnet higher derivative terms in the dual gravitational theory and apparently related to the presence of fundamental matter in the gauge theory, comes in only at order N_f/N_c, with no enhancement by λ. It is difficult at present to do more than speculate, but perhaps in the strongly coupled plasma of theories that, like QCD, have $N_f \sim N_c$ any reduction in η/s attributable to the fundamental matter may turn out not to be large in magnitude. This story remains to be written, but it seems likely that as the phenomenological determination of η/s tightens in future, the gauge/string duality will turn data on the gauge side into insight on the string side, working in the opposite direction to that which motivates much of our book today.

6.2.3 Bulk viscosity

As we have discussed in Section 2.2, while the bulk viscosity ζ is very small in the QCD plasma at temperatures larger than 1.5–2 T_c, with ζ/s much smaller than

$1/4\pi$, ζ/s rises in the vicinity of T_c, a feature which can be important for heavy ion collisions. Since the plasma of a conformal theory has zero bulk viscosity, $\mathcal{N} = 4$ SYM theory is not a useful example to study the bulk viscosity of a strongly coupled plasma. However, the bulk viscosity has been calculated both in more sophisticated examples of the gauge/string duality in which the gauge theory is not conformal [134, 207, 209, 600, 213], as well as in AdS/QCD models that incorporate an increase in the bulk viscosity near a deconfinement phase transition [409, 404, 417].

We will only briefly review what is possibly the simplest among the first type of examples, the so-called Dp-brane theory. This is a $(p+1)$-dimensional cousin of $\mathcal{N} = 4$ SYM, namely a $(p+1)$-dimensional SYM theory (with 16 supercharges) living at the boundary of the geometry describing a large number of non-extremal black Dp-branes [487] with $p \neq 3$. The case $p = 3$ is $\mathcal{N} = 4$ SYM, while the cases $p = 2$ and $p = 4$ correspond to nonconformal theories in $(2+1)$- and $(4+1)$-dimensions. We emphasize that we choose this example for its simplicity rather than because it is directly relevant for phenomenology.

The metric sourced by a stack of black Dp-branes can be written as

$$ds^2 = \alpha' \frac{(d_p \tilde{\lambda} z^{3-p})^{\frac{1}{5-p}}}{z^2} \left(-\tilde{f} dt^2 + ds_p^2 + \left(\frac{2}{5-p}\right)^2 \frac{dz^2}{\tilde{f}} + z^2 d\Omega_{8-p}^2 \right), \quad (6.38)$$

where

$$\tilde{\lambda} = g^2 N, \qquad \tilde{f} = 1 - \left(\frac{z}{z_0}\right)^{\frac{14-2p}{5-p}}, \qquad d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma\left(\frac{7-p}{2}\right) \quad (6.39)$$

and

$$g^2 = (2\pi)^{p-2} g_s \alpha'^{\frac{3-p}{2}} \quad (6.40)$$

is the Yang–Mills coupling constant, which is dimensionful if $p \neq 3$. For $p = 2$ and $p = 4$ there is also a non-trivial profile for the dilaton field but we shall not give its explicit form here. The metric above is dual to $(p+1)$-dimensional SYM theory at finite temperature.

The bulk viscosity can be computed from the dual gravitational theory via the Kubo formula (A.10). However this computation is more complicated in the bulk channel than in the shear channel and we will not reproduce it here. An alternative and simpler way to compute the bulk viscosity is based on the fact that, in the hydrodynamic limit, the sound mode has the following dispersion relation:

$$\omega = c_s q - \frac{i}{\epsilon + p} \left(\frac{p-1}{p} \eta + \frac{\zeta}{2} \right) q^2 + \cdots, \quad (6.41)$$

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with c_s the speed of sound. Thus, ζ contributes to the damping of sound. In the field theory, the dispersion relation for the sound mode can be found by examining the poles of the retarded Green's function for the stress tensor in the sound channel. As discussed in Section 5.3.1, on the gravity side these poles correspond to normalizable solutions to the equations of motion for metric perturbations, which we will describe more explicitly in Section 6.4. The explicit computation of these normalizable modes for the metric (6.38) performed in Ref. [600] showed that the sound mode has the dispersion relation

$$\omega = \sqrt{\frac{5-p}{9-p}}\,q - i\,\frac{2}{9-p}\,\frac{q^2}{2\pi T} + \cdots \tag{6.42}$$

from which one finds that (after using $\eta/s = 1/(4\pi)$)

$$c_s = \sqrt{\frac{5-p}{9-p}}, \qquad \frac{\zeta}{s} = \frac{(3-p)^2}{2\pi p(9-p)}\,. \tag{6.43}$$

The above expressions imply an interesting relation [209]

$$\frac{\zeta}{\eta} = 2\left(\frac{1}{p} - c_s^2\right) = 2\left(c_{s,\mathrm{CFT}}^2 - c_s^2\right)\,, \tag{6.44}$$

where we have used the fact that the speed of sound for a CFT in $(p+1)$-dimension is $c_{s,\mathrm{CFT}} = 1/\sqrt{p}$. This result might not seem surprising since the bulk viscosity of a theory which is close to being conformal can be expanded in powers of $c_{s,\mathrm{CFT}}^2 - c_s^2$, which is a measure of deviation from conformality. The non-trivial result is that even though the Dp-brane gauge theories are *not* close to being conformal, their bulk viscosities are nevertheless linear in $c_{s,\mathrm{CFT}}^2 - c_s^2$. While this is an interesting observation, it is not clear to what extent it is particular to the Dp-brane gauge theories or whether it is more generic.

6.2.4 Relaxation times and other second order transport coefficients

As we have described in Section 2.2.3, transport coefficients correspond to the leading order gradient expansion of an interacting theory which corrects the ideal hydrodynamic description. *A priori*, there is no reason to stop the extraction of these coefficients at first order, and higher order ones can be (and have been) computed using gauge/string duality. Of particular importance is the determination of the five second order coefficients, τ_π, κ, λ_1, λ_2, λ_3 defined in Eq. (2.24). Unlike for the first order coefficients, the gravitational computation of these second order coefficients is quite technical and we shall not review it here. We shall only describe the main points and refer the reader to Refs. [107, 155] for details.

The strategy for determining these coefficients is complicated by the fact that the three coefficients λ_i involve only nonlinear combinations of the hydrodynamic fields. Thus, even though formulae can be derived for the linear coefficients τ_π and κ [107, 627], the nonlinear coefficients cannot be determined from two-point correlators, since these coefficients are invisible in the linear perturbation analysis of the background. Their determination thus demands the small gradient analysis of nonlinear solutions to the Einstein equations[3] as performed in Ref. [155] (see also Ref. [107]) which yields

$$\tau_\pi = \tfrac{2-\ln 2}{2\pi T}\,, \quad \kappa = \tfrac{\eta}{\pi T}\,, \qquad (6.45)$$
$$\lambda_1 = \frac{\eta}{2\pi T}\,, \qquad \lambda_2 = -\tfrac{\eta \ln 2}{\pi T}\,, \qquad \lambda_3 = 0\,.$$

These results are valid in the large-N_c and strong coupling limit. Finite coupling corrections to some of these coefficients can be found in Ref. [214]. Additionally, the first and second order coefficients have been studied in a large class of nonconformal theories with or without flavor in Refs. [162, 160].

To put these results in perspective we will compare them to those extracted in the weakly coupled limit of QCD ($\lambda \ll 1$) [812]. We shall not comment on the values of all the coefficients, since, as discussed in Section 2.2.3, the only one with any impact on current phenomenological applications to heavy ion collisions is the shear relaxation time τ_π. In the weak coupling limit,

$$\lim_{\lambda\to 0} \tau_\pi \simeq \frac{5.9}{T}\frac{\eta}{s}\,, \qquad (6.46)$$

where the result is expressed in such a way as to show that τ_π and η have the same leading order dependence on the coupling λ (up to logarithmic corrections). For comparison, the strong coupling result from (6.45) may be written as

$$\lim_{\lambda\gg 1} \tau_\pi = \frac{7.2}{T}\frac{\eta}{s}\,, \qquad (6.47)$$

which is remarkably close to (6.46). But, of course, the value of η/s is vastly different in the weak and strong coupling limits. On general grounds, one may expect that relaxation and equilibration processes are more efficient in the strong coupling limit, since they rely on the interactions between different modes in the medium. This general expectation is satisfied for the shear relaxation time of the $\mathcal{N} = 4$ SYM plasma, with τ_π diverging at weak coupling and taking on the small value

$$\lim_{\lambda\to\infty} \tau_\pi \simeq \frac{0.208}{T} \qquad (6.48)$$

in the strong coupling limit. For the temperatures $T > 200$ MeV, which are relevant for the quark–gluon plasma produced in heavy ion collisions, this relaxation

[3] Kubo-like formulas involving three-point correlators (as opposed to two) can also be used to determine the coefficients λ_i [627]. At the time of writing, this approach had not been explored within the gauge/gravity context.

time is of the order of 0.2 fm/c or smaller, which is much smaller than perturbative expectations. We have recalled already at other places in this book that caveats enter if one seeks quantitative guidance for heavy ion phenomenology on the basis of calculations made for $\mathcal{N} = 4$ SYM plasma. However, the qualitative (and even semi-quantitative) impact of the result (6.48) on heavy ion phenomenology should not be underestimated: the computation of τ_π demonstrated for the first time that at least some excitations in a strongly coupled non-Abelian plasma dissipate on timescales that are much shorter than $1/T$, i.e. on time scales much shorter than 1 fm/c. Such small relaxation time scales did not have any theoretical underpinning before, and they are clearly relevant for phenomenological studies based on viscous fluid dynamic simulations. As we discussed in Section 2.2, the success of the comparison of such simulations to heavy ion collision data implies that a hydrodynamic description of the matter produced in these collisions is valid only ~ 1 fm/c after the collision. Although this equilibration time is related to out-of-equilibrium dynamics, whereas τ_π is related to near-equilibrium dynamics (only to second order), the smallness of τ_π makes the rapid equilibration time seem less surprising. We shall return to this subject in Chapter 7, where we shall describe insights coming from holographic analyses of far-from-equilibrium dynamics that corroborate the conclusions that we have drawn here. As in the case of η/s, the gauge/gravity calculation of τ_π has made it legitimate to consider values of an important parameter that had not been considered before by showing that this regime arises in the strongly coupled plasma of a quantum field theory that happens to be accessible to reliable calculation because it possesses a gravity dual.

Let us conclude this section by mentioning that the second order transport coefficients are known for the same nonconformal gauge theories whose bulk viscosity we discussed in Section 6.2.3. Since conformal symmetry is broken in these models, there are a total of 15 first and second order transport coefficients, nine more than in the conformal case (including both shear and bulk viscosities in the counting) [715]. In addition, the velocity of sound c_s is a further independent parameter that characterizes the zeroth order hydrodynamics of nonconformal plasmas, whose equations of state are not given simply by $P = \varepsilon/3$. As for the case of the bulk viscosity, the variable $\left(\frac{1}{3} - c_s^2\right)$ can be used to parametrize deviations from conformality, and all transport coefficients can indeed be written explicitly as functions of $\left(\frac{1}{3} - c_s^2\right)$ [511].

6.2.5 Transport coefficients in charged plasmas, including those with quantum anomalies

So far, in the discussion of this section we have focused on the transport properties of non-Abelian plasmas with no conserved charges. Further transport coefficients

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become relevant if we wish to characterize the viscous hydrodynamics of charged plasmas. For example, one of the new coefficients that arises at first order in a derivative expansion is the (electric) conductivity. It characterizes how much of a conserved charge is transported in the presence of gradients in some chemical potential[4] or, if the conserved charge is associated with a gauged $U(1)$ symmetry, in response to external electric fields. Also, since the rest frames that are locally comoving with the charge density and the energy density can differ, the description of transport in charged plasmas requires the introduction of the heat conductivity[5] which characterizes the transport of energy density in response to a temperature gradient relative to the frame that is locally comoving with the charge density. There are also thermoelectric coefficients that describe the transport of charge density in the presence of a temperature gradient or the transport of energy density in the presence of an electric field or a gradient in some chemical potential.

The only one of these transport coefficients that has received some attention in the context of understanding the properties of quark–gluon plasma is the electric conductivity σ, which can in principle be determined from lattice calculations, albeit calculations that face all the difficulties that, as we have seen in Section 3.2, are associated with constraining Minkowski space spectral functions and transport coefficients from Euclidean calculations. Current lattice calculations indicate that the electric conductivity of quark–gluon plasma in the quenched limit in which the $N_f = 3$ quarks are arbitrarily heavy lies in the range [323]

$$\frac{1}{9} \lesssim \frac{\sigma_{\rm electric}^{\rm QGP,\ quenched}}{2\, e^2\, T} \lesssim \frac{1}{3} \tag{6.49}$$

at $T \simeq 1.45 T_c$, where $e^2 = 4\pi/137$ is the square of the electromagnetic coupling constant and where the sum of the squares of the electric charges of the quarks is given by $\frac{2}{3} e^2 N_c = 2e^2$. The calculation in Ref. [323] was done only at one temperature but more recent calculations at one other temperature [324] support the expectation that σ is proportional to T. And, the first attempt to determine σ for a quark–gluon plasma containing light quarks (i.e. without making the quenched approximation) yields an estimate that falls within the range (6.49) [191, 192].

The authors of Ref. [239] have shown how to obtain an analog of the electric conductivity for the strongly coupled plasma of $\mathcal{N} = 4$ SYM theory by gauging a $U(1)$ subgroup of the (otherwise global) $SU(4)$ R-symmetry of the theory. In

[4] A chemical potential is an intensive thermodynamic variable which, like pressure or temperature or energy density, varies as a function of space and time in a hydrodynamic fluid. Gradients in a chemical potential drive flows of the corresponding conserved particle number. The chemical potential or the temperature at any point in a moving fluid is the same as the chemical potential or the temperature of an external bath in equilibrium with a static homogeneous fluid with the same values of all intensive thermodynamic variables.

[5] The presence of a charge density is a necessary condition for the introduction of a heat conductivity only in homogeneous and isotropic fluids. In more complicated situations, heat transport may occur in a fluid in the absence of any charge density. We will not discuss such cases in this book.

particular, they have chosen a $U(1)$ subgroup such that the sum of the squares of the charges is $2e^2$ in $\mathcal{N} = 4$ SYM theory with $N_c = 3$, as in QCD with $N_c = N_f = 3$. (It is worth noting, though, that in QCD the electric charge is carried entirely by fields that are in the fundamental representation of the gauge group while in $\mathcal{N} = 4$ SYM the R-charge is carried entirely by fields that are in the adjoint representation of the gauge group.) With an analog of electromagnetism defined, the computation of the conductivity σ then proceeds along the lines of the holographic calculations of other transport coefficients that we have described in Section 6.2.1 since σ is obtained from the zero-frequency limit of the current–current correlator at vanishing three-momentum. The authors of Ref. [239] obtain

$$\frac{\sigma_R^{\mathcal{N}=4\ \mathrm{SYM}}}{2\,e^2\,T} = \frac{N_c^2}{32\pi}\,, \tag{6.50}$$

which for $N_c = 3$ lies just below the range (6.49). We also note that the authors of Ref. [604] have shown how to gauge a $U(1)$ symmetry whose charge is carried only by fundamental degrees of freedom in a model in which $N_f \ll N_c$ flavors of fundamental matter, with the sum of the squares of their charges given by $e^2 N_f N_c$, have been added to the $\mathcal{N} = 4$ SYM plasma. They have calculated the conductivity in this case, finding

$$\frac{\sigma_{\mathrm{fundamental}}^{\mathcal{N}=4\ \mathrm{SYM}}}{e^2\,N_c N_f\,T} = \frac{1}{4\pi}\,, \tag{6.51}$$

which again lies just below the range (6.49). We shall return to this model at some length in Chapter 9. Although it is not clear how best to make the comparison between these theories and QCD, perhaps these results indicate that the quark–gluon plasma of QCD is not quite as strongly coupled as the $\mathcal{N} = 4$ SYM plasma in the infinite coupling limit.

The transport coefficients involving a temperature gradient or an energy current or both have received less attention in the QCD context but, motivated by considerations from condensed matter physics, they have been calculated holographically in Ref. [427].

For the rest of this section we shall focus on a particularly interesting class of charged plasmas, namely those with quantum anomalies. Such systems have been studied using the techniques of gauge/gravity duality [341, 117, 750], and these calculations illustrate how the first order dissipative hydrodynamics of non-Abelian plasmas in theories with anomalies features novel transport coefficients that are not present in traditional textbook presentations of hydrodynamics like that of Ref. [567]. We hasten to remark that the special role of quantum anomalies in hydrodynamics was observed already in Refs. [791], and related phenomena involving parity violating currents in the presence of rotation or in a magnetic field, which we will refer to below as the chiral vortical and magnetic effects, were discovered even earlier in the pioneering work of Refs. [787, 789, 788]. However, the recent rediscovery of how anomalies influence hydrodynamic flows using the techniques of gauge/gravity duality has led to a deeper and more systematic understanding of how quantum anomalies can have macroscopic consequences at length scales much larger than any mean free path or any other microscopic length scale. In this sense, what we discuss in the following is another case where gauge/gravity duality has contributed important qualitative insights into the behavior of non-Abelian plasmas.

To be specific, consider a system with one global $U(1)$ symmetry which is anomalous. As discussed in Section 5.1.4, the boundary $U(1)$ symmetry is mapped to a $U(1)$ gauge symmetry in the bulk with the boundary $U(1)$ current J^μ mapped to a bulk gauge field A_M. That the boundary $U(1)$ symmetry is anomalous is reflected on the gravity side through the presence of a Chern–Simons term, the coefficient of which determines the anomaly coefficient. In Chapter 7, we shall explain in detail how the hydrodynamics of a neutral fluid can be derived in a derivative expansion of Einstein's equations for AdS_5. The techniques described there can be generalized to a charged fluid with a quantum anomaly by finding long wavelength solutions to an Einstein–Maxwell–Chern–Simons theory in AdS_5. In contrast with the derivative expansion of the equations of motion discussed in Section 7.2.1, such a calculation incorporates variations in space and time of not only $T(x^\mu)$ and $u^\mu(x^\nu)$ but also of the chemical potential $\mu(x^\mu)$ corresponding to the anomalous global charge. One finds that up to first order in the derivative expansion, the anomalous charge current $j^\mu \equiv \langle J^\mu \rangle$ can be written in the form [341, 117, 750]

$$j^\mu = \rho u^\mu - \sigma T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \xi \omega^\mu , \tag{6.52}$$

where ρ is the charge density and σ is the charge conductivity that appears at first order in a derivative expansion. The last term implies a contribution to the current that is directed parallel to, and is induced by, the vorticity $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho$. This is called the chiral vortical effect. For non-anomalous currents, such a term is forbidden by the second law of thermodynamics, and that is the reason why it does not appear in traditional textbooks on hydrodynamics [750]. However, such currents can in fact arise in rotating systems [787, 789, 788, 791, 792]. It should therefore not have been a surprise when such a term was found in the hydrodynamics of the charged fluid described above [341, 117] and, as argued most generally in the analysis of these calculations in Ref. [750], such a term must be present if the current in question corresponds to an anomalous global symmetry. More precisely, if the anomaly of J^μ is given by

$$\partial_\mu J^\mu = -\frac{1}{8} C \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\alpha\beta} , \tag{6.53}$$

with F the field strength of an external gauge field coupled to the current J^μ itself, then the new transport coefficient ξ entering (6.52) is completely determined by the anomaly coefficient C and is given in the simplest case by [750]

$$\xi = C\left(\mu^2 - \frac{2}{3}\frac{\mu^3\rho}{\varepsilon + P}\right) . \tag{6.54}$$

We note that, in addition to the contribution written here, if the $U(1)$ current features a mixed gravitational anomaly, i.e. if there is an additional term on the right-hand side of (6.53) proportional to $\epsilon^{\mu\nu\lambda\rho}R^{\alpha}_{\beta\mu\nu}R^{\beta}_{\alpha\lambda\rho}$ with R the Riemann tensor, then ξ includes a term proportional to T^2 that is present even when $\mu = 0$, as seen in quantum field theoretical derivations of the chiral vortical effect [791, 569, 568, 381, 570, 497] as well as in derivations of the effect from kinetic transport theory [366].

The specific instance of a charged non-Abelian plasma that we have discussed above provides a good illustration of the generic relation between a quantum anomaly and the vorticity-induced contribution to the corresponding current that it induces, namely the chiral vortical effect. Further qualitatively new and interesting effects are seen if one considers such charged plasmas embedded in an external field coupling to the current. In classical textbook presentations of hydrodynamics, the current (6.52) will acquire in an external electromagnetic field a term proportional to the electric field strength $E^\mu \equiv F^{\mu\nu}u_\nu$. The proportionality constant in front of E_μ is not an independent transport coefficient; it is the same charge conductivity σ that determines the magnitude of the electric current that flows in the presence of gradients in the chemical potential. In addition, in the presence of quantum anomalies there is also a contribution to the current (6.52) that is proportional to the magnetic field strength $B^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$ denoted by $\xi_B\, B^\mu$ with

$$\xi_B = C\left(\mu - \frac{1}{2}\frac{\mu^2\rho}{\varepsilon + P}\right) , \tag{6.55}$$

meaning that ξ_B is again proportional to the strength C of the quantum anomaly. This means that a quantum anomaly can induce an electric current in the direction of an applied external magnetic field. This is called the chiral magnetic effect.

In order to apply these ideas to the QCD plasma they must be generalized because in these applications the electric and magnetic field strengths of interest are those of ordinary electromagnetism, whose gauge field couples to the non-anomalous vector (i.e. electric) current $J^{V\mu}$, not to the anomalous axial current $J^{A\mu}$. In Ref. [750], the analysis is generalized even further to a theory in which there are arbitrarily many $U(1)$ currents, some or all of which are anomalous, with the anomaly equation (6.53) replaced by

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$$\partial_\mu J^{a\mu} = -\frac{1}{8} C^{abc} \epsilon^{\mu\nu\lambda\rho} F^b_{\mu\nu} F^c_{\lambda\rho}\,, \tag{6.56}$$

where the different currents are enumerated by a, b and c and where C^{abc} is symmetric under permutations of its indices. In this context, the chiral vortical effect for each of the currents is controlled by a coefficient ξ^a given by a generalization of (6.54):

$$\xi^a = C^{abc} \mu^b \mu^c - \frac{2}{3} \rho^a C^{bcd} \frac{\mu^b \mu^c \mu^d}{\varepsilon + P}\,. \tag{6.57}$$

And, in the presence of a magnetic field for the bth $U(1)$ the current $J^{a\mu}$ for the ath $U(1)$ receives a contribution $\xi_B^{ab} B^{b\mu}$ with

$$\xi_B^{ab} = C^{abc} \mu^c - \frac{1}{2} \rho^a C^{bcd} \frac{\mu^c \mu^d}{\varepsilon + P}, \tag{6.58}$$

which is the generalization of (6.55). If we now specialize to the case that is relevant for QCD, we have two $U(1)$ currents, $J^{V\mu}$ and $J^{A\mu}$, the only nonzero anomaly coefficients are C^{AVV} and permutations, with $C^{AVV} = N_c e^2/(2\pi^2)$, and $C^{AAA} = C^{AVV}/3$ (see, e.g., [488, 684]), and the only magnetic field strength of interest to us is $B^{V\mu}$. In QCD, therefore, the chiral vortical effect coefficients are

$$\xi^A = C^{AVV} \mu^V \mu^V + C^{AAA} \mu^A \mu^A - \frac{2}{3} \rho^A \frac{3C^{AVV} \mu^A \mu^V \mu^V + C^{AAA} \mu^A \mu^A \mu^A}{\varepsilon + P} \tag{6.59}$$

and

$$\xi^V = 2C^{AVV} \mu^V \mu^A - \frac{2}{3} \rho^V \frac{3C^{AVV} \mu^A \mu^V \mu^V + C^{AAA} \mu^A \mu^A \mu^A}{\varepsilon + P}\,. \tag{6.60}$$

Note that (6.59) reduces to (6.54) if $\mu^V = 0$, as it should. The chiral magnetic effect coefficients are

$$\xi_B^{AV} = C^{AVV} \mu^V - \rho^A C^{AVV} \frac{\mu^A \mu^V}{\varepsilon + P} \tag{6.61}$$

and

$$\xi_B^{VV} = C^{AVV} \mu^A - \rho^V C^{AVV} \frac{\mu^A \mu^V}{\varepsilon + P} \tag{6.62}$$

in QCD. Because the derivation of the chiral vortical effect does not require a gauge field coupled to $J^{V\mu}$, in (6.59) and (6.60) the vector current can be taken to be either the baryon number current or the electric current, meaning that μ^V could be either μ_B or the chemical potential for electric charge, which is to say the electrostatic potential. In (6.61) and (6.62), μ_V is the electrostatic potential.

We note that both the chiral vortical and the chiral magnetic effects are of potential phenomenological interest. For example, the first term on the right-hand side of

(6.59) tells us that in a rotating lump of cold dense quark matter in which $\mu_B > 0$, as may be found within the core of a neutron star, an axial current will develop along the rotation axis, meaning that quarks with opposite chirality will move in opposite directions, parallel and antiparallel to the rotation vector. In this way, an anomalous current in the direction of the rotation vector will be induced. Similarly, in a region of quark–gluon plasma in which there is an external magnetic field, for example sourced by the positively charged spectators in a heavy ion collision with nonzero impact parameter, the first terms on the right-hand sides of (6.61) and (6.62) both have striking implications. From (6.62) we see that in a region of the plasma that is in a magnetic field and in which the density of axial charge happens to be nonzero there will be a tendency toward developing an electric current parallel (or antiparallel, depending on the sign of the axial charge density) to the magnetic field, with positively charged and negatively charged particles moving in opposite directions, parallel and antiparallel to the magnetic field [527, 526, 528, 365, 467]. And, from (6.61) we see that in a region of the plasma that is in a magnetic field and in which the density of electric charge happens to be nonzero there will be a tendency toward developing an axial current parallel (or antiparallel) to the magnetic field, with quarks with opposite chirality moving in opposite directions, parallel and antiparallel to the magnetic field [222, 220]. The observable consequences of these anomalous transport phenomena are currently under active investigation and, although various authors have employed gauge/string duality in their investigations of the chiral magnetic effect, because the phenomenological side of the story is still being written we will not present it in this book.

In summary, this discussion of charged non-Abelian plasmas with anomalous currents illustrates that beyond the by now rather complete understanding of the effects and importance of shear viscosity in non-Abelian plasmas, there are a significant number of phenomenologically relevant transport properties to which studies based on gauge/gravity duality are likely to contribute further in the coming years.

6.3 Quasiparticles and spectral functions

In Sections 6.1 and 6.2 we have illustrated the power of gauge/string duality by performing, in a remarkably simple way, computations that via standard field theoretical methods either take teraflop-years of computer time or are not accessible. However, to someone familiar with gauge theory calculations in other contexts it may seem that the surprising simplicity of the calculations we have done comes with a price. Because we do the calculations in the dual gravitational description of the theory, the reliable results that we obtain are not accompanied by the kinds of intuition about what is happening in the gauge theory that we would get automatically from a field theory calculation done with Feynman diagrams or could get with effort from one done on the lattice. The gravitational calculation yields answers, and new kinds of intuition, but since by using it we are abandoning the description of the plasma in terms of quark and gluon quasiparticles interacting with each other, we are losing our prior sense of how the dynamics of the gauge theory works. There are two salient responses to this reaction. First, any description based upon Feynman diagrams and interacting quarks and gluons was inherently weakly coupled, meaning that once we discover that the quark–gluon plasma produced in heavy ion collisions is a strongly coupled liquid we must abandon our prior intuition. In this sense, the price referred to above is one that we must pay whether or not we explore calculations done via the dual gravitational description. Second, as we have already begun to see and as we will see again and again throughout the remainder of this book, the new intuition that comes from the gravitational calculations, intuition based upon strings and horizons and metric perturbations and such, is extraordinarily powerful as a source of insights into strongly coupled, liquid, plasma. A reasonable skeptic, however, may still ask whether the liquid that we are describing via the new gravitational language could in fact also be described on the gauge theory side in familiar terms. In other words, is the dynamics within a strongly coupled plasma different in a qualitative way from that in a weakly coupled plasma, or does it merely differ quantitatively? We have given up the description in terms of quasiparticles, but maybe the familiar quasiparticles or some new kind of quasiparticles are in fact nevertheless present and, without our knowing it, are what the gravitational dual is describing. We rule out this possibility in this section, illustrating that a strongly coupled non-Abelian gauge theory plasma really is qualitatively different from a weakly coupled one: while in perturbation theory the degrees of freedom of the plasma are long-lived quasiparticle excitations which carry momentum, color and flavor, there are no quasiparticles in the strongly coupled plasma. The pictures that frame how we think about a weakly coupled plasma are simply invalid for the strongly coupled case.

Determining whether a theory possesses quasiparticles with a given set of quantum numbers is a conceptually well defined task: it suffices to analyze the spectral function of operators with that set of quantum numbers and look for narrow peaks in momentum space. In weakly coupled Yang-Mills theories, the quasiparticles (gluons and quarks in QCD) are colored and are identified by studying operators that are *not* gauge invariant. Within perturbation theory, it can be shown that the poles of these correlators, which determine the physical properties of the quasiparticles, are gauge invariant [186]. However, nonperturbative gauge-invariant operators corresponding to these excitations are not known, which complicates the search for these quasiparticles at strong coupling. Note, however, that even if such operators were known, demonstrating the absence of quasiparticles with the same quantum numbers as in the perturbative limit does not guarantee the absence of quasiparticles, since at strong coupling the system could reorganize itself into quasiparticles with different quantum numbers. Thus, proving the absence of quasiparticles along these lines would require exploring all possible spectral functions in the theory. Fortunately, there is an indirect method which can answer the question of whether any quasiparticles that carry some conserved "charge" (including momentum) exist, although this method cannot determine the quantum numbers of the long-lived excitations if any are found to exist. The method involves the analysis of the small frequency structure of the spectral functions of those conserved currents of the theory which do not describe a propagating hydrodynamic mode like sound. As we will see, the presence of quasiparticles leads to a narrow structure (the transport peak) in these spectral functions [10, 777]. In what follows we will use this method to demonstrate that the strongly coupled $\mathcal{N} = 4$ SYM plasma does not possess any colored quasiparticles that carry momentum. In order to understand how the method works, we first apply it at weak coupling where there are quasiparticles to find.

6.3.1 Quasiparticles in perturbation theory

We start our analysis by using kinetic theory to predict the general features of the low frequency structure of correlators of conserved currents in a weakly coupled plasma. Kinetic theory is governed by the Boltzmann equation, which describes excitations of a quasiparticle system at scales which are long compared to the inter-particle separation. The applicability of the kinetic description demands that there is a separation of scales such that the duration of interactions among particles is short compared to their mean free path ($\lambda_{\rm mfp}$) and that multiparticle distributions are consequently determined by the single particle distributions. In Yang–Mills theories at nonzero temperature and weak coupling, kinetic theory is important since it coincides with the Hard Thermal Loop description [439, 187, 360, 774, 524, 167], which is the effective field theory for physics at momentum scales of order gT, and the Boltzmann equation can be derived from first principles [167, 232, 233, 168, 169, 77]. In Yang–Mills theory at weak coupling and nonzero temperature, the necessary separation of scales arises by virtue of the small coupling constant g, since $\lambda_{\rm mfp} \sim 1/(g^4 T)$ and the time scale of interactions is $1/\mu_D \sim 1/(gT)$, where $1/\mu_D$ is the Debye screening length of the plasma.[6] The small value of the coupling constant also leads to the factorization of higher-point correlation functions.

In the kinetic description, the system is characterized by a distribution function

$$f(x, \mathbf{p}) \,, \tag{6.63}$$

[6] Strictly speaking, $\lambda_{\rm mfp} \sim 1/(g^4 T)$ is the length-scale over which an order 1 change of the momentum-vector of the quasiparticles occurs. Over the shorter length scale $1/\mu_D$, soft exchanges (of order gT; not enough to change the momenta which are $\sim T$ significantly) occur. These soft exchanges are not relevant for transport.

which determines the number of particles of momentum $\mathbf{p}$ at spacetime position x. Note that this position should be understood as the center of a region in spacetime with a typical size much larger, at least, than the de Broglie wavelength of the particles, as demanded by the uncertainty principle. As a consequence, the Fourier transform of x, which we shall denote by $K = (\omega, \mathbf{q})$, must be much smaller than the typical momentum scale of the particles, $K \ll |\mathbf{p}| \sim T$. (Here and below, when we write a criterion like $K \ll |\mathbf{p}|$ we mean that both ω and $|\mathbf{q}|$ must be $\ll |\mathbf{p}|$.) Owing to this separation in momentum scales, the x-dependence of the distribution functions is said to describe the soft modes of the gauge theory while the momenta $\mathbf{p}$ are those of the hard modes. If K is sufficiently small (smaller than the inverse inter-particle separation $\sim T$), the mode with four-momentum K is a collective excitation that involves the motion of many particles, while $\mathbf{p}$ is the momentum of those particles. In this case, the Fourier-transformed distribution $f(K, \mathbf{p})$ can be interpreted approximately as the number of particles within the wavelength of the excitation. At the long distances at which the kinetic theory description is valid, particles are on mass shell, as determined by the position of the peaks in the correlation functions of the relevant operators ($p^0 = E_{\mathbf{p}}$), and these hard modes describe particles that follow classical trajectories, at least between the microscopic collisions. All the properties of the system can be extracted from the distribution function. In particular, the stress tensor is given by

$$T^{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p} f(x, \mathbf{p})\,. \tag{6.64}$$

Since all quasiparticles carry energy and momentum, we will concentrate only on the kinetic theory description of stress tensor correlators. Our analysis is analogous to the one performed for the determination of the Green–Kubo formulae in Appendix A, and proceeds by studying the response of the system to small metric fluctuations. The dynamics are, then, governed by the Boltzmann equation which states the continuity of the distribution function f up to particle collisions [578]:

$$E_p \frac{d}{dt} f(x, \mathbf{p}) = p^\mu \partial_{x^\mu} f(x, \mathbf{p}) + E_p \frac{d\mathbf{p}}{dt} \frac{\partial}{\partial \mathbf{p}} f(x, \mathbf{p}) = \mathcal{C}\left[f\right]\,, \tag{6.65}$$

where $\mathcal{C}\left[f\right]$ is the collision term which encodes the microscopic collisions among the plasma constituents and vanishes for the equilibrium distribution $f_{\rm eq}(E_p)$ (which does not depend on x and which does not depend on the direction of $\mathbf{p}$). In writing (6.65), we are assuming that $\mathbf{p} = E_p\, \mathbf{v}_p$, where $\mathbf{v}_p$ is the velocity of the particle. In curved space, in the absence of external forces, the Boltzmann equation becomes

$$p^\mu \partial_{x^\mu} f(x, \mathbf{p}) - \Gamma^\lambda_{\mu\nu} p^\mu p^\nu \partial_{p^\lambda} f(x, \mathbf{p}) = \mathcal{C}\left[f\right]\,, \tag{6.66}$$

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where $\Gamma^{\lambda}_{\mu\nu}$ are the Christoffel symbols of the background metric. As in Appendix A, we shall determine the stress tensor correlator by introducing a perturbation in which the metric deviates from flat space by a small amount, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and studying the response of the system. Even though the analysis for a generic perturbation can be performed, it will suffice for our purposes to restrict ourselves to fluctuations which in Fourier space have only one non-vanishing component $h_{xy}(K)$. We choose the directions x and y perpendicular to the wave vector $\mathbf{q}$, which lies in the z direction. For this metric, the only Christoffel symbols that are non-vanishing at leading order in h_{xy} are $\Gamma^{t}_{xy} = \Gamma^{x}_{ty} = \Gamma^{y}_{tx} = -i\omega h_{xy}/2$ and $\Gamma^{x}_{zy} = \Gamma^{y}_{zy} = -\Gamma^{z}_{xy} = iq\, h_{xy}/2$.

We will assume that prior to the perturbation the system is in equilibrium. In response to the external disturbance the equilibrium distribution changes

$$f(x,\mathbf{p}) = f_{\rm eq}(E_p) + \delta f(x,\mathbf{p})\,. \tag{6.67}$$

In the limit of a small perturbation, the modified distribution function $\delta f(x,\mathbf{p})$ is linear in the perturbation h_{xy}. We will also assume that the theory is rotationally invariant so that the energy of the particle E_p is only a function of the modulus of $p^2 = g_{ij}p^i p^j$. As a consequence, the metric perturbation also changes the on-shell relation, and the equilibrium distribution must also be expanded to first order in the perturbation, yielding

$$f_{\rm eq} = f_0 + f_0' p^x p^y \frac{|v_p|}{p} h_{xy} \approx f_0 + f_0' \frac{p^x p^y}{E_p} h_{xy}\,, \tag{6.68}$$

where f_0 is the equilibrium distribution in flat space, $f_0'(E) = df(E)/dE$, and the velocity is given by $v_p = dE_p/dp$. In the last equality we have again approximated $v_p \approx p/E_p$.

The solution of the Boltzmann equation requires the computation of the collision term $\mathcal{C}$. In general this is a very complicated task since it takes into account the interactions among all the system constituents, which are responsible for maintaining equilibrium. However, since our only goal is to understand generic features of the spectral function, it will be sufficient to employ the relaxation time approximation

$$\mathcal{C} = -E_p \frac{f - f_{\rm eq}}{\tau_R} \tag{6.69}$$

for the collision term, in which the parameter τ_R is referred to as the relaxation time.[7] Since small perturbations away from equilibrium are driven back to equilibrium by particle collisions, the relaxation time must be of the order of the mean

[7] In this approximation, this relaxation time coincides with the shear relaxation time: $\tau_R = \tau_\pi$ [106]. However, since τ_π is a property of the theory itself (defined as the appropriate coefficient in the effective field theory, also known as the hydrodynamic expansion) whereas τ_R is a parameter specifying a simplified approximation to the collision kernel, which in general is not of the form Eq. (6.69), we will maintain the notational distinction between τ_π and τ_R.

free path $\lambda_{\rm mfp}$ (which is long compared to the inter-particle distance). The relaxation time approximation is a very significant simplification of the full dynamics, but it will allow us to illustrate the main points that we wish to make. A complete analysis of the collision term within perturbation theory for the purpose of extracting the transport coefficients of a weakly coupled plasma can be found in Refs. [75, 78, 459, 812].

Within the approximation (6.69), upon taking into account that the distribution function in Eq. (6.66) depends on the energy of the particles only through their spatial momenta, the solution to the linearized Boltzmann equation is given by

$$\delta f(K, \mathbf{p}) = \frac{-i\omega p^x p^y f_0'(p)}{-i\omega + i\mathbf{v}_p\mathbf{q} + \frac{1}{\tau_R}} \frac{h_{xy}(K)}{E_p} . \tag{6.70}$$

Substituting this into Eq. (6.64) we learn that the perturbation of the distribution function leads to a perturbation of the stress tensor given by

$$\delta T^{\mu\nu}(K) = \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p} \delta f(K, \mathbf{p}) = -G_R^{xy,xy}(K) h_{xy}(K) , \tag{6.71}$$

where the retarded correlator is given by

$$G_R^{xy,xy}(K) = -\int \frac{d^3p}{(2\pi)^3} v^x v^y \frac{\omega\, p^x p^y\, f_0'(p)}{\omega - \mathbf{q}\mathbf{v}_p + \frac{i}{\tau_R}} . \tag{6.72}$$

From the definition (3.13), the spectral function associated with this correlator is

$$\rho^{xy,xy}(K) = -\omega \int \frac{d^3p}{(2\pi)^3} \frac{(p^x p^y)^2}{E_p^2} f_0'(p) \frac{\frac{2}{\tau_R}}{(\omega - \mathbf{q}\mathbf{v}_p)^2 + \frac{1}{\tau_R^2}} . \tag{6.73}$$

Obtaining this spectral function was our goal, because as we shall now see it has qualitative features that indicate the presence (in this weakly coupled plasma) of quasiparticles.

To clarify the structure of the spectral function (6.73) we begin by describing the free theory limit, in which $\tau_R \to \infty$ since the collision term vanishes. In this limit, the Lorentzian may be replaced by a δ-function, yielding

$$\rho^{xy,xy}(K) = -\omega \int \frac{d^3p}{(2\pi)^3} \frac{(p^x p^y)^2}{E_p^2} f_0'(p)\, 2\pi\, \delta\left(\omega - \mathbf{q}\cdot\mathbf{v}_p\right) . \tag{6.74}$$

The δ-function arises because, in this limit, the external perturbation (the gravity wave) interacts with free particles. The δ-function encodes the conservation of the energy of the free particles in the plasma that absorb the energy and momentum of the gravity wave. Thus, in the free theory limit, this δ-function encodes the existence of free particles in the plasma. For an isotropic distribution of particles, such as the thermal distribution, at any $\mathbf{q} \neq 0$ the integration over angles washes out the δ-function and one is left with some function of ω that is characterized by

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the typical momentum scale of the particles ($\sim T$) and that is not of interest to us here.[8] On the other hand, at $\mathbf{q} = 0$ we find that $\frac{1}{\omega}\rho^{xy,xy}(\omega, 0)$ is proportional to $\delta(\omega)$. This δ-function at $\omega = 0$ in the low momentum spectral function is a direct consequence of the presence of free particles in the plasma. As we now discuss, the effect of weak interactions is to dress the particles into quasiparticles and to broaden the δ-function into a narrow, tall, peak at $\omega = 0$.

When the interactions do not vanish, we can proceed by relating the relaxation time to the shear viscosity. To do so, we work in the hydrodynamic limit in which all momenta must be smaller than any internal scale. This means that we can set $\mathbf{q}$ to zero, but we must keep the relaxation time τ_R finite. The spectral density at zero momentum is then given by

$$\rho^{xy,xy}(\omega, 0) = -\omega \int \frac{d^3p}{(2\pi)^3} \frac{(p^x p^y)^2}{E_p^2} f_0'(p) \frac{\frac{2}{\tau_R}}{\omega^2 + \frac{1}{\tau_R^2}} . \tag{6.75}$$

Note that the spectral density at zero momentum has a peak at $\omega = 0$, and note in particular that the width in ω of this peak is $\sim 1/\tau_R \ll T$. The spectral density has vanishing strength for $\omega \gg 1/\tau_R$. This low frequency structure in the zero-momentum spectral function is called the "transport peak". It is clear that in the $\tau_R \to \infty$ limit it becomes the δ-function that characterizes the spectral density of the free theory that we described above. Here, in the presence of weak interactions, this peak at $\omega = 0$ is a direct consequence of the presence of momentum-carrying quasiparticles whose mean free time is $\sim \tau_R$.

The expression (6.75) is only valid for $\omega \ll T$ where the modes are correctly described by the Boltzmann equation. For $\omega \gg T$, since the quasiparticles can be resolved, the structure of the spectral density is close to that in vacuum. The separation of scales in the spectral density is directly inherited from the separation of scales which allows the Boltzmann description. Finally, using the Green–Kubo formula for the shear viscosity (A.9), we find

$$\eta = -\tau_R \int \frac{d^3p}{(2\pi)^3} \frac{(p^x p^y)^2}{E_p^2} f_0'(p) . \tag{6.76}$$

Thus, since η is determined by the collisions among the quasiparticles, we can understand $1/\tau_R$ as the width that arises because the quasiparticles do not have well-defined momenta due to the collisions among them. In particular, in perturbation theory [75, 78]

[8] A distinct peak at in the spectral density at some $\omega \neq 0$ could be observed if the initial distribution were very anisotropic. This can arise if the theory has a (gauged) conserved charge and if the system is analyzed in the presence of a constant force that acts on this charge – i.e. an electric field.

$$\frac{1}{\tau_R} \sim \frac{1}{T} g^4 \ln \frac{1}{g} \sim \frac{1}{\lambda_{\rm mfp}} . \tag{6.77}$$

However, independent of the value of the weak coupling or the details of the underlying theory, Eq. (6.76) shows that the presence of quasiparticles in the system, as assumed in kinetic theory, imposes a strong relation between the shear viscosity of the plasma and the width of the transport peak. This relation can be further simplified by assuming that the quasiparticles of the system are massless, which leads to the conformal equation of state with pressure $p = \epsilon/3$. Recalling that for a weakly coupled plasma in equilibrium $\int d^3p \; p f_0(p) = 6\pi^3 Ts$, we see that Eq. (6.76) can be recast as

$$\tau_R = \frac{5}{T}\frac{\eta}{s} . \tag{6.78}$$

While this relation is based on an oversimplified relaxation time approach, a more complete perturbative computation, which takes into account the explicit form of the interaction kernel as well as the thermal mass corrections to the equilibrium distributions, leads, at most, to a 20% correction of this result, as we have quoted in Eq. (6.46) [812].

Let us summarize the main points. The zero-momentum spectral densities of a plasma with quasiparticles have a completely distinctive structure: there is a separation of scales between the scale T (the typical momentum of the quasiparticles in the plasma) and the much lower scale $1/\lambda_{\rm mfp}$. In particular, there is a narrow peak in $\rho(\omega, 0)/\omega$ around $\omega = 0$ of width $\tau_R \sim 1/\lambda_{\rm mfp}$ and height 2η. At larger frequencies, the strength of the spectral function is very small. At the scale of the mass of the quasiparticles, the spectral function grows again. For massless particles or those with mass much smaller than any temperature-related scale, the role of the mass threshold is played by the thermal mass of the particles, gT, which is much higher than the scale $1/\lambda_{\rm mfp} \sim g^4 T$ associated with the mean free path due to the weakness of the coupling. Finally, above the scale T the structure of the spectral function approaches what it would be in vacuum. A sketch of this behavior can be found in the top panel of Fig. 6.1. These qualitative features are independent of any details of the theory, and do not even depend on its symmetries. All that matters is the existence of momentum-carrying quasiparticles. In the presence of quasiparticles, no matter what their quantum numbers are, these qualitative features must be present in the spectral density.

6.3.2 Absence of quasiparticles at strong coupling

We return now to the strongly coupled $\mathcal{N} = 4$ SYM plasma, with its dual gravitational description, in order to compare the expectation (6.75) for how the spectral

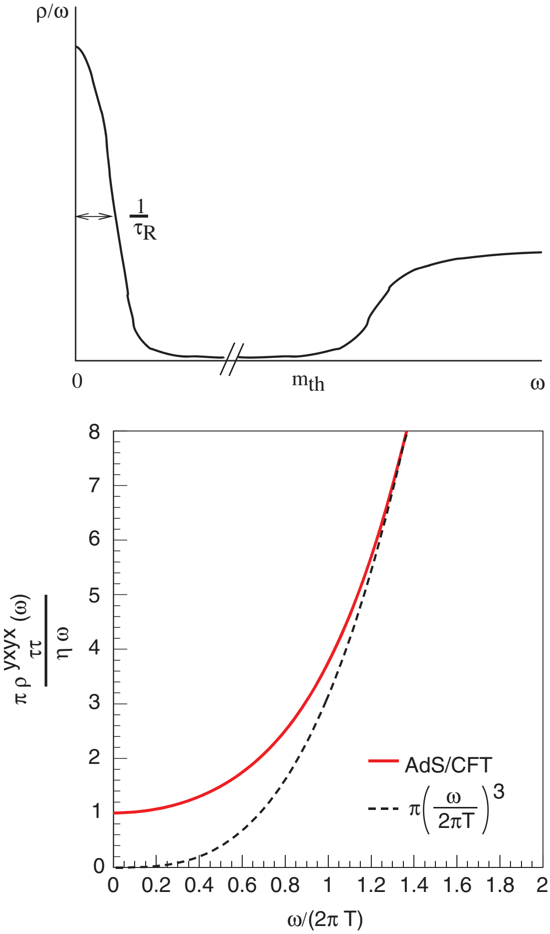

Figure 6.1 Top: sketch of the spectral function at zero momentum as a function of frequency for a weakly coupled plasma, as obtained from kinetic theory. The narrow structure at small frequency is the transport peak with a width $1/\tau_R$ that is suppressed by the coupling $(1/\tau_R \sim g^4 T)$. The thermal mass is $m_{\text{th}} \propto gT$. Bottom: spectral function for the shear channel in the strongly coupled plasma of $\mathcal{N} = 4$ SYM theory computed via gauge/string duality [777] (solid red) and a comparison with the vacuum spectral function (dashed black) which it approaches at high frequencies. The vertical axis of this figure has been scaled by the shear viscosity $\eta = s/4\pi$ of the strongly coupled plasma. Note that the definition of $\rho^{xyxy} = -\text{Im}G_R/\pi$ used in Ref. [777] is different from that in Eq. (3.13) by a factor of $\pi/2$.

density should look if the plasma contains any momentum-carrying quasiparticles to an explicit computation of the retarded correlator at strong coupling, of course done via gauge/string duality. In this section we will benefit from the general analyses of Sections 6.2.1 and 6.2.2. As in the kinetic theory computation, we

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study the response to a metric fluctuation $h_{xy}(\omega, \mathbf{q})$ in the boundary theory, with the same conventions as before. As in Section 6.2.2, the fluctuation in the boundary leads to a metric perturbation in the bulk, δg_x^y, of the form

$$\delta g_x^y(\omega, q, z) = \phi(\omega, q, z) e^{-i\omega t + iqz} \, . \tag{6.79}$$

The field ϕ is governed by the classical action (6.31) which yields an equation of motion for $\phi(\omega, q, z)$ that is given by

$$\phi''(\omega, q, u) - \frac{1+u^2}{uf}\phi'(\omega, q, u) + \frac{\mathfrak{w}^2 - \mathfrak{q}^2(1-u^2)}{uf^2}\phi(\omega, q, u) = 0 \, , \tag{6.80}$$

where $u = z^2$, $\mathfrak{w} = \omega/(2\pi T)$ and $\mathfrak{q} = q/(2\pi T)$. We may now use the general program outlined in Section 5.3.3 to determine the retarded correlator. It is given by Eq. (5.64) which, together with Eqs. (6.18) and (6.32), leads to

$$G_R^{xy,xy} = - \lim_{u \to 0} \frac{1}{16\pi G_N} \frac{\sqrt{-g}\, g^{uu} \partial_u \phi(\mathfrak{w}, \mathfrak{q}, u)}{\phi_0(\mathfrak{w}, \mathfrak{q}, u)} \, , \tag{6.81}$$

where $\phi(\mathfrak{w}, \mathfrak{q}, u)$ is the solution to the equation of motion (6.80) with infalling boundary conditions at the horizon. For arbitrary values of $\mathfrak{w}$ and $\mathfrak{q}$, Eq. (6.80) must be solved numerically [777, 553]. From the correlator (6.81), the spectral function is evaluated using the definition (3.13). The result of this computation at zero spatial momentum $\mathbf{q} = 0$ is shown in the bottom panel of Fig. 6.1, where we have plotted ρ/ω which should have a peak at $\omega = 0$ if there are any quasiparticles present.

In stark contrast to the kinetic theory expectation, there is no transport peak in the spectral function at strong coupling. In fact, the spectral function has no interesting structure at all at small frequencies. The numerical computation whose results are plotted in the bottom panel of Fig. 6.1 also shows that there is no separation of scales in the spectral function. In the strong coupling calculation, quite unlike in perturbation theory, the small and large frequency behaviors join smoothly and the spectral density is only a function of $\mathfrak{w} = \omega/2\pi T$. This could perhaps have been expected in a conformal theory with no small coupling constant, but note that a free massless theory is conformal and that theory does have a δ-function peak in its spectral function at zero frequency. So, having the explicit computation that gauge/string duality provides is necessary to give us confidence in the result that there is no transport peak in the strongly coupled plasma. The absence of the transport peak shows unambiguously that there are no momentum-carrying quasiparticles in the strongly coupled plasma. Thus, the physical picture of the system is completely different from that in perturbation theory.

The considerations we have discussed motivate the expectation that the absence of quasiparticles is a generic property of strong coupling and is not specific to any particular theory with any particular symmetries or matter content. To do so, let us recall that in the kinetic theory calculation the separation of scales required for its consistency are a consequence of the weak coupling; this is so in perturbative QCD or in perturbative $\mathcal{N} = 4$ or in any weakly coupled plasma. Now, imagine increasing the coupling. According to kinetic theory, independent of the symmetries or the matter content of the theory, the width of the transport peak grows and its height decreases as the coupling increases. This reflects the fact that as the coupling grows so does the width of the quasiparticles. Extrapolating this trend to larger and larger couplings leads to the disappearance of the transport peak which, at a qualitative level, agrees nicely with the strong coupling result for the $\mathcal{N} = 4$ SYM plasma obtained by explicit computation and shown in the right panel of Fig. 6.1. As we will argue in the next section, this observation is one of the most salient motivations for the phenomenological applications of AdS-based techniques.

6.3.3 Are there quasiparticles in the QGP?

As we have argued extensively in Section 2.2, Chapter 3 and Section 6.2, the quark–gluon plasma of QCD at temperatures a few times its T_c is strongly coupled. As a consequence, the quasiparticle picture that has conventionally been used to think about its dynamics is unlikely to be valid in this regime. Taking advantage of the general discussion of the previous sections, in this section we will provide further evidence in support of the absence of quasiparticles excitations in the QCD plasma.

Our first observation is that the quantitative relation (6.78) imposes, in fact, a very strong constraint on the minimum value of η/s consistent with a quasiparticle approach. Since the width of the transport peak, $1/\tau_R$, must be small for a consistent quasiparticle description, the relaxation time must be long compared to the inverse temperature, $T\tau_R \gg 1$, which, together with Eq. (6.78), implies

$$\frac{\eta}{s} \gg \frac{1}{5} . \tag{6.82}$$

As we have stressed, this lower bound on η/s arises solely by demanding the presence of quasiparticles and is independent of the underlying dynamics of the system. It is instructive to express this bound in units of $1/4\pi$, which shows that any value of $\eta/s < 2.5/4\pi$ is incompatible with a quasiparticle description, which is consistent with the absence of quasiparticles in strongly coupled $\mathcal{N} = 4$ SYM discussed in the previous section. Furthermore, the phenomenological fits to flow data described in Section 2.2 favor η/s values which are smaller than the bound (6.82), indicating that the relevant degrees of freedom of the QCD plasma in this region of parameter space cannot be described in terms of quasiparticles.

The argument above is somewhat indirect, since it utilizes the results of a complicated phenomenological analysis and suffers from the systematic uncertainties in extracting η/s from experimental data. A much cleaner approach is to try to directly extract the relevant spectral functions of conserved currents from lattice QCD. This is a very complicated procedure which suffers from the same numerical complications that affect the extraction of η/s from the lattice, which we briefly reviewed in Section 3.2. These difficulties notwithstanding, the authors of Ref. [323] have made the attempt to extract the spectral function of the spatial electromagnetic current–current correlator $\langle J_i(x) J_i(0)\rangle$ with $J_i(x)$ the electromagnetic current of the different quark fields, from a lattice QCD calculation performed in the quenched approximation. This spectral function, which is different from the spectral functions of stress tensor components we studied in the previous section, is sensitive to the transport of electric charge in the plasma. For a theory with charged quasiparticles, similar arguments to those in Section 6.3.1 show that this spectral function must have a transport peak in the low frequency region, signaling the presence of charged quasiparticles. On the contrary, a strong coupling computation of this spectral function [777] leads to a structureless behavior with the same qualitative features as those shown in the lower panel of Fig. 6.1.

The spectral function that best fits the lattice correlator at a fixed temperature $T = 1.45\, T_c$ is shown in Fig. 6.2. In contrast to the general expectation of the quasiparticle picture, no narrow structure was found at small frequencies. This is a strong indication that at least charge carriers in the plasma do not behave like well-defined quasiparticles with lifetimes longer than $1/T$ and that the charged plasma components must be strongly coupled. While these lattice results clearly disfavor a quasiparticle description of the QGP, they are also qualitatively different from the results obtained for the same correlator via gauge/gravity duality calculations in strongly coupled $\mathcal{N} = 4$ SYM, since some wide structure does remains at low frequency. Whether this structure is a hint of the presence of some broad excitations in the plasma or whether it is due to the many differences between QCD and $\mathcal{N} = 4$ is hard to gauge without further studies. In either case, the failure of the quasiparticle picture makes it very important to have new techniques at our disposal that allow us to study strongly coupled plasmas with no quasiparticles, seeking generic consequences of the absence of quasiparticles for physical observables. Gauge/gravity duality is an excellent tool for these purposes, as we have already seen in Sections 6.1 and 6.2 and as we will further see in the remaining chapters of this book. Indeed, as we use gauge/gravity duality to calculate more, and more different, physical observables we will discover that the calculations done in the dual gravitational description begin to yield a new form of physical intuition, phrased in the dual language rather than in the gauge theory language, in addition to yielding reliable results.

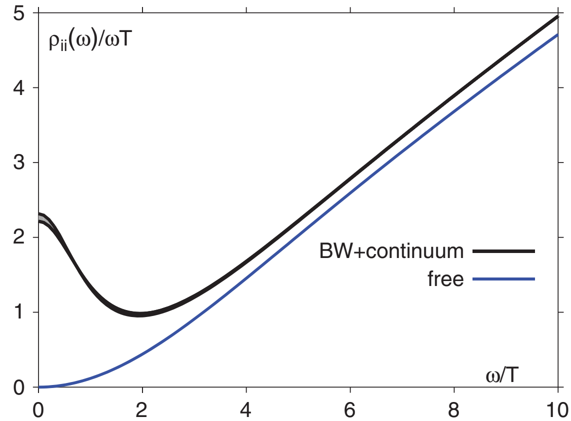

Figure 6.2 Spectral function of the electromagnetic current–current correlator in QCD at $T = 1.45\, T_c$ extracted from the lattice computation in Ref. [323]. The black band reflects the uncertainty in the spectral function due to the numerical error of the lattice results within a fixed parametrization of the spectral function. This spread does not include the effect of different parametrizations, which lead to further systematic uncertainties (see Ref. [323] for details). The spectral function for noninteracting quark–gluon plasma is also shown for comparison. This free gas spectral function is given by the blue curve for $\omega > 0$ and also includes a delta function at $\omega = 0$ that is not shown. This δ-function reflects the presence of noninteracting quasiparticles (particles, in fact) in the free gas, as discussed in the context of the stress–energy correlator after Eq. (6.74).

6.4 Quasinormal modes and plasma relaxation

As we have argued in Section 6.3.2, there are no colored quasiparticles in strongly coupled $\mathcal{N} = 4$ SYM. These correlators nevertheless possess an interesting analytic structure. Inspection of Eq. (6.81) reveals that this particular retarded correlator can have poles whenever the boundary value field $\phi_0(\mathfrak{w}, \mathfrak{q}, u \to 0)$ vanishes. This observation is not restricted to the particular stress tensor channel described by Eq. (6.81). It is true for the retarded correlator of any operator in the gauge theory since, as explained in Section 5.3, the general expression for the retarded Green's function Eq. (5.64) is inversely proportional to the amplitude of the non-normalizable mode, $A(k)$, of the field dual to the particular operator of interest. Since, as outlined in Section 5.3, the retarded correlator is obtained from solutions to the classical equations of motion in the gravity theory with infalling boundary conditions at the horizon, this field theory correlator has poles for those values of $\mathfrak{w}$ and $\mathfrak{q}$ for which a normalizable and infalling solution can be found. For the particular case of the scalar mode described in Section 6.3.2, this amounts to finding solutions to Eq. (6.80) that satisfy the boundary conditions

$$\phi(\mathfrak{w}, \mathfrak{q}, u \to 0) = 0\,, \tag{6.83}$$

$$\phi(\mathfrak{w}, \mathfrak{q}, u \to 1) = (1-u)^{-i\mathfrak{w}/2}\,, \tag{6.84}$$

where the second equation corresponds to the infalling boundary condition (5.61).

A solution to the above boundary value problem cannot be found for arbitrary values of $\mathfrak{w}$ and $\mathfrak{q}$. Nevertheless, since the problem of finding these solutions is formally identical to that of finding the energy levels of a hamiltonian in quantum mechanics, it is easy to see that for a fixed value of $\mathfrak{q}$ there will be a discrete and generally infinite set of values $\mathfrak{w}_n(\mathfrak{q})$ for which these solutions exist. However, differently from the quantum mechanical problem, the absorptive boundary condition (6.84) forces these $\mathfrak{w}_n(\mathfrak{q})$ values to be complex with a negative imaginary part. For this reason, this discrete set of solutions are called quasinormal modes, and the complex function $\mathfrak{w}_n(\mathfrak{q})$ can be thought of as the dispersion relation of the corresponding mode. Thus, at strong coupling the retarded two point functions are analytic in the upper half frequency-plane, as expected from general considerations, but with a discrete set of poles in the lower half plane, which correspond to the quasinormal modes.

In general, there are no closed form expressions for the quasinormal mode spectrum of a given operator and the frequencies $\mathfrak{w}_n(\mathfrak{q})$ must be found numerically. For the field ϕ, the first few quasinormal modes are plotted in the top panel of Fig. 6.3 at fixed $\mathfrak{q} = 1$ [555]. These complex frequencies have imaginary parts which are as large as their real parts. Thus, the poles of the associated stress tensor correlator do not describe quasiparticles. Furthermore, since the widths of these modes are of order T or larger, the lifetimes of the associated excitations are of order $1/T$ or shorter.

Although the low momentum modes described by these quasinormal modes all have short lifetimes, we shall see in Section 8.6 that in some channels the imaginary parts of their complex frequencies are proportional to $(\pi T)^{4/3} q^{-1/3}$ and so vanish in the limit in which $\mathfrak{q} \to \infty$ and $\mathfrak{w}/\mathfrak{q} \to 1$ [349]. In this regime, they describe short wavelength collective modes moving at close to the speed of light. Following Ref. [295], we shall use this feature to construct a model of a jet moving through the strongly coupled plasma in Section 8.6.

The interpretation of these quasinormal mode excitations on the gravity side is straightforward. Since the field ϕ describes a particular set of metric fluctuations (6.79), these modes describe the relaxation of small perturbations of the thermal black hole, which lead to disturbances of the black hole metric. Similarly, since these modes correspond to the poles of the retarded Green's function, they describe the relaxation of the strongly coupled plasma as it responds to external disturbances. At sufficiently long times, this relaxation process is dominated by the lowest mode, since it possesses the smallest imaginary part and, thus, the

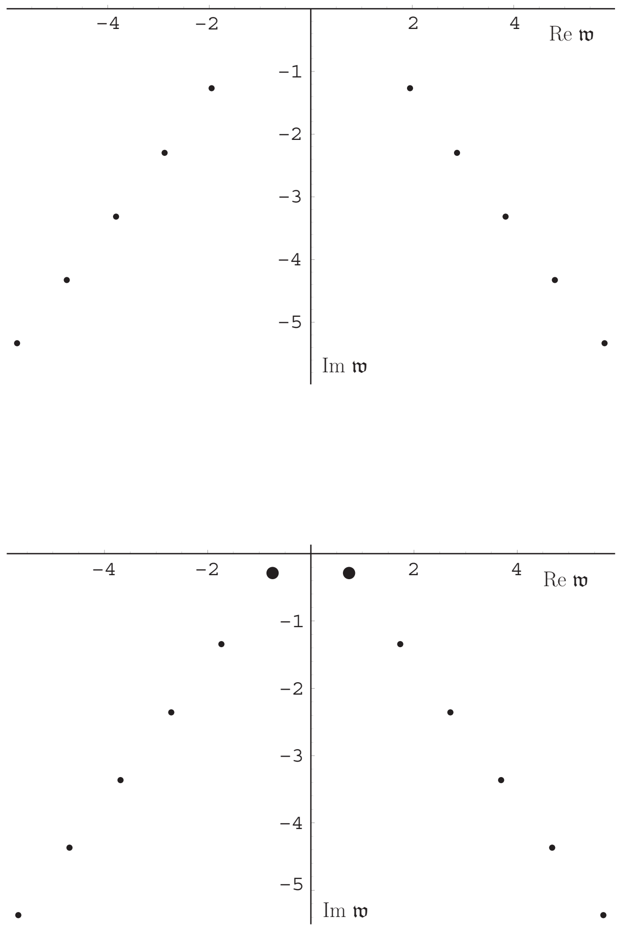

Figure 6.3 Location of the quasinormal mode poles in the complex $\mathfrak{w}$ plane at fixed $\mathfrak{q} = 1$ for the scalar (top panel) and sound (bottom panel) components of the stress tensor. Figures taken from Ref. [555].

longest lifetime. For this particular channel, the imaginary part of the latter mode is always of order T and all excitations relax within a time $1/T$, which is the generic strong coupling prediction for all plasma excitations which do not involve conserved currents. (The component of the stress tensor associated with ϕ decouples from the conservation equation.)

In contrast to the typical correlators that describe the response of the plasma (or thermal black hole) to generic disturbances, such as those described above, the relaxation of disturbances in the conserved currents must be described in the long time and long distance limit by hydrodynamics, as we have described in Section 2.2. Thus, the structure of the retarded correlator for the associated operator must reflect the general expectations from hydrodynamics in the $\mathfrak{q} \to 0$, $\mathfrak{w} \to 0$ limit. Since in this limit the retarded correlators for these currents are Green's functions of the conservation equations, they must have a pole solely determined by hydrodynamics. Thus, the low momentum and frequency limit of the quasinormal mode spectrum of the gravitational field dual to these operators must reflect the hydrodynamic behavior. In the bottom panel of Fig. 6.3 we show the quasinormal spectrum for the stress tensor component associated with sound waves, which will be defined precisely in Section 8.3. All but the lowest one of the quasinormal modes in this channel are similar to the quasinormal modes in the left panel, with real and imaginary parts of comparable magnitude. We shall refer to all modes such as these as non-hydrodynamic quasinormal modes. The lowest mode in the bottom panel of Fig. 6.3 is clearly distinct from the others as it has a much smaller imaginary part. Furthermore, the frequency of this mode $\mathfrak{w}_0(q) \to 0$ as $\mathfrak{q} \to 0$. In contrast, all the higher non-hydrodynamic modes have $\mathfrak{w} \neq 0$ at $\mathfrak{q} = 0$. This means that the lowest mode controls the dynamics of the system at late times and long distances. Furthermore, in this limit the dispersion relation of $\mathfrak{w}_0(\mathfrak{q})$ can be found analytically and is given by [555, 107]

$$\mathfrak{w}_0 = \pm \frac{1}{\sqrt{3}}\mathfrak{q} - \frac{i\mathfrak{q}^2}{3} + \mathcal{O}\left(\mathfrak{q}^3\right), \tag{6.85}$$

which coincides with the sound dispersion relation (6.41) with $c_s^2 = 1/3$, $\eta/s = 1/4\pi$ and $\zeta = 0$, consistent with our previous derivation of the shear viscosity to entropy ratio in Section 6.2. This analysis can be used to determine additional transport coefficients, as has been done in the case of the nonconformal model described in Section 6.2.3 [600]. As we will elaborate further in the next chapter, in a context in which an initially far-from-equilibrium state evolves in time and comes at late time to be described hydrodynamically, the dynamics of the lowest quasinormal mode controls the late time hydrodynamic behavior of the fluid while all the other non-hydrodynamic quasinormal modes describe the relaxation of the initially far-from-equilibrium state.