Solution with Axial Symmetry of Einstein's Equations of Teleparallelism

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§ 1. *Introduction.*

Einstein[1] has recently adopted a new set of field-equations in his Unified Field-Theory of Gravitation and Electricity, the so-called theory of parallelism at a distance or Teleparallelism, and has given[2] a solution of these equations with spherical symmetry, corresponding to the field of a charged mass-particle. In the present paper we discuss the solution of these equations with axial symmetry, which corresponds to a statical field whose field-variables depend on a single coordinate only, viz. the coordinate which is measured along the axis of symmetry. We begin by finding this solution and showing that it is the only one of this type possible on the theory of teleparallelism. This result contrasts with that of the hitherto accepted relativity theory of 1916, in which a number of solutions of this type are known, corresponding to different values, assigned *a priori*, of the energy tensor. In particular the gravitational field of a uniform electric force[3] has, on the 1916 theory, the axial type of symmetry defined above. Bearing this in mind, we then show that the single solution with axial symmetry yielded by the theory of teleparallelism has the following three properties. Firstly, it contains no electromagnetic force, according to the definition of this force in the theory of teleparallelism. Secondly, it is not one of the fields of electromagnetic force already found on the 1916 theory. Thirdly, it corresponds, on this latter theory, to a distribution of matter which, although possible in theory, cannot be said to have any physical counterpart.

§ 2. *The Field-Equations.*

The field-variables in a four dimensional manifold are, according to the theory of teleparallelism, sixteen quantities ${}_s h^{\alpha}$. The x_r are

[1] A. Einstein, *Berlin Akad. Sitz.*, 1 (1930), 18.

[2] A. Einstein and Mayer, *ibid.*, 6 (1930), 110.

[3] G. C. McVittie, *Proc. Roy. Soc.* (*A*), 124 (1929), 366.

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Gaussian coordinates, and the manifold is taken to be Riemannian so that its metric is

$$ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu. \tag{1}$$

The geometrical interpretation of the ${}_s h^a$ is this. Consider a point whose coordinates are (x_1, x_2, x_3, x_4), then for a given a and for $s = 1, 2, 3, 4$ the four ${}_s h^a$ are the projections on the a-axis of the Gaussian coordinates of four orthogonal unit vectors in a tangent Euclidean manifold, touching the Riemannian manifold at the point considered. It can be shown[1] to follow from this that

$$g^{\mu\nu} = {}_s h^\mu\, {}_s h^\nu, \qquad g_{\mu\nu} = {}^s h_\mu\, {}^s h_\nu, \tag{2}$$

$${}^s h_\mu\, {}_s h^\nu = \delta^\nu_\mu, \qquad {}^s h_\mu\, {}_t h^\mu = \delta^s_t,$$

where

$${}^s h_\mu = (\text{minor of } {}_s h^\mu \text{ in } |\,{}_s h^\mu|\,)\,/\,|\,{}_s h^\mu| \tag{3}$$

$$\sqrt{g} = h = |\,{}_s h^\mu|,$$

$$\delta^\nu_\mu = \text{Kronecker's delta.}$$

A further restriction is placed on the ${}_s h^a$ as follows. Imagine the four unit vectors defined by them set up at each point of the Riemannian manifold. We shall call this a "set of 4-vectors." Then every set of 4-vectors which can be obtained from a given set by a rotation, the same at every point, of the given set is to be considered equivalent to that set. This enables Einstein to define a connection with respect to the set of 4-vectors, for which teleparallelism exists. The coefficients of the connection are

$$\Delta^a_{\mu\nu} = {}_s h^a\, \frac{\partial\, {}^s h_\mu}{\partial x_\nu}, \tag{4}$$

and since they are not symmetrical in μ and ν, we put

$$\Lambda^a_{\mu\nu} = \Delta^a_{\mu\nu} - \Delta^a_{\nu\mu}, \tag{5}$$

$$\phi_a = \Lambda^a_{\mu a}. \tag{6}$$

The field equations given by Einstein are then

$$g^{\mu\rho} g^{\nu\sigma} (\Lambda^a_{\rho\sigma;\,\nu} - \Lambda^\tau_{\rho\sigma} \Lambda^a_{\tau\nu}) = 0, \tag{7}$$

$$\Lambda^\sigma_{\mu a;\,\sigma} = 0. \tag{8}$$

In (7) and (8) the semi-colon denotes that the covariant derivative with respect to the connection (4) has been taken.

[1] A. Einstein, *Berlin Akad. Sitzb.*, **17-19** (1928), 217. It should be observed that we use the summation convention regarding repeated suffixes, whether these are in Latin or Greek type.

The group (8) of equations can be replaced by

$$\frac{\partial \phi_\mu}{\partial x_\alpha} - \frac{\partial \phi_\alpha}{\partial x_\mu} = 0. \tag{9}$$

The ${}^s h_\mu$ are interpreted physically by Einstein (in the first approximation only) as follows:—

If ${}^s h_\mu = \delta^s_\mu + \bar{h}_{s\mu}$, where $\bar{h}_{s\mu}$ is small compared to unity, then $a_{s\mu} = \bar{h}_{s\mu} - \bar{h}_{\mu s}$ is the electromagnetic force tensor in the field, and the $g_{s\mu} = \bar{h}_{s\mu} + \bar{h}_{\mu s}$ are the gravitational potentials in the field.

§ 3. *The form of the Field-Variables for Axial Symmetry.*

Let us denote by x_1 the coordinate along the axis of symmetry of the field and by x_2, x_3 the coordinates along the other two directions of space. Let x_4 denote the time. We consider fields which are statical and where, moreover, the ${}_s h^\alpha$ are functions of x_1 alone. In consequence of this, the metrical tensor $g_{\mu\nu}$, is, by (2), a function of x_1 alone. We may therefore take the geometry of the (x_2, x_3) "planes" to be Euclidean and consider these two coordinates as analogous to Cartesians in plane geometry, so that x_2 and x_3 will enter symmetrically into our equations.

Furthermore, we contemplate fields containing continuous distributions of matter or energy and assume that no singularities of our field variables will occur at the origin. We also take coordinates such that, at the origin, the ${}_s h^\alpha$ have Euclidean values.

We now proceed to show that under these conditions only six of the sixteen ${}_s h^\alpha$ are non-zero and of these only five are independent.

Consider firstly a spatial section of the four-dimensional manifold representing the field. Such a section is a three-dimensional continuum which is invariant under the transformation

$$\bar{x}_1 = x_1, \qquad \bar{x}_\alpha = a_{\alpha\beta}\, x_\beta \qquad (\alpha, \beta = 2, 3) \tag{10}$$

where $((a_{\alpha\beta}))$ is any orthogonal matrix.

By hypothesis, all the field-variables are functions of x_1 only; hence we put

$${}_s h^\alpha (x_1, x_2, x_3) = {}_s h^\alpha (x_1).$$

Since the geometry of the (x_2, x_3) planes is to be Euclidean, ${}_s h^\alpha$ must, for a fixed value of x_1 and for $s, \alpha = 2, 3$, be a constant multiple of δ^α_s; hence

$${}_2 h^3 (x_1) = {}_3 h^2 (x_1) = 0 \qquad \text{and} \qquad {}_2 h^2 (x_1) = {}_3 h^3 (x_1).$$

Since our field-variables are to have Euclidean values at the origin, we have

$$ {}_s h^a (0) = \delta_s^a \qquad (s, a = 1, 2, 3). \tag{11} $$

We now apply the condition that all sets of 3-vectors obtained from each other by simultaneous rotations at all points are to be equivalent. Perform the transformation (10) on a set of ${}_s h^a$: we get

$$ \left. \begin{aligned} {}_s\bar{h}^1 (\bar{x}_1) &= {}_s h^1 (x_1) \qquad & (s = 1, 2, 3), \\ {}_s\bar{h}^a (\bar{x}_1) &= a_{a\beta}\, {}_s h^\beta (x_1) \qquad & (a, \beta = 2, 3). \end{aligned} \right\} \tag{12} $$

If the new ${}_s h^a$ are to be equivalent to the old, there must exist a unique orthogonal transformation $((A_{st}))$, the same for each point of the three-space, such that the new set of 3-vectors, specified by the ${}_s\bar{h}^a (\bar{x}_1)$, at $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ can be rotated into the set of 3-vectors, specified by the ${}_s h^a (\bar{x}_1)$ at the point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. That is to say

$$ {}_s\bar{h}^\gamma (\bar{x}_1) = A_{st}\, {}_t h^\gamma (\bar{x}_1) \qquad (s, t, \gamma = 1, 2, 3). $$

Hence the functional equations for the ${}_s h^\nu$ are, by (10) and (12)

$$ \left. \begin{aligned} A_{st}\ {}_t h^1 (x_1) &= {}_s h^1 (x_1) \qquad & (s, t = 1, 2, 3), \\ A_{st}\ {}_t h^a (x_1) &= a_{a\beta}\, . {}_s h^\beta (x_1) \qquad & (a, \beta = 2, 3). \end{aligned} \right\} \tag{13} $$

Since the $((A_{st}))$ is the same for each point it is sufficient to calculate its value at one point. We take the origin. Applying (11) we get from the first group of equations (13)

$$ \begin{aligned} &\delta_s^1 = A_{st}\, \delta_t^1, \\ &A_{11} = 1, \quad A_{s1} = 0 \quad \text{if } s \neq 1, \end{aligned} $$

and from the second group of (13),

$$ a_{a\beta}\, \delta_s^\beta = A_{st}\, \delta_t^a \qquad (a, \beta = 2, 3) \quad (s, t = 1, 2, 3). $$

Hence

$$ \begin{aligned} a_{as} &= A_{sa} \qquad & (s, a = 2, 3), \\ 0 &= A_{1a} \qquad & (a \ \ = 2, 3). \end{aligned} $$

By substituting these values of the A_{st} into the first group of equations (13) they become

$$ \begin{aligned} {}_1 h^1 (x_1) &= {}_1 h^1 (x_1), \\ {}_2 h^1 (x_1) &= a_{22}\ {}_2 h^1 (x_1) + a_{32}\ {}_3 h^1 (x_1), \\ {}_3 h^1 (x_1) &= a_{23}\ {}_2 h^1 (x_1) + a_{33}\ {}_3 h^1 (x_1), \end{aligned} $$

but $a_{22}, a_{23}, a_{32}, a_{33}$ are the elements of any orthogonal matrix. Hence we can only satisfy the last two equations if

$$ {}_2 h^1 (x_1) = {}_3 h^1 (x_1) = 0. $$

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By the same reasoning applied to the second group of (13) we get

$$ {}_1h^2(x_1) = {}_1h^3(x_1) = 0. $$

Hence we can describe any spatial section of our field by means of the three quantities ${}_1h^1(x_1)$, ${}_2h^2(x_1)$ and ${}_3h^3(x_1)$.

To extend this to four dimensions. The 4-space must now be invariant under the transformation

$$ \bar{x}_4 = x_4, \quad \bar{x}_1 = x_1, \quad \bar{x}_\alpha = a_{\alpha\beta}\, x_\beta \quad (\alpha, \beta = 2, 3), \tag{14} $$

and such that

$$ {}_sh^\alpha(x_1, x_2, x_3, x_4) = {}_sh^\alpha(x_1, x_2, x_3) \quad (s, \alpha = 1, 2, 3). $$

Hence we have that the only non-zero ${}_sh^\alpha$ $(s, \alpha = 1, 2, 3)$ are ${}_1h^1(x_1)$, ${}_2h^2(x_1)$, ${}_3h^3(x_1)$. As before,

$$ {}_sh^\alpha(x_1, x_2, x_3, x_4) = {}_sh^\alpha(x_1) \text{ for } s, \alpha = 1, 2, 3, 4. $$

Applying (14) to the ${}_sh^\alpha$ we get

$$ {}_s\bar{h}^4(x_1) = {}_sh^4(x_1), \quad {}_s\bar{h}^1(x_1) = {}_sh^1(x_1) \qquad (s = 1, 2, 3, 4), $$
$$ {}_s\bar{h}^\alpha(x_1) = a_{\alpha\beta}\ {}_sh^\beta(x_1) \qquad (\alpha, \beta = 2, 3), $$

and, as before, there must be a unique orthogonal transformation $((B_{st}))$ for all points, such that

$$ {}_s\bar{h}^\alpha(\bar{x}_1) = B_{st}\ {}_th^\alpha(\bar{x}_1) \quad (s, t, \alpha = 1, 2, 3, 4). $$

Thus the functional equations for the ${}_sh^\alpha$ are now

$$ \left. \begin{aligned} &{}_sh^4(x_1) = B_{st}\ {}_th^\alpha(x_1), \qquad {}_sh^1(x_1) = B_{st}\ {}_th^1(x_1), \\ &\qquad a_{\alpha\beta}\ {}_sh^\beta(x_1) = B_{st}\ {}_th^\alpha(x_1) \qquad (s, t = 1, 2, 3, 4;\ \alpha, \beta = 2, 3) \end{aligned} \right\} \tag{15} $$

Applying (11) we get

$$ B_{44} = 1, \quad B_{s4} = 0\ (s \neq 4), \quad B_{11} = 1, \quad B_{s1} = 0\ (s \neq 1), $$

and

$$ a_{\alpha\beta}\, \delta^\beta_s = B_{st}\, \delta^\alpha_t \qquad (\alpha, \beta = 2, 3;\ s, t = 1, 2, 3, 4) $$

Substituting these into the equations (15) we prove, in the same manner as for ${}_1h^2$, ${}_2h^1$, ${}_3h^1$, ${}_1h^3$ that

$$ {}_4h^2 = {}_4h^3 = {}_2h^4 = {}_3h^4 = 0. $$

Hence finally:—

The ${}_sh^\nu$ appropriate to a field with axial symmetry are

$$ \left. \begin{aligned} &{}_1h^1(x_1),\ {}_4h^4(x_1),\ {}_4h^1(x_1),\ {}_1h^4(x_1),\ {}_2h^2(x_1),\ {}_3h^3(x_1), \\ &\textit{where} \qquad {}_2h^2(x_1) = {}_3h^3(x_1) \\ &\textit{and} \quad {}_1h^4(0) = {}_4h^1(0) = \delta_{14} = 0 \quad \textit{at the origin;} \\ &\textit{all the other } {}_sh^\nu \textit{ are zero.} \end{aligned} \right\} \tag{16} $$

§ 4. *The Solution of the Field-Equations.*

Before proceeding with the actual solution, we shall make the further restriction that the form (1) is indefinite, and to avoid the use of imaginaries in our calculations we shall introduce the numbers e_a, which are such that

$$e_4 = 1, \ e_1 = e_2 = e_3 = -1$$

in our case.

The formulae (2) and (3) then become

$$g^{\mu\nu} = e_s \, {}_sh^\nu \, {}_sh^\nu, \qquad g_{\mu\nu} = e_s \, {}^sh_\mu \, {}^sh_\nu, \tag{17}$$

$$\left.\begin{aligned} &e_s \, {}^sh_\mu \, {}_sh^\nu = \delta^\nu_\mu, \qquad {}^sh_\mu \, {}_th^\mu = \delta^s_t, \qquad h = | \, e_s \, {}_sh^\mu \, |, \\ &{}^sh_\mu = (minor \ of \ {}_sh^\mu \ in \ | \, e_s \, {}_sh^\mu |) \, / \, | \, e_s \, {}_sh^\mu \, |, \end{aligned}\right\} \tag{18}$$

whilst

$$\Delta^a_{\mu\nu} = e_s \, {}_sh^a \frac{\partial \, {}^sh_\mu}{\partial x^\nu} \tag{19}$$

and (5) and (6) remain unchanged in form.

Also, in virtue of (14), the form (1) may now be written as

$$ds^2 = g_{44}(x_1) \, dx_4^2 + g_{14}(x_1) \, dx_1 \, dx_4 - g_{11}(x_1) \, dx_1^2 - g_{22}(x_1) \, (dx_2^2 + dx_3^2). \tag{20}$$

Since we require both the ${}_sh^\mu$ and the ${}^sh_\mu$ we calculate the former in terms of the latter by means of

$${}_sh^\mu = (minor \ of \ {}^sh_\mu \ in \ | \, e_s \, {}^sh_\mu \, |) \, / \, | \, e_s \, {}^sh_\mu \, | \, .$$

We get

$$\left.\begin{aligned} &{}_4h^4 = {}^1h_1 \, / \, H, \qquad {}_1h^1 = - \, {}^4h_4 \, / H, \qquad {}_4h^1 = - \, {}^1h_4 \, / H, \\ &{}_1h^4 = {}^4h_1 \, / \, H, \qquad {}_2h^2 = {}_3h^3 = 1 \, / \, {}^2h_2, \\ &\qquad h = - \, {}^3h_3 \, {}^2h_2 \, H, \\ \text{where} \qquad &H \equiv \ {}^4h_4 \, {}^1h_1 - {}^4h_1 \, {}^1h_4. \end{aligned}\right\} \tag{21}$$

The non zero Δ^a_{kl} are

$$\left.\begin{aligned} \Delta^4_{41} &= {}_4h^4 \frac{d \, {}^4h_4}{dx_1} - {}_1h^4 \frac{d \, {}^1h_4}{dx_1} = \left({}^1h_1 \frac{d \, {}^4h_4}{dx_1} - {}^4h_1 \frac{d \, {}^1h_4}{dx_1} \right) \Big/ H, \\ \Delta^4_{11} &= {}_4h^4 \frac{d \, {}^4h_1}{dx_1} - {}_1h^4 \frac{d \, {}^1h_1}{dx_1} = \left({}^1h_1 \frac{d \, {}^4h_1}{dx_1} - {}^4h_1 \frac{d \, {}^1h_1}{dx_1} \right) \Big/ H, \\ \Delta^3_{31} &= \Delta^2_{21} = {}_3h^3 \frac{d \, {}^3h_3}{dx_1} = \frac{d \log {}^3h_3}{dx_1}, \\ \Delta^1_{11} &= {}_4h^1 \frac{d \, {}^4h_1}{dx_1} - {}_1h^1 \frac{d \, {}^1h_1}{dx_1} = \left({}^4h_4 \frac{d \, {}^1h_1}{dx_1} - {}^1h_4 \frac{d \, {}^4h_1}{dx_1} \right) \Big/ H, \\ \Delta^1_{41} &= {}_1h^1 \frac{d \, {}^1h_4}{dx_1} + {}_4h^1 \frac{d \, {}^4h_4}{dx_1} = \left({}^4h_4 \frac{d \, {}^1h_4}{dx_1} - {}^1h_4 \frac{d \, {}^4h_4}{dx_1} \right) \Big/ H. \end{aligned}\right\} \tag{22}$$

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Hence the non-zero Λ^{α}_{kl} are

$$\Lambda^4_{41} = -\Lambda^4_{14} = \Delta^4_{41}$$
$$\Lambda^3_{31} = \Lambda^2_{21} = -\Lambda^3_{13} = -\Lambda^2_{12} = \Delta^3_{31} = \Delta^2_{21}$$
$$\Lambda^1_{41} = -\Lambda^1_{14} = \Delta^1_{41}.$$

The functions ϕ_α are, by equation (6),

$$\left.\begin{array}{ll} \phi_4 = \Delta^4_{41}, & \phi_1 = -\Delta^4_{41} - \Delta^2_{21} - \Delta^3_{31} \\ \phi_2 = \phi_3 = 0. & \end{array}\right\} \tag{23}$$

We now proceed to substitute these values into the field-equations (7) and (8). Take first the equations (8) or their equivalents (9): they reduce to the single one

$$\frac{d\phi_4}{dx_1} = \frac{d}{dx_1}\Delta^1_{41} = 0.$$

Hence $\Delta^1_{41} = a \qquad (a = \text{constant}).$ (24)

The equations (7) may be written in full as

$$g^{\nu\rho}\left[\frac{\partial\Lambda^{\alpha}_{\sigma\rho}}{\partial x^{\nu}} - \Delta^{j}_{\sigma\nu}\,\Lambda^{\alpha}_{j\rho} - \Delta^{j}_{\rho\nu}\,\Lambda^{\alpha}_{\sigma j} + \Delta^{\alpha}_{\nu j}\,\Lambda^{j}_{\sigma\rho}\right] = 0.$$

Hence the ones which do not vanish identically in our case are

$$g^{11}\left[\frac{d}{dx_1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\,\Delta^4_{41} + \Delta^4_{11}\,\Delta^1_{41}\right] + g^{14}\,\Delta^1_{41}\,\Delta^4_{41} = 0, \tag{25}$$

$$g^{14}\left[\frac{d}{dx_1}\Delta^4_{41} - (\Delta^4_{41})^2 - \Delta^1_{11}\,\Delta^4_{41} + \Delta^4_{11}\,\Delta^1_{41}\right] + g^{44}\,\Delta^4_{41}\,\Delta^1_{41} = 0, \tag{26}$$

$$g^{11}\left[\frac{d}{dx_1}\Delta^4_{41} - \Delta^1_{41}\,\Delta^4_{41}\right] + g^{14}\,(\Delta^1_{41})^2 = 0, \tag{27}$$

$$g^{14}\left[\frac{d}{dx_1}\Delta^1_{41} - \Delta^1_{41}\,\Delta^4_{41}\right] + g^{44}\,\Delta^4_{41}\,\Delta^1_{41} = 0, \tag{28}$$

$$g^{11}\left[\frac{d}{dx_1}\Delta^2_{21} - (\Delta^2_{21})^2 - \Delta^2_{21}\,\Delta^1_{11}\right] - g^{14}\,\Delta^1_{41}\,\Delta^2_{21} = 0. \tag{29}$$

The six equations (24) to (29) now determine the five unknown ${}^{s}h_{\mu}$. These six equations are not, of course independent: the identities existing between them have been given by Einstein.[1]

We have, by (24) and (28), either

$$g^{14} - g^{44} = 0,$$
$$\text{or} \qquad \Delta^4_{41} = 0.$$

[1] *Berlin Akad. Sitzb.*, 1 (1930), 18.

But the first alternative is impossible: for (by (11))

$$g^{14} = {}_4h^4\,{}_4h^1 - {}_1h^1\,{}_1h^4 = ({}^4h_4\,{}^4h_1 - {}^1h_1\,{}^1h_4)\,/\,H^2 \to 0 \qquad \text{at the origin,}$$
$$g^{44} = ({}_4h^4)^2 - ({}_1h^4)^2 = \{({}^1h_1)^2 - ({}^4h_1)^2\}\,/\,H^2 \to 1 \qquad \text{at the origin.}$$

Hence we have

$$\Delta^4_{41} = 0.$$

(27) now gives $\alpha^2 g^{14} = 0,$

whilst (25) and (26) give $\Delta^4_{11} = 0.$

Hence the equations (24) to (29) are equivalent to

$$\Delta^1_{41} = \alpha, \tag{30}$$
$$\Delta^4_{41} = 0, \tag{31}$$
$$\alpha^2 g^{14} = 0, \tag{32}$$
$$\Delta^4_{11} = 0, \tag{33}$$
$$g^{11}\left[\frac{d}{dx_1}\Delta^2_{21} - (\Delta^2_{21})^2 - \Delta^2_{21}\,\Delta^1_{11}\right] - g^{14}\,\alpha\,\Delta^2_{21} = 0. \tag{29}$$

By (22) the equation (33) is

$${}^1h_1\,\frac{d\,{}^4h_1}{dx_1} - {}^4h_1\,\frac{d\,{}^1h_1}{dx_1} = 0.$$

Hence ${}^4h_1$ is a constant multiple of ${}^1h_1$.
But ${}^4h_1 \to 0$ at the origin whilst ${}^1h_1 \to 1$, so that the multiplier must be zero.

Hence ${}^4h_1$ is zero, whilst ${}^1h_1$ is arbitrary. (34)

Again by (22) and (34) the equation (31) reduces to

$$\frac{d\,{}^4h_4}{dx_1} = 0.$$

Hence ${}^4h_4 = 1.$ (35)

With regard to the equation (32), we have three possibilities:

$$\begin{array}{lll} (a) & \alpha \neq 0 & g^{14} = 0 \\ (b) & \alpha = 0 & g^{14} \neq 0 \\ (c) & \alpha = 0 & g^{14} = 0. \end{array}$$

Consider (*a*):

$$0 = g^{14} = ({}^4h_4\,{}^4h_1 - {}^1h_4\,{}^1h_1)\,/\,H^2 = -\,{}^1h_4\,/\,({}^4h_4)^2.\,({}^1h_1) \text{ by (34).}$$

Hence ${}^1h_4 = 0.$ (36)

But this is impossible if $\alpha \neq 0$, since by (24) and (22)

$$\alpha = \left({}^4h_4\,\frac{d\,{}^1h_4}{dx_1} - {}^1h_4\,\frac{d\,{}^4h_4}{dx_1}\right)\Big/\,H,$$

and, by (36) and (34), the right-hand side of this equation is zero whilst the left-hand is not.

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Hence the alternative (*a*) is impossible. Similarly it may be shown that (*b*) is impossible. We are thus left with (*c*), which by (24) and the value of g^{14} given above, leads to

$$ {}^1h_4 = 0. \tag{36} $$

Again the equation (29), by (22), (34), (36), becomes

$$ \frac{d^2}{dx_1^2}(\log {}^2h_2) - \left[\frac{d}{dx_1}(\log {}^2h_2)\right]^2 - \frac{d}{dx_1}(\log {}^1h_1) \cdot \frac{d}{dx_1}(\log {}^2h_2) = 0. \tag{37} $$

Now by (34) ${}^1h_1$ is arbitrary. Hence change the variable from x_1 to z by means of

$$ {}^1h_1 \, dx_1 = dz. $$

(37) becomes

$$ \frac{d^2}{dz^2}(\log {}^2h_2) - \left[\frac{d}{dz}(\log {}^2h_2)\right]^2 = 0. $$

The solution of this equation with suitable adjustment of the constants is

$$ {}^2h_2 = \frac{1}{c(1-z)} \qquad (c = constant). \tag{38} $$

Hence finally, putting $z = x_1 / c$, *we may write our solution in the form :*

$$ \left.\begin{aligned} & ds^2 = dx_4^2 - c^{-2}\,dx_1^2 - (c - x_1)^{-2}(dx_2^2 + dx_3^2) \\ \text{with} \quad & {}^4h_1 = {}^1h_4 = 0 \\ & {}^4h_4 = 1, \ {}^1h_1 = c^{-1}, \ {}^2h_2 = {}^3h_3 = (c - x_1)^{-1}. \end{aligned}\right\} \tag{39} $$

The condition ${}^4h_1 = {}^1h_4 = 0$ is important, since it enables us to say that there is no electromagnetic force in this field, according to the definition of this force in the theory of teleparallelism. For, referring to this definition given at the end of § 2, we see that for (39), in the first approximation, all the $a_{s\mu}$ are zero.

We see also that the equations (24) to (29) are just sufficient to determine the field (39). This field is therefore the only one with the type of axial symmetry considered, which can be obtained from the theory of teleparallelism and it is a field not containing electromagnetic forces.

We should add that the metric given in (39) is that of a curved four-space, as may be seen by calculating a few components of the Riemann-Christoffel tensor belonging to it.

§ 5. *Comparison with General Relativity Theory.*

It is interesting to note that the gravitational field of a uniform electric force[1], on the 1916 theory, has just the type of axial symmetry considered in this paper. The field is

$$\left.\begin{array}{ll} & ds^2 = e^{ax_1}\, dx_4^2 - e^{-2ax_1}\, dx_1^2 - e^{-ax_1}\,(dx_2^2 + dx_3^2) \\ \text{with} & F_{41} = \tfrac{1}{4}\, a e^{\frac{1}{2}ax_1}\, \pi^{-\frac{1}{2}} = -F_{14}, \quad F_{\mu\nu} = 0 \quad (\mu, \nu \neq 1, 4) \end{array}\right\} \tag{40}$$

where $F_{\mu\nu}$ is the electromagnetic force tensor.

Since (39) is the only solution of this type which will satisfy the equations of teleparallelism, the solution (40), which is not reducible to (39), will not satisfy them. The gravitational fields of electromagnetic forces on the two theories do not therefore agree.

If we calculate the energy tensor,

$$-8\pi\, T_\mu^\nu = G_\mu^\nu - \tfrac{1}{2}\, G\, \delta_\mu^\nu$$

where $G_{\mu\nu}$ is the contracted Riemann-Christoffel tensor, for (39), we get

$$\left.\begin{array}{r} -8\pi\, T_4^4 = 5\left(1 - \dfrac{x_1}{c}\right)^{-2} \\ -8\pi\, T_1^1 = \left(1 - \dfrac{x_1}{c}\right)^{-2} \\ -8\pi\, T_2^2 = -8\pi\, T_3^3 = 2\left(1 - \dfrac{x_1}{c}\right)^{-2} \end{array}\right\} \tag{41}$$

Since

$$-8\pi\, T = -8\pi\, T_\nu^\nu = 10\left(1 - \frac{x_1}{c}\right)^{-2} \neq 0,$$

the energy cannot be solely electromagnetic.[2] The energy tensor (41) corresponds, in fact, to a distribution of matter whose density is zero at $x_1 = \pm\infty$ and infinite at $x_1 = c$. The hydrostatic pressure in the matter is such that, at any point, the pressure in the x_1-direction is half that in the x_2- and x_3-directions. Although theoretically possible, such a distribution can hardly be said to have any physical counterpart.

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[1] G. C. McVittie, *loc. cit.*

[2] See A. S. Eddington, "*The Math. Theory of Relativity*" (1924), *Ch. VI*, § 77.

§ 6. *Conclusion.*

The disagreement between the results, for the fields of electromagnetic forces, on the general relativity theory and the theory of teleparallelism, pointed out in the last paragraph, provides one reason for rejecting the latter in favour of the former. It is true, of course that there is no direct experimental evidence in favour of the field (40), but this result was arrived at on the basis of general relativity, for which experimental evidence can be found in other directions. The theory of teleparallelism, on the other hand, has provided no results, as yet, which are in accordance with experiment.

Another disadvantage of this latter theory is its rigidity: one set of mathematical assumptions with regard to the field-variables leads to one result only: on general relativity the same set of assumptions leads to more than one, corresponding to the solutions of more than one physical problem.

As far as the investigations in this paper go, we therefore conclude that the theory of teleparallelism is unsatisfactory.