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AN INEQUALITY SATISFIED BY THE GAMMA FUNCTION

BY
J. B. SELLIAH

1. **Introduction.** Gurland [1] by making use of the Cramer-Rao lower bound for the variance of an unbiased estimator obtained the following inequality

$$\frac{\Gamma^2(\delta+\alpha)}{\Gamma(\delta)\Gamma(\delta+2\alpha)} \leq \frac{\delta}{\delta+\alpha^2}, \tag{1}$$

for real values of α and δ satisfying $\alpha+\delta>0$, $\alpha\neq 0$, $\delta>0$. He used the fact that $(\Gamma(\delta)/\Gamma(\delta+\alpha))x^\alpha$ is an unbiased estimator of θ^α, where θ is the parameter for the density function

$$f(x) = \frac{1}{\theta^\delta\Gamma(\delta)} x^{\delta-1} \exp\left(\frac{-x}{\theta}\right), \qquad x > 0, \theta > 0, \delta > 0.$$

Olkin [2] used the Wishart distribution

$$f(S) = \frac{|\Lambda|^\delta\, |S|^{\delta-(p+1)/2} e^{-1/2 tr \Lambda S}}{2^{\delta p} \pi^{(p(p-1))/4} \prod_0^{p-1} \Gamma\left(\delta-\frac{i}{2}\right)}, \qquad \Lambda = \Sigma^{-1} > 0, \tag{2}$$

and considered the unbiased estimate of $|\Sigma|^\alpha$. He used a bound for

$$E\left[\frac{\partial \log f(S)}{\partial |\Sigma|^\alpha}\right]^2$$

to obtain the inequality

$$\prod_{i=0}^{p-1} \frac{\Gamma^2(\delta+\alpha-i/2)}{\Gamma(\delta-i/2)\Gamma(\delta+2\alpha-i/2)} \leq \frac{\delta^2(p^2-1)^2+\delta p^4}{\delta^2(p^2-1)^2+\delta p^4+\alpha^2} \tag{3}$$

for all real values of α and δ satisfying $\delta+\alpha>(p-1)/2$, $\delta>0, p>0$, the inequality being strict for $\alpha=p=1$ or $\alpha=0$.

In this paper, we use the multiparameter version of the Cramer Rao lower bound using the information matrix for the same problem, and obtain an inequality which is sharper than that obtained by Olkin [2].

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2. **Main Result.** The main result of this note is the following: Theorem 1. For all real values of α and δ satisfying $\delta+\alpha>(p-1)/2$, $\delta>0$, $p>0$

$$\prod_{i=0}^{p-1} \frac{\Gamma^2(\delta+\alpha-i/2)}{\Gamma(\delta-i/2)\Gamma(\delta+2\alpha-i/2)} \leq \frac{\delta}{\delta+p\alpha^2}. \tag{4}$$

Proof. If t is an unbiased estimate of $\tau(\theta_1, \theta_2, \ldots, \theta_k)$, a function of k parameters $\theta_1, \theta_2, \ldots \theta_k$, then we have the multiparameter Cramer-Rao inequality

$$\text{var } t \geq \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial \tau}{\partial \theta_i} \frac{\partial \tau}{\partial \theta_j} J^{ij} \tag{5}$$

where J^{ij} is the ijth element of J^{-1}, and where the ijth element of the matrix J is given by

$$J_{ij} = E\left(\frac{\partial \log L}{\partial \theta_i} \cdot \frac{\partial \log L}{\partial \theta_j}\right)$$

where L is the likelihood function.

We are interested in estimating $|\Sigma| = |\Lambda|^{-1}$. While it is possible to obtain the bound using the $p(p+1)/2$ parameters $\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1p}, \lambda_{22}, \lambda_{23}, \ldots, \lambda_{2p}, \ldots, \lambda_{pp}$, we can simplify the problem to that of consideration of the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_p$ of Σ.

Since $\Sigma>0$, there is an orthogonal matrix Γ such that $\Gamma'\Sigma\Gamma = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$. If $V = \Gamma' S \Gamma$ then V has a Wishart distribution

$$f(V) = \frac{[\prod_{i=1}^{p} \lambda_i]^{-\delta} |V|^{\delta-(p+1)/2} \exp\left(-\frac{1}{2} \sum \frac{v_{ii}}{\lambda_i}\right)}{2^{\delta p} \pi^{p(p-1)/4} \prod_0^{p-1} \Gamma\left(\delta - \frac{i}{2}\right)}, \qquad V>0, \lambda_i>0 \tag{6}$$

It is clear that

$$E[C_\alpha |V|^\alpha] = |\Sigma|^\alpha, \quad \text{where} \quad C_\alpha = 2^{-p\alpha} \prod_{i=0}^{p-1} \frac{\Gamma(\delta-i/2)}{\Gamma(\delta+\alpha-i/2)}. \tag{7}$$

Thus $C_\alpha |V|^\alpha$ is an unbiased estimate of $|\Sigma|^\alpha$ and its variance satisfies the inequality

$$\text{Var}(C_\alpha |V|^\alpha) = |\Sigma|^{2\alpha}\left(\frac{C_\alpha^2}{C_{2\alpha}} - 1\right) \geq \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{\partial |\Sigma|^\alpha}{\partial \lambda_i} \cdot \frac{\partial |\Sigma|^\alpha}{\partial \lambda_j} \cdot J^{*ij} \tag{8}$$

where J^{*ij} is the ijth element of J^{*-1} and where the ijth element of J^* is given by

$$J^*_{ij} = E\left(\frac{\partial \log f(V)}{\partial \lambda_i} \cdot \frac{\partial \log f(V)}{\partial \lambda_j}\right).$$

Now for $i=1, 2, \ldots p$,

$$\frac{\partial}{\partial \lambda_i} \log f(V) = -\frac{\delta}{\lambda_i} + \frac{v_{ii}}{2\lambda_i^2}$$

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Hence

$$E\left\{\frac{\partial}{\partial\lambda_i}\log f(V)\right\}^2 = \frac{1}{4\lambda_i^4}[E(v_{ii}^2)-4\delta\lambda_i E(v_{ii})+4\delta^2\lambda_i^2]$$

$$= \frac{1}{4\lambda_i^4}[(4\delta^2+4\delta)\lambda_i^2-8\delta^2\lambda_i^2+4\delta^2\lambda_i^2]$$

(9) $$= \delta\lambda_i^{-2}.$$

Since the covariance matrix of V is diagonal, v^{ij} are independent and hence

(10) $$E\left\{\frac{\partial}{\partial\lambda_i}\log f(V)\cdot\frac{\partial}{\partial\lambda_j}\log f(V)\right\} = 0 \quad \text{for} \quad i \neq j.$$

From (9) and (10) we have

$$J^* = \operatorname{diag}(\delta\lambda_1^{-2}, \delta\lambda_2^{-2}, \ldots, \delta\lambda_p^{-2})$$

and thus

(11) $$J^{*-1} = \operatorname{diag}\left(\frac{\lambda_1^2}{\delta}, \frac{\lambda_2^2}{\delta}, \ldots, \frac{\lambda_p^2}{\delta}\right).$$

We also have

(12) $$\frac{\partial\,|\Sigma|^\alpha}{\partial\lambda_i} = \alpha\,|\Sigma|^{\alpha-1}\prod_{j\neq i}^{p}\lambda_j = \frac{\alpha\,|\Sigma|^\alpha}{\lambda_i}.$$

Substituting (11) and (12) in the right hand side of (8), we have

(13) $$|\Sigma|^{2\alpha}\left(\frac{C_\alpha^2}{C_{2\alpha}}-1\right) \geq |\Sigma|^{2\alpha}\frac{\alpha^2}{\delta}\cdot p, \quad \text{i.e.} \quad \frac{C_\alpha^2}{C_{2\alpha}} \geq 1+\frac{\alpha^2 p}{\delta},$$

and the inequality (4) follows immediately.

References

1. J. Gurland, *An inequality satisfied by the Gamma Function*, Skand Akt. (1956), 171–172.
2. I. Olkin, *An inequality satisfied by the Gamma Function*, Skand Akt. (1958), 37–39.

University of Toronto
Toronto, Ontario