The Operational Representations of $D_n(x)$ and $(D^2_{-(n+1)}(ix) - D^2_{-(n+1)}(-ix))$

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(*Received* 13*th April*, 1933; *and in revised form* 31*st July*, 1933. *Read* 5*th May*, 1933.)

1. A given function $f(x)$ is said to be represented operationally by another function $\phi(p)$, if

$$\phi(p) = p\int_0^\infty e^{-px} f(x)\,dx, \tag{1}$$

provided that the integral converges.

The relation (1) between $f(x)$ and $\phi(p)$ may be denoted as

$$\phi(p) \doteqdot f(x).^{1} \tag{2}$$

When $\phi(p)$ is known, the original $f(x)$ is recovered by means of the Bromwich-Wagner theorem

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi(p)}{p} e^{px}\,dp, \tag{3}$$

which follows from the Mellin-Fourier theorem.[2]

If $f(x) = x$, then $\phi(p) = \dfrac{1}{p}$ and we have the relations

$$\frac{1}{p} \doteqdot x \quad \text{or} \quad x \doteqdot \frac{1}{p}\,. \tag{4}$$

2. Let n be a positive integer.

The differential equation[3] satisfied by $D_{2n+1}(x)$ is

$$\frac{d^2y}{dx^2} + (2n + \tfrac{3}{2} - \tfrac{1}{4}x^2)\,y = 0. \tag{5}$$

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[1] B. V. d. Pol and K. F. Nissen, *Phil. Mag.*, 8 (1929), 13 (1932).
S. Goldstein, *Proc. London Math. Soc.*, 2, **34** (1932).
Carson, *Electric Circuit Theory and the Operational Calculus* (1926).

[2] Courant und Hilbert, "*Methoden der Mathematischen Physik*," I, 90.

[3] Whittaker and Watson, *Modern Analysis* (*third edition*), 347.

If we put $x^2 = 4u$, then (5) becomes

$$u \frac{d^2y}{du^2} + \tfrac{1}{2} \frac{dy}{du} + (2n + \tfrac{3}{2} - u)\, y = 0. \tag{6}$$

Let

$$X = \int_0^\infty e^{-pu}\, y(u)\, du. \tag{7}$$

Multiplying (6) by e^{-pu} and integrating and noticing that $D_{2n+1}(0)$ is zero, we obtain the differential equation satisfied by X, viz.

$$(p^2 - 1) \frac{dX}{dp} = \{(2n + \tfrac{3}{2}) - \tfrac{3}{2}p\}\, X, \tag{8}$$

of which the solution is

$$X = C \frac{(p-1)^n}{(p+1)^{n+\frac{3}{2}}}. \tag{9}$$

Adjusting C properly, we find that

$$(-1)^n \sqrt{\pi} \frac{\Gamma(2n+2)}{\Gamma(n+1)\, 2^n} \frac{p(p-1)^n}{(p+1)^{n+\frac{3}{2}}} \doteqdot D_{2n+1}(2\sqrt{x}). \tag{10}$$

By forming the differential equation,[1] we can prove in a similar manner, that

$$\frac{2\pi}{\Gamma(n+1)} \sqrt{p} \frac{(p-1)^n}{(p+1)^{n+1}} \doteqdot i\{D^2_{-(n+1)}(i\sqrt{2x}) - D^2_{-(n+1)}(-i\sqrt{2x})\}. \tag{11}$$

Now consider the series

$$\sqrt{2} \frac{p(p-1)^n}{(p+1)^{n+\frac{3}{2}}} \left\{1 - \frac{\frac{1}{2}}{1!}\left(\frac{p-1}{p+1}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{p-1}{p+1}\right)^2 - \ldots\right\} = \sqrt{p} \frac{(p-1)^n}{(p+1)^{n+1}}.$$

Term by term interpretation gives

$$\begin{aligned}
&(-1)^n i\{D^2_{-(n+1)}(i\sqrt{2x}) - D^2_{-(n+1)}(-i\sqrt{2x})\} \\
&= \frac{\sqrt{2\pi}\, 2^{n+1}}{\Gamma(2n+2)} \Big\{ D_{2n+1}(2\sqrt{x}) + \frac{1}{2(2n+3)} D_{2n+3}(2\sqrt{x}) \\
&+ \frac{1.3}{2.4(2n+3)(2n+5)} D_{2n+5}(2\sqrt{x}) \\
&+ \frac{1.3.5}{2.4.6(2n+3)(2n+5)(2n+7)} D_{2n+7}(2\sqrt{x}) + \ldots \Big\}.
\end{aligned} \tag{12}$$

[1] *Bull. Calcutta Math. Soc.*, 17, 34. The transformation is on the same lines as before.

The relation (11) holds for all positive values of n also.

Again consider the series

$$\sqrt{p}\,\frac{(p-1)^n}{(p+1)^{n+1}}\left\{1+\frac{\frac{1}{2}}{1!}\left(\frac{p-1}{p+1}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{p-1}{p+1}\right)^2+\ldots\right\}$$
$$=\sqrt{2}\,\frac{p\,(p-1)^n}{(p+1)^{n+\frac{3}{2}}}\,. \qquad (13)$$

Term by term interpretation gives

$$D_{2n+1}(2x)=\frac{\Gamma(2n+2)}{\sqrt{2\pi}\,2^{n+1}}\left[\left(e^{(n+\frac{1}{2})\pi i}D^2_{-(n+1)}(ix\sqrt{2})+e^{-(n+\frac{1}{2})\pi i}D^2_{-(n+1)}(-ix\sqrt{2})\right)\right.$$
$$+\sum_{s=1}^{\infty}\frac{-1\,.\,1\,.\,3\,.\,5\ldots\ldots(2s-3)\,(n+1)\,(n+2)\ldots\ldots(n+s)}{\Gamma(s+1)\,2^s}\times$$
$$\left.\left(e^{(n+s+\frac{1}{2})\pi i}D^2_{-(n+s+1)}(ix\sqrt{2})+e^{-(n+s+\frac{1}{2})\pi i}D^2_{-(n+s+1)}(-ix\sqrt{2})\right)\right]. \qquad (14)$$

https://doi.org/10.1017/S0013091500024159 Published online by Cambridge University Press