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SPACES WHICH CANNOT BE WRITTEN AS A COUNTABLE DISJOINT UNION OF CLOSED SUBSETS

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It is well known (see [3](1)) that no continuum (i.e. compact, connected, Hausdorff space) can be written as a countable disjoint union of its (nonvoid) closed subsets. This result can be generalized in two ways into the setting of locally compact, connected, Hausdorff spaces. Using the one point compactification of a locally compact, connected, Hausdorff space X one can easily show that X cannot be written as a countable disjoint union of compact subsets. If one makes the further assumption that X is locally connected, then one can show that X cannot be written as a countable disjoint union of closed subsets.(2)

The object of this note is to generalize the latter of these two results. Specifically, we prove the following theorem:

THEOREM. *If X is a connected, Hausdorff space with the property that every point in X is contained in the interior of a subcontinuum of X, then X cannot be written as a countable disjoint union of its nonvoid closed subsets.*

Proof. Let X be as above and just suppose that $X=\bigcup_{i=1}^{\infty} F_i$ where the F_i's are closed, disjoint and nonvoid. Note that since X is connected, each F_i must have nonvoid boundary (otherwise it would be both open and closed). Now let $x \in Fr(F_1)$ and let K be a subcontinuum of X such that $x \in \text{Int}(K)$. Then $K \subset F_1$ (since $K= \bigcup_{i=1}^{\infty} (K \cap F_i)$ and $K \cap F_i \neq \varnothing$). Thus $x \in \text{Int}(K) \subset \text{Int}(F_1)$, which implies that $x \notin Fr(F_i)$. This contradiction establishes the theorem. Q.E.D.

Note that any space X satisfying the hypotheses of the above theorem is necessarily locally compact. Thus the theorem does not move outside the setting of locally compact spaces. However, it does embrace more than the class of locally connected spaces, as can be seen from the fact that every (connected, Hausdorff) aposyndetic space satisfies the hypotheses of the theorem.

DEFINITION. A space X is said to be *aposyndetic* if given any two distinct points x and y in X, there is a subcontinuum K of X such that $x \in \text{Int}(K)$ and $y \notin K$.

(1) A more accessible reference is [4, p. 209].

(2) The authors have been unable to find a reference for this result. For a proof in the metric setting see [1, p. 229].

COROLLARY. *If X is an aposyndetic, connected, Hausdorff space; then X cannot be written as a countable, disjoint union of nonvoid closed subsets.*

That *some* additional hypothesis is needed for connected, locally compact, Hausdorff spaces in order to conclude that they cannot be written as a countable disjoint union of closed subsets is shown by the following example of such a space

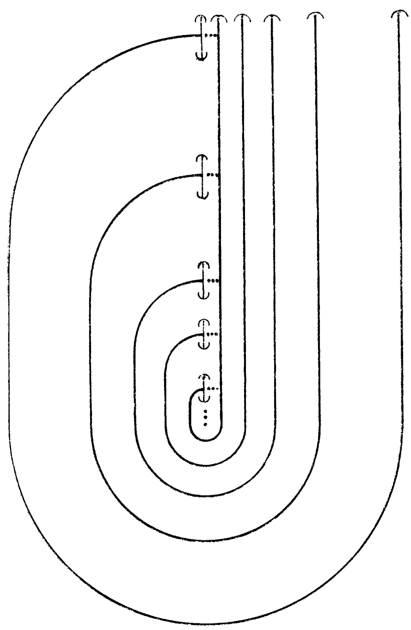

Figure 1.

which *can* be written as such a union:(3) By crossing the above example with the open unit interval one obtains a locally compact, connected, Hausdorff space X such that given any two distinct points x and y in X, there are connected neighborhoods U and V of x and y respectively such that $U \cap V = \varnothing$. But this space can also be written as a countable disjoint union of its nonvoid closed subsets.

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(3) This example is due to A. Lelek and is a simplification of another example of the same phenomenon due to one of the authors and Steve Silverman. Other examples of the same phenomena are already known. See e.g. [2, p. 259, Example 13].

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