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MULTIPLICATION OPERATORS AND DYNAMICAL SYSTEMS

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Abstract

Let X be a completely regular Hausdorff space, let V be a system of weights on X and let T be a locally convex Hausdorff topological vector space. Then $CV_b(X, T)$ is a locally convex space of vector-valued continuous functions with a topology generated by seminorms which are weighted analogues of the supremum norm. In the present paper we characterize multiplication operators on the space $CV_b(X, T)$ induced by operator-valued mappings and then obtain a (linear) dynamical system on this weighted function space.

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Introduction

Let X be a non-empty set, let T be a topological algebra and let $L(X, T)$ be the linear space of all functions from X to T. Let $F(X, T)$ be a topological vector subspace of $L(X, T)$. Let ψ be a mapping on X such that $\psi f \in L(X, T)$ whenever $f \in F(X, T)$. This gives rise to a linear transformation $M_\psi: F(X, T) \to L(X, T)$ defined as $M_\psi f = \psi f$, where the product of functions is defined pointwise. In case M_ψ takes $F(X, T)$ into itself and is continuous, it is called a multiplication operator on $F(X, T)$ induced by the mapping ψ.

This paper is a continuation of our earlier paper [8] in which we have studied multiplication operators on weighted spaces of vector-valued continuous functions induced by scalar-valued and vector-valued mappings. In the present paper we concentrate on the study of multiplication operators on weighted spaces of vector-valued mappings induced by operator-valued mappings and then we endeavor to study a (linear) dynamical system on these function spaces.

Preliminaries

Let X be a completely regular Hausdorff space, let T be a Hausdorff locally convex topological vector space over $\mathbb{C}$ and let $C(X, T)$ be the vector space of all continuous functions from X into T. By $\mathrm{cs}(T)$ we mean the set of all continuous seminorms on T, and $B(T)$ denotes the set of all continuous linear operators on T. By a system of weights we mean a set V of non-negative upper-semicontinuous functions on X such that, given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$ and for every pair $u, v \in V$ and $\alpha > 0$, there exists $w \in V$ so that $\alpha u \leq w$ and $\alpha v \leq w$ (point wise on X).

Now we consider the following vector space of vector-valued continuous functions:

$$CV_b(X, T) = \{f \in C(X, T) : vf(X) \text{ is bounded in } T \text{ for all } v \in V\}.$$

Now, let $v \in V$, $q \in \mathrm{cs}(T)$ and $f \in C(X, T)$. If we put $\|f\|_{v,q} = \mathrm{Sup}\{v(x)q(f(x)) : x \in X\}$, then $\|\cdot\|_{v,q}$ is a seminorm on $CV_b(X, T)$ and the family $\{\|\cdot\|_{v,q} : v \in V \;\; q \in \mathrm{cs}(T)\}$ defines a locally convex topology on $CV_b(X, T)$.

In case $T = \mathbb{C}$, we shall omit T from our notation and write $CV_b(X)$ in place of $CV_b(X, \mathbb{C})$. We also write $\|\cdot\|_v$ in place of $\|\cdot\|_{v,q}$ for each $v \in V$, where $q(z) = |z|$, $z \in \mathbb{C}$. We shall denote by $B_{v,q}$ the closed unit ball corresponding to the seminorm $\|\cdot\|_{v,q}$. In case $T = (T, q)$, any normed linear space, we simply write B_v. We refer to the papers of Bierstedt [1, 2] and Prolla [7] for more details and examples of these function spaces.

Let $\mathscr{F}$ be the family of all bounded subsets of T and let $M \in \mathscr{F}$ and $p \in \mathrm{cs}(T)$. If we define the function

$$S_{M,p} : B(T) \to \mathbb{R}^+ \text{ as } S_{M,p}(A) = \mathrm{Sup}\{p(A(y)) : y \in M\}$$

then $S_{M,p}$ is a seminorm on $B(T)$ and the family $\{S_{M,p} : M \in \mathscr{F}, p \in \mathrm{cs}(T)\}$ defines a locally convex topology on $B(T)$ which we call the topology of uniform convergence on bounded sets and denote by $\mathscr{U}$. Thus $(B(T), \mathscr{U})$ is a locally convex topological vector space of continuous linear operators on

T. For more details of these topologies on the spaces of linear operators we refer to Grothendieck [4] and Kothe [5].

2. Functions inducing multiplication operators

Throughout this section we will work under the following modest requirements, while developing our characterisation of an operator-valued mapping $\psi: X \to B(T)$ which induces a multiplication operator on $CV_b(X, T)$:

(2.a) X is a completely regular Hausdorff space;

(2.b) T is a Hausdorff locally convex topological vector space;

(2.c) V is a system of weights on X.

In the following theorem we characterise operator-valued mappings which induce multiplication operators on $CV_b(X, T)$.

2.1. THEOREM. *Let $\psi: X \to B(T)$ be an operator-valued continuous function. Then $M_\psi: CV_b(X, T) \to CV_b(X, T)$ is a multiplication operator if and only if for every $v \in V$ and $p \in \text{cs}(T)$, there exist $u \in V$ and $q \in \text{cs}(T)$ such that $v(x)p(\psi(x)y) \leq u(x)q(y)$, for every $x \in X$ and $y \in T$.*

PROOF. First, let us suppose that for every $v \in V$ and $p \in \text{cs}(T)$, there exist $u \in V$ and $q \in \text{cs}(T)$ such that

$$v(x)p(\psi(x)y) \leq u(x)q(y), \quad \text{for every } x \in X \text{ and } y \in T.$$

Then we shall show that M_ψ is a continuous linear operator on $CV_b(X, T)$. First of all, we show that M_ψ is an into map. Let $\{x_\alpha : \alpha \in \Delta\}$ be a net in X such that $x_\alpha \to x$. To show that $\psi(x_\alpha)f(x_\alpha) \to \psi(x)f(x)$ in T, it suffices to show that for every $p \in \text{cs}(T)$ and $\varepsilon > 0$, there exists $\alpha_0 \in \Delta$ such that

$$p(\psi(x_\alpha)f(x_\alpha) - \psi(x)f(x)) < \varepsilon, \quad \text{for every } \alpha \geq \alpha_0.$$

Now,

$$\text{(i)} \qquad p(\psi(x_\alpha)f(x_\alpha) - \psi(x)f(x)) \leq p[(\psi(x_\alpha) - \psi(x))(f(x_\alpha))] + p[\psi(x)(f(x_\alpha) - f(x))].$$

Since the set $\{f(x_\alpha) : \alpha \in \Delta\}$ is bounded in T, for every $p \in \text{cs}(T)$ and $\varepsilon > 0$, there exists $\alpha_1 \in \Delta$ such that

$$\text{(ii)} \qquad p[(\psi(x_\alpha) - \psi(x))(f(x_\alpha))] < \varepsilon/2, \quad \text{for every } \alpha \geq \alpha_1.$$

Again, since $\psi(x)$ is a continuous linear operator on T, for every $p \in \text{cs}(T)$ and $\varepsilon > 0$, there exists a neighbourhood W of the origin in T such that

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$p(\psi(x)y) < \varepsilon/2$ for every $y \in W$. Since f is continuous, there exists $\alpha_2 \in \Delta$ such that $f(x_\alpha) - f(x) \in W$, for $\alpha \geq \alpha_2$ and consequently

(iii) $$p[\psi(x)f(x_\alpha) - f(x))] < \varepsilon/2, \quad \text{for every } \alpha \geq \alpha_2.$$

Let $\alpha_0 \in \Delta$ be such that $\alpha_1 \leq \alpha_0$ and $\alpha_2 \leq \alpha_0$. Then from (ii) and (iii) it follows that

$$p(\psi(x_\alpha)f(x_\alpha) - \psi(x)f(x)) \leq \varepsilon, \qquad \text{for every } \alpha \geq \alpha_0.$$

This proves the continuity of ψf. Further, let $v \in V$, $p \in \mathrm{cs}(T)$ and $f \in CV_b(X, T)$. The

$$\|\psi f\|_{v,p} = \mathrm{Sup}\{v(x)p(\psi(x)f(x))\ x \in X\} \leq \mathrm{Sup}\{u(x)q(f(x)) \colon x \in X\} < \infty.$$

This implies that $\psi f \in CV_b(X, T)$. Clearly M_ψ is linear on $CV_b(X, T)$. In order to prove the continuity of M_ψ on $CV_b(X, T)$, it is enough to show that M_ψ is continuous at the origin. For this, suppose $\{f_\alpha\}$ is a net in $CV_b(X, T)$ such that $\|f_\alpha\|_{v,p} \to 0$, for every $v \in V$ and $p \in \mathrm{cs}(T)$.

$$\begin{aligned}\|M_\psi f_\alpha\|_{v,p} &= \mathrm{Sup}\{v(x)p(\psi(x)f_\alpha(x)) \colon x \in X\} \leq \mathrm{Sup}\{u(x)q(f_\alpha(x)) \colon x \in X\} \\ &= \|f_\alpha\|_{u,q} \to 0.\end{aligned}$$

This proves the continuity of M_ψ at the origin and hence M_ψ is continuous on $CV_b(X, T)$.

Conversely, suppose M_ψ is a continuous linear operator on $CV_b(X, T)$. We shall show that for every $v \in V$ and $p \in \mathrm{cs}(T)$, there exist $u \in V$ and $q \in \mathrm{cs}(T)$ such that

$$v(x)p(\psi(x)y) \leq u(x)q(y), \quad \text{for every } x \in X \text{ and } y \in T.$$

Let $v \in V$ and $p \in \mathrm{cs}(T)$. Since M_ψ is continuous at the origin, there exist $u \in V$ and $q \in \mathrm{cs}(T)$ such that $M_\psi(B_{u,q}) \subseteq B_{v,p}$. We claim that

$$v(x)p(\psi(x)y) \leq 2u(x)q(y), \quad \text{for every } x \in X \text{ and } y \in T.$$

Take $x_0 \in X$, $y_0 \in T$ and set $u(x_0)q(y_0) = \varepsilon$. In case $\varepsilon > 0$, the set $G = \{x \in X \colon u(x)q(y_0) < 2\varepsilon\}$ is an open neighbourhood of x_0. Thus, according to [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every $p \in \mathrm{cs}(T)$, $0 \leq (p \circ g) \leq p(y_0)$, $(p \circ g)(x_0) = p(y_0)$ and $(p \circ g)(X - G) = 0$. Let $h = (2u(x_0)q(y_0))^{-1}g$. Then clearly $h \in B_{u,q}$ and this yields that $\psi h \in B_{v,p}$. Hence $v(x)p(\psi(x)h(x) \leq 1$, for every $x \in X$. From this, it follows that

$$v(x)p(\psi(x)g(x)) \leq 2u(x_0)q(y_0), \quad \text{for every } x \in X.$$

This implies that

$$v(x_0)p(\psi(x_0)y_0) \leq 2u(x_0)q(y_0).$$

On the other hand, suppose $u(x_0)q(y_0) = 0$. Then the following three cases arise:

(i) $u(x_0) = 0, \quad q(y_0) \neq 0$;
(ii) $u(x_0) \neq 0, \quad q(y_0) = 0$;
(iii) $u(x_0) = 0, \quad q(y_0) = 0$.

Let us suppose that (i) holds and let $v(x_0)p(\psi(x_0)y_0) > 0$. Put $\varepsilon = v(jx_0)p(\psi(x_0)y_0)/2$. Then $G = \{x \in X : u(x)q(y_0) < \varepsilon\}$ is an open neighbourhood of x_0 and hence again by [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Again, define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every $p \in \text{cs}(T)$, $0 \leq (p \circ g) \leq p(y_0)$, $(p \circ g)(x_0) = p(y_0)$ and $(p \circ g)(X - G) = 0$. Consider $h = \varepsilon^{-1}g$. Then $h \in B_{u,q}$ and therefore $\psi h \in B_{v,p}$. Hence $v(x)p(\psi(x)h(x)) \leq 1$ for every $x \in X$. This implies that

$$v(x)p(\psi(x)g(x)) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2}, \qquad \text{for every } x \in X.$$

From this, it follows that

$$v(x_0)p(\psi(x_0)y_0) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2}$$

which is impossible and hence in this case our claim is established.

CASE (ii). Suppose $u(x_0) \neq 0$, $q(y_0) = 0$ and $v(x_0)p(\psi(x_0)y_0) > 0$. Put $\varepsilon = v(x_0)p(\psi(x_0)y_0)/2$. Then $G = \{x \in X : u(x) < \varepsilon + u(x_0)\}$ is an open neighbourhood of x_0 and therefore by [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every $p \in \text{cs}(T)$, $0 \leq (p \circ g) \leq p(y_0)$, $(p \circ g)(x_0) = p(y_0)$ and $(p \circ g)(X - G) = 0$. Consider $h = \varepsilon^{-1}g$. Then $h \in B_{u,q}$ and this yields that $\psi h \in B_{v,p}$. This implies that $v(x)p(\psi(x)h(x)) \leq 1$, for every $x \in X$. From this, it follows that

$$v(x)p(\psi(x)g(x)) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2}, \qquad \text{for every } x \in X.$$

Further, it implies that

$$v(x_0)p(\psi(x_0)y_0) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2}$$

which is impossible and hence in this case too our claim is established.

CASE (iii). Finally, suppose $u(x_0) = 0$ and $q(y_0) = 0$. Let $v(x_0)p(\psi(x_0)y_0) > 0$ and put $\varepsilon = v(x_0)p(\psi(x_0)y_0)/2$. Then $G = \{x \in X : u(x) < \varepsilon\}$ is an open neighbourhood of x_0 and again by [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every

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$p \in \mathrm{cs}(T)$, $0 \leq (p \circ g) \leq p(y_0)$, $(p \circ g)(x_0) = p(y_0)$ and $(p \circ g)(X - G) = 0$. Consider $h = \varepsilon^{-1} g$. Then $h \in B_{u,q}$ and this implies that $\psi h \in B_{v,p}$. Hence $v(x)p(\psi(x)h(x)) \leq 1$, for every $x \in X$. From this, it follows that

$$v(x)p(\psi(x)g(x)) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2}, \qquad \text{for every } x \in X.$$

Further, it implies that

$$v(x_0)p(\psi(x_0)y_0) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2},$$

which is a contradiction and with this our claim is established. This completes the proof of the theorem.

2.2 Remark (i). Every constant map $\psi: X \to B(T)$ induces a multiplication operator on $CV_b(X, T)$. For, if we define $\psi: X \to B(T)$ as $\psi(x) = A$, for every $x \in X$ where A is any continuous linear operator on T. Let $v \in V$, and $p \in \mathrm{cs}(T)$. Since A is a continuous linear operator, there exist $m > 0$ and $q \in \mathrm{cs}(T)$ such that

$$p(Ay) \leq mq(y), \qquad \text{for every } y \in T.$$

This implies that $p(\psi(x)y) \leq mq(y)$, for every $x \in X$ and $y \in T$. Further, it follows that

$$\begin{aligned} v(x)p(\psi(x)y) &\leq mv(x)q(y) \quad (\text{for every } x \in X \text{ and } y \in T) \\ &\leq u(x)q(y) \quad ((\text{for every } x \in X \text{ and } y \in T). \end{aligned}$$

Hence by Theorem 2.1, M_ψ is a multiplication operator on $CV_b(X, T)$.

(ii) Let X be a completely regular Hausdorff space and let $T = Y$ be any Banach space. Then every continuous bounded operator-valued mapping induces a multiplication operator on $CV_b(X, Y)$. For, let $\psi: X \to B(Y)$ be a bounded operator-valued mapping. Then there exists $m > 0$ such that $\|\psi(x)\| \leq m$, for every $x \in X$, Let $v \in V$, $x \in X$ and $y \in Y$. Then

$$\begin{aligned} v(x)\|\psi(x)y\| &\leq v(x)\|\psi(x)\|\,\|y\| \leq mv(x)\|y\| \\ &\leq u(x)\|y\| \quad (\text{for every } x \in X \text{ and } y \in Y). \end{aligned}$$

Hence by Theorem 2.1, M_ψ is a multiplication operator on $CV_b(X, Y)$.

If $T = Y$ is any Banach space and V is the system of weights generated by the characteristic functions of all compact subsets, then it turns out that every continuous operator-valued mapping induces a multiplication operator on $CV_b(X, Y)$. This we shall establish in the following proposition.

2.3 Proposition. *Let X be a completely regular Hausdorff space and let*

$$V = \{\lambda \chi_K : \lambda \geq 0,\ K \subset X \text{ and } K \text{ is a compact set}\}.$$

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Then every continuous mapping $\psi: X \to B(Y)$, induces a multiplication operator M_ψ on $CV_b(X, Y)$.

PROOF. In order to show that M_ψ is a continuous linear operator on $CV_b(X, Y)$, in the light of Theorem 2.1 it is enough to show that for every $v \in V$, there exists $u \in V$ such that

$$v(x)\|\psi(x)y\| \leq u(x)\|y\|, \qquad \text{for every } x \in X \text{ and } y \in Y.$$

If $v \in V$, then $v = \lambda\chi_K$, for some compact subset K of X. Since $\psi: X \to B(Y)$ is continuous, $\psi(K)$ is a compact subset in $B(Y)$. Let $m = \operatorname{Sup}\{\|\psi(x)\|: x \in K\}$. Put $u(x) = \lambda m\chi_K(x)$. Then $u \in V$. Let $x \in K$ and $y \in Y$. Then

$$\|\psi(x)y\| \leq \|\psi(x)\|\,\|y\| \leq m\|y\|.$$

From this, it follows that

$$\lambda\chi_K(x)\|\psi(x)y\| \leq \lambda\chi_K(x)m\|y\|.$$

This implies that

$$v(x)\|\psi(x)y\| \leq u(x)\|y\|, \quad \text{for every } x \in K \text{ and } y \in Y.$$

On the other hand, if $x \in X\backslash K$, then obviously

$$v(x)\|\psi(x)y\| \leq u(x)\|y\|.$$

Thus $v(x)\|\psi(x)y\| \leq u(x)\|y\|$, for every $x \in X$, $y \in Y$ and hence M_ψ is a multiplication operator on $CV_b(X, Y)$. This completes the proof of the theorem.

2.4 REMARK (i). From the above proposition, we note that if $\psi: X \to B(T)$ where T is any Banach space, is an unbounded continuous operator-valued mapping, even then ψ gives rise to a multiplication operator M_ψ on $CV_b(X, T)$, where V is the system of weights generated by the characteristic functions of all compact subsets of X.

(ii) In the above proposition, if we replace the system of weights

$$V = \{\lambda\chi_K: \lambda \geq 0,\ K \subset X \text{ and } K \text{ is a compact set}\}$$

by $C_c^+(X)$, the set of all positive continuous functions having compact supports, even then the conclusion holds.

2.5 COROLLARY. *Let X have the discrete topology and*

$$V = \{\lambda\chi_K: \lambda \geq 0,\ K \subset X \text{ and } K \text{ is a compact set}\}.$$

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Then every map $\psi: X \to B(T)$, where T is a Banach space, induces a multiplication operator M_ψ on $CV_b(X, T)$.

Now, we shall give certain examples of operator-valued mappings which induce and do not induce multiplication operators on $CV_b(X, T)$.

2.6 EXAMPLE. Let $X = \mathbb{N}$ with discrete topology and let $T = l^2$, the Hilbert space of all square summable sequences of complex numbers. If we define $\psi: \mathbb{N} \to B(l^2)$ by $\psi(n) = U^n$, where U is the unilateral shift operator on l^2, then

$$\|\psi(n)\| = \|U^n\| \leq \|U\|^n \leq 1, \qquad \text{for every } n \in \mathbb{N}.$$

This shows that ψ is a bounded operator-valued mapping and hence by Remark 2.2 (ii), M_ψ is a multiplication operator on $CV_b(X, T)$.

2.7 EXAMPLE. Let $X = \mathbb{N}$, with discrete topology and $T = \mathbb{R}^2$, the real Banach space. Define $\psi: \mathbb{N} \to B(\mathbb{R}^2)$ by $\psi(n) = P^n$, where P is a projection operator on $\mathbb{R}^2$. Then $\|\psi(n)\| = \|P^n\| \leq \|P\|^n \leq 1$. This implies that ψ is a bounded operator-valued mapping and hence by Remark 2.2(ii), M_ψ is a multiplication operator on $CV_b(X, T)$.

2.8 EXAMPLE. Let $X = \mathbb{N}$ be the set of natural numbers with discrete topology and let $V = K^+(\mathbb{N})$, the system of all positive constant weights. Let $T = C_b(\mathbb{N}) = l^\infty$ be the Banach space of all bounded sequences of complex numbers and $B(l^\infty)$, the Banach algebra of bounded operators on l^∞. Define $\psi: \mathbb{N} \to B(l^\infty)$ as $\psi(n) = C_{\phi^n}$, where $C_\phi: l^\infty \to l^\infty$ is the composition operator induced by a map $\phi: \mathbb{N} \to \mathbb{N}$. Then it can be seen that for every $v \in V$, there exists $u \in V$ such that

$$v(n)\|\psi(n)f\| \leq u(n)\|f\|, \qquad \text{for every } n \in \mathbb{N} \text{ and } f \in l^\infty$$

and hence by Theorem 2.1, M_ψ is a multiplication operator on $CV_b(X, T)$.

2.9 EXAMPLE. Let $X = \mathbb{N}$, the set of natural numbers with discrete topology, $T = l^2$ and let $B(l^2)$ be the Banach space of bounded linear operators on l^2. Let $v(n) = n$, for every $n \in \mathbb{N}$. Then $V = \{\lambda v: \lambda \geq 0\}$ is a system of weights on $\mathbb{N}$. Let us define $\psi: \mathbb{N} \to B(l^2)$ as $\psi(n) = A^n$, where A is the multiplication operator on l^2 induced by the constant function 2, that is, $A: l^2 \to l^2$ is defined as

$$A(x_1, x_2, \ldots) = 2(x_1, x_2 \ldots).$$

Then clearly one can check that

$$v(n)\|\psi(n)x\| \not\leq u(n)\|x\|.$$

Thus ψ does not induce a multiplication operator M_ψ on $CV_b(\mathbb{N}, l^2)$. In fact M_ψ is not even an into map. For, take $f: \mathbb{N} \to l^2$ as $f(n) = 1/n^2$. Then obviously $f \in CV_b(\mathbb{N}, l^2)$ but $\psi f(n) = \psi(n)f(n) = A^n(1/n^2) = 2^n/n^2 \to \infty$ as $n \to \infty$ and therefore $\psi f \notin CV_b(\mathbb{N}, l^2)$. In this example, if we take V as the system of positive constant weights on $\mathbb{N}$, even then ψ does not induce a multiplication operator M_ψ on $CV_b(\mathbb{N}, l^2)$. If fact, if $f(n) = 1/n$, then $f \in CV_b(\mathbb{N}, l^2)$ but $\psi f \notin CV_b(\mathbb{N}, l^2)$.

3. Dynamical systems induced by multiplication operators

Throughout this section we shall take X to be the real line $\mathbb{R}$ (with the usual topology) and T to be a Banach space. We shall denote by $B(T)$, the Banach algebra of all bounded linear operators on T and by $F_b(\mathbb{R})$, the set of all continuous bounded functions on $\mathbb{R}$. Let V be a system of weights on $\mathbb{R}$. Then clearly $CV_b(\mathbb{R}, T)$ is a locally convex Hausdorff topological vector space with the weighted topology defined in the last section. Now let U be a countable set of non-negative upper semicontinuous functions on $\mathbb{R}$ such that $W = \{\lambda u : \lambda \geq 0,\ u \in U\}$ is a system of weights on $\mathbb{R}$ with $W \approx V$. Then one can easily prove that the weighted space $CV_b(\mathbb{R}, T)$ is metrizable. In case $T = \mathbb{C}$, the metrizable weighted space $CV_b(\mathbb{R})$ is a special case of the result proved by Summers [10, Theorem 2.10].

Now, fix $g \in F_b(\mathbb{R})$ and $A \in B(T)$. For each $t \in \mathbb{R}$, we define $\psi_t : \mathbb{R} \to B(T)$ as $\psi_t(w) = e^{tg(w)A}$, for every $w \in \mathbb{R}$. We can easily see that ψ_t is a bounded operator-valued mapping from $\mathbb{R} \to B(T)$ and hence by Remark 2.2(ii), ψ_t induces a multiplication operator M_{ψ_t} on the weighted metrizable locally convex Hausdorff space $CV_b(\mathbb{R}, T)$.

3.1 THEOREM. *Let $g \in F_b(\mathbb{R})$, $A \in B(T)$ and let $\Pi_{A,g} : \mathbb{R} \times CV_b(\mathbb{R}, T) \to C(\mathbb{R}, T)$ be the function defined by $\Pi_{A,g}(t, f) = M_{\psi_t} f$ for $t \in \mathbb{R}$ and $f \in CV_b(\mathbb{R}, T)$. Then $\Pi_{A,g}$ is a dynamical system on $CV_b(\mathbb{R}, T)$.*

PROOF. Since M_{ψ_t} is a multiplication operator on $CV_b(\mathbb{R}, T)$ for every $t \in \mathbb{R}$, we can conclude that $\Pi_{A,g}(t, f)$ belongs to $CV_b(\mathbb{R}, T)$ whenever $t \in \mathbb{R}$ and $f \in CV_b(\mathbb{R}, T)$. Thus $\Pi_{A,g}$ is a function from $\mathbb{R} \times CV_b(\mathbb{R}, T)$ to $CV_b(\mathbb{R}, T)$. It can be easily seen that $\Pi_{A,g}(0, f) = f$, and

$$\Pi_{A,g}(t+s, f) = \Pi_{A,g}(t, \Pi_{A,g}(s, f))$$

for all $t, s \in \mathbb{R}$ and $f \in CV_b(\mathbb{R}, T)$.

In order to show that $\Pi_{A,g}$ is a dynamical system on $CV_b(\mathbb{R}, T)$, it is enough to show that $\Pi_{A,g}$ is a separately continuous map since joint continuity follows from [3, Theorem 1]. Let us first prove the continuity of

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$\Pi_{A,g}$ in the first argument. Let $t_n \to t$ in $\mathbb{R}$. Then $|t_n - t| \to 0$ as $n \to \infty$. We shall show that $\Pi_{A,g}(t_n, f) \to \Pi_{A,g}(t, f)$ in $CV_b(\mathbb{R}, T)$. Let $v \in V$. Then

$$\begin{aligned}
&\|\Pi_{A,g}(t_n, f) - \Pi_{A,g}(t, f)\|_v\| = \|\psi_{t_n} f - \psi_t f\|_v \\
&\quad = \operatorname{Sup}\{v(w)\|(\psi_{t_n}(w) - \psi_t(w))(f(w))\| : w \in \mathbb{R}\} \\
&\quad = \operatorname{Sup}\{v(w)\|(e^{t_n g(w)A - tg(w)A} - I)e^{tg(w)A}(f(w))\| : w \in \mathbb{R}\} \\
&\quad \leq \operatorname{Sup}\{v(w)\|(e^{t_n g(w)A - tg(w)A} - I)\| \, \|e^{tg(w)A}(f(w))\| : w \in \mathbb{R}\} \\
&\quad \leq \operatorname{Sup}\{\|(e^{t_n g(w)A - tg(w)A} - I)\| : w \in \mathbb{R}\} \operatorname{Sup}\{v(w)\|e^{tg(w)A}(f(w))\| : w \in \mathbb{R}\} \\
&\quad \leq (e^{|t_n - t|M\|A\|} - 1)\|f\|_u \to 0 \text{ as } |t_n - t| \to 0.
\end{aligned}$$

This proves the continuity of $\Pi_{A,g}$ in the first argument. Now, we shall prove the continuity of $\Pi_{A,g}$ in the second argument. Let $\{f_\alpha\}$ be a net in $CV_b(\mathbb{R}, T)$ such that $f_\alpha \to f$ in $CV_b(\mathbb{R}, T)$. Then $\|f_\alpha - f\|_v \to 0$ for every $v \in V$. We shall show that

$$\Pi_{A,g}(t, f_\alpha) \to \Pi_{A,g}(t, f) \quad \text{in} \quad CV_b(\mathbb{R}, T).$$

For this, let $v \in V$. Then

$$\begin{aligned}
\|\Pi_{A,g}(t, f_\alpha) - \Pi_{A,g}(t, f)\|_v &= \|\psi_t f_\alpha - \psi_t f\|_v \\
&= \operatorname{Sup}\{v(w)\|\psi_t(w)(f_\alpha(w) - f(w))\| : w \in \mathbb{R}\} \\
&\leq \operatorname{Sup}\{u(w)\|f_\alpha(w) - f(w)\| : w \in \mathbb{R}\} \\
&= \|f_\alpha - f\|_u \to 0.
\end{aligned}$$

This proves the continuity of $\Pi_{A,g}$ in the second argument and hence $\Pi_{A,g}$ is a (linear) dynamical system on the weighted space $CV_b(\mathbb{R}, T)$.

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