SOME GENERALIZATIONS OF THE PROBLEM OF DISTINCT REPRESENTATIVES

N. S. MENDELSOHN AND A. L. DULMAGE

1. Introduction. If $S_1, S_2, S_3, \ldots, S_n$ are subsets of a set M then it is known that a necessary and sufficient condition that it is possible to choose representatives a_i such that a_i is in S_i for $(i = 1, 2, 3, \ldots, n)$ and such that $a_i \neq a_j$ for $i \neq j$, is that for $k = 1, 2, 3, \ldots, n$, the union of any k of the sets $S_1, S_2, \ldots, S_n$, contains at least k elements. The theorem has a number of consequences amongst which we list the following.

(1) If a set M containing rs elements be broken up in two different ways into r disjoint subsets each containing s elements then it is possible to find elements $a_1, a_2, \ldots, a_r$ which will serve as representatives of the sets of both decompositions **(7)**, **(15)**.

(2) If A is an r by r matrix whose entries are all either one or zero and if each row and each column contains exactly s ones then A can be expressed as a sum of s permutation matrices **(15)**.

(3) An n by n Latin square can always be completed when m of its rows $(m < n)$ are specified **(5)**.

(4) Any doubly stochastic matrix A lies in the convex closure of the permutation matrices. More particularly, any doubly stochastic matrix is a weighted average of at most $n^2 - n + 1$ permutation matrices **(2)**.

Ryser and Mann **(15)** generalized the representative problem to obtain sufficient conditions that specified elements $a_1, a_2, a_3, \ldots, a_r$ may appear in a system of distinct representatives and Hoffman and Kuhn **(9)** replaced these by conditions which are both necessary and sufficient. These generalized results can be used to prove a theorem due to Ryser **(18)** which states necessary and sufficient conditions in order that a specified r by s subrectangle be completable to an n by n latin square.

In this paper we obtain a simple combinatorial proof of a generalization of the theorem of Hoffman and Kuhn. We also obtain generalizations in other directions which enable us to extend some of the results enumerated above.

For our purpose it is more convenient to use an alternative equivalent formulation of the distinct representative theorem. We first define a few terms.

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_s$ $(n \leqslant s)$ be two sets of elements. Let R be a dyadic relation connecting an a with a b. A pair (a, b) will be called an incidence if the relation R holds for a and b. A set S of incidences

$$(a_{q_1}, b_{r_1}), (a_{q_2}, b_{r_2}), \ldots (a_{q_k}, b_{r_k})$$

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will be called regular if no a or b appears more than once amongst the components of the pairs of S. The distinct representative theorem can now be reworded as follows:

A necessary and sufficient condition that a regular set of incidences which includes all of the $a_1, a_2, \ldots, a_n$ exist, is that for each k, $(k = 1, 2, 3, \ldots, n)$ every subset of k of the a_i are incident with at least k distinct b_j.

In connection with the sets $\{a_i\}$, $\{b_j\}$ and the incidence relation R, we define an incidence matrix A to be an n by s matrix whose entry in the ith row and jth column is 1 if (a_i, b_j) is an incidence and 0 otherwise.

An m by p matrix will be called a sub permutation matrix of rank r if it satisfies the following conditions:

(1) Its entries are all 0 or 1.

(2) Each row and each column contains at most one 1.

(3) The matrix contains exactly r 1's.

The set of places at which the 1's appear in a sub permutation matrix of rank r will be called a sub permutation set of places of rank r.

With this new notation the distinct representative theorem simply states that if the stated conditions on the a's and b's are satisfied the incidence matrix will contain 1's at a sub permutation set of places of rank n.

2. A generalization of the Hoffman-Kuhn Theorem.

The following theorem generalizes the Hoffman-Kuhn theorem which was given in (9). It has the advantage that the result is completely symmetric in the a's and b's.

THEOREM 1. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$ be two sets of elements connected by an incidence relation R. A necessary and sufficient condition that a regular set S of incidences exist in which $a_1, a_2, \ldots, a_r$ and $b_1, b_2, \ldots, b_s$ appear is:*

(1) *For $k = 1, 2, \ldots, r$, any subset of k of the elements of $a_1, a_2, \ldots, a_r$ are incident with at least k distinct elements of $b_1, b_2, \ldots, b_m$.*

(2) *For $p = 1, 2, \ldots, s$, any subset of p of the elements $b_1, b_2, \ldots, b_s$ are incident with at least p distinct elements of $a_1, a_2, \ldots, a_n$.*

Proof. Form the incidence matrix A. By condition (1), using the distinct representative theorem, the first r rows of A contain at least one sub permutation matrix of rank r. Put the letter R in each of the places occupied by 1 in any one such sub permutation matrix. Similarly the first s columns of A contain at least one sub permutation matrix of rank s. Put the letter C in the places occupied by 1 in any such sub permutation matrix. (It is possible for the same place to be marked with both R and C.) The matrix A is now said to be marked by a set of R-places and a set of C-places. It will now be shown how to choose a subset of the R and C places which will produce the set of incidences S required by the theorem.

(1) If a place is marked by both R and C it is to be included in the subset.

(2) If a marked element is alone in its row or column it is the beginning of

a connected chain which ends with an element which is alone in its row or column. A connected chain is defined as a sequence of marked places formed as follows. Start with any marked place which is alone in its row or column and proceed to a marked place in the same row (column), continue to a further marked place in the same column (row) and continue as far as possible. The set of marked places visited by this procedure is called a connected chain. By our definition a single marked element alone in its row and column is a chain. There are four types of connected chain.

Type 1. The chain begins and ends with an R. (See Figure 1.)

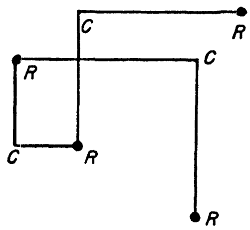

FIGURE 1

In this case we must include the R's and omit the C's in the required subset.

Type 2. The chain begins with an R and ends with a C and the first step is horizontal. (See Figure 2.)

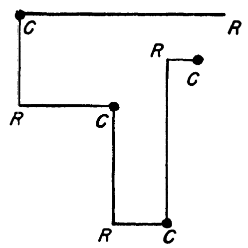

FIGURE 2

In this case we must include the C's and omit the R's from the required subset.

Type 3. The chain begins with an R and ends with a C and the first step is vertical. (See Figure 3.)

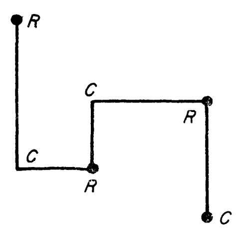

FIGURE 3

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In this case the required subset must include the R's and omit the C's.

Type 4. The chain begins and ends with a C.

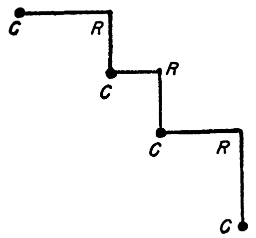

FIGURE 4

In this case the required subset must include the C's and omit the R's.

In view of the fact that the beginning and end of a chain are interchangeable, no other types of chain are possible.

(3) After removing from the set of marked places all doubly marked places and all chains, either there are no further marked places or the remaining marked places (which we will call the core) have the following property. Each marked place in the core has another marked place in its row and another marked place in its column. Hence, the marked places in the core all lie in a square sub-matrix of the incidence matrix which is included in the r by s sub-matrix of A which lies in the upper left corner. The required subset can now be completed by choosing from the core either all the places marked R or all the places marked C. (There are other ways of making the choice as our next theorem will show.)

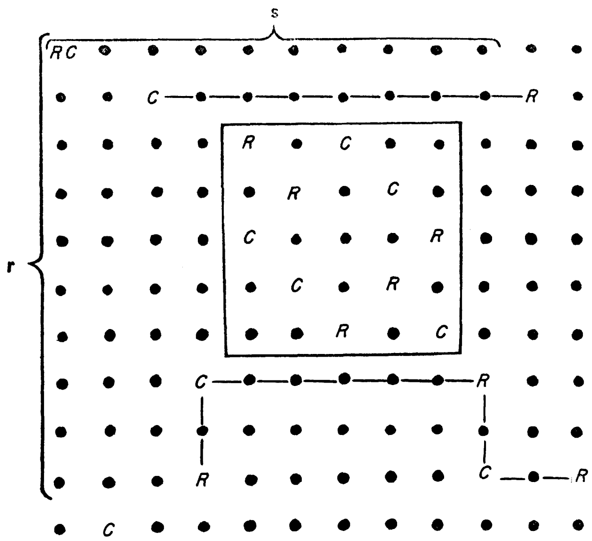

FIGURE 5

It is clear that the chosen subset of marked places yields a set of incidences S which satisfies the conditions of the theorem. In Figure 5 is shown an R and C marking of an incidence matrix. Here $m = 12$, $s = 10$, $n = 11$, $r = 10$. The chains are marked by lines and the core is surrounded by a singly lined square.

A question of some interest is the following. From an incidence matrix A marked by a set of R-places and a set of C-places in how many ways is it possible to choose a subset of places which yield a regular set S of incidences? From our proof of Theorem 1 it is immediate that the inclusion or exclusion of a place which is doubly marked or which belongs to a connected chain is uniquely determined. With regard to the core, its places lie in a square. By permuting the rows and the columns of this square it is possible to arrange that the marked squares can be confined to a set of square blocks along the main diagonal of the square as in Figure 6.

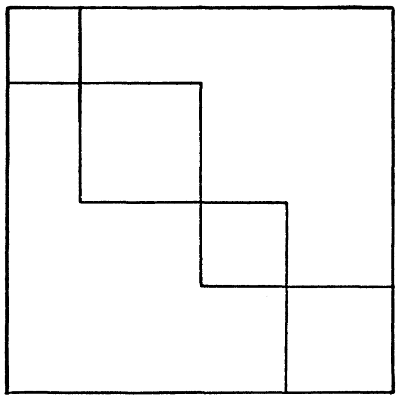

FIGURE 6

Let k be the maximal number of such blocks. In each such block we can choose for our subset either all the R's or all the C's. Hence, the number of ways of making a choice is 2^k.

There is another way of determining the number k which does not require the rearrangement of the rows and columns of the square containing the core. Start with any marked place of the core and proceed to the other marked place in its row, continue to the remaining marked place in the column of the place last visited and proceed in this way alternately along rows and columns. Ultimately the starting point is reached. The marked places visited in this manner form a re-entrant cycle and the number k described above is precisely the number of re-entrant cycles in the core. In Figure 7 there is a diagram of a core which is included in a 7 by 7 square and which decomposes into two cycles.

The theorem just proved can be described as follows.

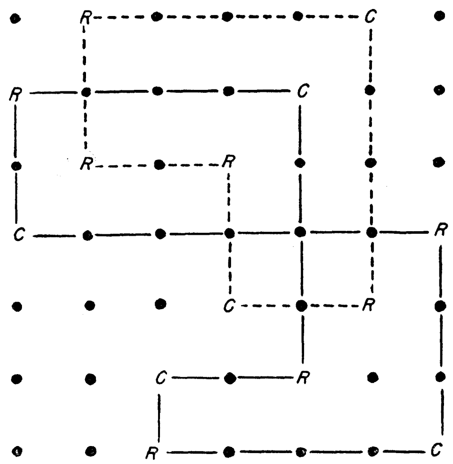

FIGURE 7

THEOREM 2. *The number of ways of choosing a set of places which yield a regular set of incidences from an R and C marking of an incidence matrix is 2^k, where k is the number of re-entrant cycles in the core of the marking.*

3. Further generalizations. We now consider generalizations of a different character. Roughly stated our problem is this. Let A be a matrix with entries which are either positive numbers or zero. What conditions can be placed on the entries of A in order to assure that A has non-zero entries in a sub permutation set of places or rank r? We confine ourselves to the case where A is a square matrix, since rectangular matrices may be completed into squares by adding rows (or columns) of zero entries. All resulting theorems for matrices so augmented will hold for the original matrix.

Throughout the remainder of this section the following notation will be used. $A = (a_{ij})$ will represent an n by n matrix with entries $a_{ij} \geqslant 0$. R_i will denote the sum of the entries in the ith row, $R_i = \sum_j a_{ij}$; C_j will denote the sum of the entries in the jth column, $C_j = \sum_i a_{ij}$; M will denote the maximum row or column sum, $M = \max(R_i, C_j)$; and S will denote the sum of all the entries,

$$S = \sum_{i,j} a_{ij} = \sum_i R_i = \sum_j C_j.$$

We will also use the following consequence of the distinct representative theorem: if A is a square matrix with positive or zero entries such that each row sum and each column sum has the same non-zero value, then any non-zero entry of A lies in a permutation set of places all of which have non-zero entries in A.

THEOREM 3. *If $(n - 1)\, M < S$ then A has non-zero entries in at least one permutation set of places.*

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Proof. Augment the matrix A by adding an $(n+1)$th row and an $(n+1)$th column where $a_{i,n+1} = M - R_i$ $(i = 1, 2, \ldots n)$

$$a_{n+1,j} = M - C_j (j = 1, 2, \ldots n) \quad \text{and} \quad a_{n+1,n+1} = S - (n-1)M.$$

Since all of $M - R_i$, $M - C_j$, $S - (n-1)M$ are positive or zero and since in the augmented matrix all row and column sums are the same positive number, the augmented matrix has non-zero entries in a permutation set of places which includes the place occupied by $a_{n+1,n+1}$. The remaining places are thus a permutation set of places of the matrix A.

Theorem 3 is the best possible in the following sense: if the condition $(n-1)M < S$ is not satisfied there are matrices A which do not have non-zero entries in all the places of any permutation set. Indeed, if A be any sub permutation matrix of rank $n - 1$, then $(n-1)M = S$ and the theorem is obviously false for A. Theorem 3 has the following corollary which is an improvement of the result due to Hall **(7)**.

COROLLARY. *Let T be a set containing S elements and suppose T is broken up into n disjoint subsets in two different ways;*

$$T = A_1 + A_2 + \ldots + A_n,$$
$$T = B_1 + B_2 + \ldots + B_n.$$

Let M be the maximum number of elements in any of the sets $A_1, A_2, \ldots, A_n$; $B_1, B_2, B_3 \ldots, B_n$. If $(n-1)M < S$ then it is possible to choose n elements $a_1, a_2, \ldots, a_n$ which will represent both the sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, B_3, \ldots, B_n$ (each a_i being a member of the sets which it represents).

Proof. Form the matrix A whose entry a_{ij} is the number of elements in the intersection of A_i and B_j. $(i, j = 1, 2, \ldots, n)$ A satisfies the condition of Theorem 3 and the corollary follows immediately.

THEOREM 4. *If*

$$\frac{1}{n-1} \leqslant \frac{M}{S} < \frac{n-1}{n^2 - 2n}, \tag{3.1}$$

then the matrix A has non-zero entries in the places of at least one sub permutation set of rank $(n-1)$.

Proof. Let $B_2 = n M - S - M$ and $T_2 = M + B_2 - n B_2$. The inequality $S < (n-1)M$ implies $B_2 \geqslant 0$ while the inequality

$$(n^2 - 2n)M < (n-1)S$$

implies $T_2 > 0$. Augment the matrix A by the addition of two rows and two columns as follows. Put $a_{i,n+1} = M - R_i$, $a_{i,n+2} = B_2$ for $i = 1, 2, \ldots, n$; put $a_{n+1,n+1} = a_{n+1,n+2} = a_{n+2,n+1} = 0$; $a_{n+2,n+2} = T_2$; $a_{n+1,j} = M - C_j$, $a_{n+1,j+1} = B_2$ for $j = 1, 2, \ldots, n$. The augmented matrix now consists of non-negative entries whose row and column sums are all equal. Hence, since $a_{n+2,n+2} \neq 0$, there is a permutation set of places which includes the place $n+2$, $n+2$

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containing non-zero elements of the augmented matrix. The places of this permutation set which are in the matrix A form a sub permutation set of rank at least $n - 1$.

The conditions of Theorem 4 are sufficient but may not be the best possible. Since sub permutation matrices of rank $n - 2$ satisfy the condition $(n - 2)M = S$, it is natural to conjecture that the term $(n - 1)/(n^2 - 2n)$ could be replaced by $1/(n - 2)$. For large n the improvement is small.

A corollary analogous to that of Theorem 3 reads as follows:

Let T be a set containing S elements and suppose T is broken up into n disjoint subsets in two different ways;

$$T = A_1 + A_2 + \ldots + A_n = B_1 + B_2 + \ldots + B_n.$$

Let M be the maximum number of elements in any of the sets $A_1, A_2, \ldots, A_n$; $B_1, B_2, \ldots, B_n$. If (3.1) holds then $(n - 1)$ elements may be chosen, each of which represents one set of each decomposition, and two elements may be chosen to represent the remaining two sets provided they are non-null.

We now proceed to the general case by induction. We now define numbers $B_2, B_3, B_4, \ldots$; $T_1, T_2, T_3, T_4, \ldots$ as follows:

$$\begin{aligned}
B_2 &= (n - 1)M - S,\\
B_3 &= n\,B_2 - B_2 - M,\\
B_4 &= n\,B_3 - B_2 - B_3 - M,\\
B_5 &= n\,B_4 - B_2 - B_3 - B_4 - M,\\
B_r &= n\,B_{r-1} - (B_2 + B_3 + \ldots + B_{r-1}) - M,
\end{aligned}$$

and for $i = 1, 2, 3, \ldots, T_i = -\mathrm{B}_{i+1}$. It is clear that if T_r is the first of the numbers $T_1, T_2, T_3, \ldots$ which is positive, then all the numbers $B_2, B_3, \ldots, B_r$ are positive.

The B_i may be expressed in terms of n, S, M as follows:

$$\begin{aligned}
B_2 &= (n - 1)M - S,\\
B_3 &= (n^2 - 2n)M - (n - 1)S,\\
B_i &= P_{i-1}(n)M - Q_{i-2}(n)S,
\end{aligned}$$

where $P_i(n)$ and $Q_i(n)$ of polynomials in n with integral coefficients of degree i.

By subtracting two successive B's one obtains $B_i = n(B_{i-1} - B_{i-2})$. This in turn yields the following recurrence relationships amongst the P_i and the Q_i:

$$P_i = n(P_{i-1} - P_{i-2});\, Q_i = n(Q_{i-1} - Q_{i-2}).$$

Also, since $Q_0 = 1$, $Q_1 = (n - 1)$, $Q_2 = n^2 - 2n$; $P_1 = n - 1$, $P_2 = n^2 - 2n$, it follows that $P_i(n) = Q_i(n)$. Furthermore, the difference equation for P_i together with P_1 and P_2 uniquely determines P_i. It is directly verifiable by induction that

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$$P_i(n) = n^i - \binom{i}{1} n^{i-1} + \binom{i-1}{2} n^{i-2} - \ldots$$
$$= \sum_{k=0}^{[\frac{1}{2}i+\frac{1}{2}]} (-1)^k \binom{i-k+1}{k} n^{i-k}.$$

Before we can proceed to the main theorem we need some further properties of the $P_i(n)$.

THEOREM 5. *For $r > 2$ and $n \geqslant r$, $P_r(n) > 0$.*

Proof. We can bracket the terms of $P_r(n)$ as follows:

$$P_r(n) = \left\{n^r - \binom{r}{1} n^{r-1}\right\} + \left\{\binom{r-1}{2} n^{r-2} - \binom{r-2}{3} n^{r-3}\right\} + \ldots$$
$$+ \left\{\binom{r-2k+1}{2k} n^{r-2k} - \binom{r-2k}{2k+1} n^{r-2k-1}\right\} + \ldots$$

If each bracketed pair is non-negative for any positive value of n, it remains non-negative when n is replaced by a larger value. Hence, it is sufficient to prove that each bracketed pair of $P_r(r) \geqslant 0$ (and at least one pair has a value > 0). The theorem is trivially verified for $r = 3$ and $r = 4$ so we will assume $r \geqslant 5$.

Now

$$\binom{r-2k+1}{2k} r^{r-2k} - \binom{r-2k}{2k+1} r^{r-2k-1}$$
$$= r^{r-2k-1}\left\{\binom{r-2k+1}{2k} r - \binom{r-2k}{2k+1}\right\}.$$

The expression

$$\binom{r-2k+1}{2k} r - \binom{r-2k}{2k+1}$$

will be non-negative provided

$$(2k-1)r^2 - (4k^2 - 10k + 1)r > 4k(4k-1).$$

Here r is an integer $\geqslant 5$ and k is any integer such that $4k + 1 \leqslant r$. Again:

$$(2k-1)r^2 - (4k^2 - 10k + 1)r$$
$$\geqslant r\{(2k-1)(4k+1) - (4k^2 - 10k + 1)\}$$
$$= r\{4k^2 + 8k - 2\} \geqslant 5\{4k^2 + 8k - 2\} > 4k(4k-1)$$

for every positive integral k.

THEOREM 6. *For every integral $n > 2$,*

$$\frac{P_0(n)}{P_1(n)} < \frac{P_1(n)}{P_2(n)} < \frac{P_2(n)}{P_3(n)} \ldots < \frac{P_{n-2}(n)}{P_{n-1}(n)} < \frac{P_{n-1}(n)}{P_n(n)}.$$

Proof. By Theorem 5 all the denominators are positive. Also, since $P_i = n(P_{i-1} - P_{i-2})$ for every i, it follows that

$$\begin{aligned} P_r^2 - P_{r-1}P_{r+1} &= P_r^2 - P_{r-1}(n\,P_r - n\,P_{r-1}) \\ &= P_r^2 - n\,P_r\,P_{r-1} + n\,P_{r-1}^2 \\ &= (n\,P_{r-1} - n\,P_{r-2})\,P_r - n\,P_r\,P_{r-1} + n\,P_{r-1}^2 \\ &= n(P_{r-1}^2 - P_{r-2}\,P_r). \end{aligned}$$

Also $P_1^2 - P_0\,P_2 = 1$, so that $P_r^2 - P_{r-1}\,P_{r+1} = n^{r-1} > 0$, for $r > 2$, $n \geqslant r$.

THEOREM 7. *Let A be an n by n matrix with non-negative entries. Let S be the sum of all entries in A and let M be the maximum sum of any row or column of A. For $r > 2$ and $n \geqslant r$, if*

$$\frac{P_{r-2}(n)}{P_{r-1}(n)} \leqslant \frac{M}{S} < \frac{P_{r-1}(n)}{P_r(n)}, \tag{3.2}$$

then A has non-negative entries in a subpermutation set of places of rank $n - r + 1$.

Proof. By (3.2), $M\,P_r(n) - S\,P_{r-1}(n)$ is negative. That is, B_{r+1} is negative or T_r is positive. Also, since

$$\frac{P_0}{P_1} < \frac{P_1}{P_2} \cdots \frac{P_{r-2}}{P_{r-1}} \leqslant \frac{M}{S},$$

each of the numbers $B_2, B_3, B_4 \ldots B_r$ is non-negative. Augment the matrix A by the addition of r rows and r columns as follows: put

$$\begin{gathered} a_{i,\,n+1} = M - R_i \text{ for } i = 1, 2, 3, \ldots, n; \\ a_{i,\,n+t} = B_t \text{ for } i = 1, 2, \ldots n \text{ and } t = 2, 3, \ldots, r; \\ a_{n+1,\,j} = M - C_j \text{ for } j = 1, 2, 3, \ldots, n; \\ a_{n+u,\,j} = B_u \text{ for } u = 2, 3, \ldots r \text{ and } j = 1, 2, \ldots, n; \\ a_{vs} = 0 \text{ for } v > n,\ s > n \text{ and } v \neq s \\ a_{n+k,\,n+k} = B_{k+2} + B_{k+3} + \ldots + B_r \text{ for } k = 1, 2, 3, \ldots, r-2; \\ a_{n+r-1,\,n+r-1} = 0,\ a_{n+r,\,n+r} = T_r. \end{gathered}$$

Figure 8 illustrates the case $r = 5$.

$$\begin{array}{cccc|ccccc} \multicolumn{4}{c|}{} & M - R_1, & B_2, & B_3, & B_4, & B_5 \\ \multicolumn{4}{c|}{} & M - R_2, & B_2, & B_3, & B_4, & B_5 \\ \multicolumn{4}{c|}{A} & & & & & \\ \multicolumn{4}{c|}{} & M - R_n, & B_2, & B_3, & B_4, & B_5 \\ \hline M - C_1, & M - C_2, & \ldots, & M - C_n, & B_3 + B_4 + B_5, & 0 & ,0 & ,0 & ,0 \\ B_2 & , B_2 & ,\ldots, & B_2 & , 0 & , B_4 + B_5, & 0 & ,0 & ,0 \\ B_3 & , B_2 & ,\ldots, & B_2 & , 0 & , 0 & , B_5, & 0 & ,0 \\ B_4 & , B_4 & ,\ldots, & B_4 & , 0 & , 0 & ,0 & ,0 & ,0 \\ B_5 & , B_5 & ,\ldots, & B_5 & , 0 & , 0 & ,0 & ,0 & , T_5 \end{array}$$

FIGURE 8

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The augmented matrix has all its entries non-negative with the same non-zero sum for each row and column. Hence, it contains a permutation set of places which includes the place occupied by T_r. Those places which lie in A form a sub permutation set of places of rank at least $n - r + 1$.

4. Concluding Remarks. The condition (3.2) for our main theorem, 7, while sufficient to assure the existence of a sub permutation set of places of non-zero entries of rank $n - r + 1$, may not be the best possible. Since a sub permutation matrix of rank k satisfies the condition $kM = S$, it seems reasonable to conjecture that the above condition may be replaced by the condition

$$\frac{1}{n-r+1} \leqslant \frac{M}{S} < \frac{1}{n-r}.$$

If this latter condition is correct the result must be the best possible. For large n (that is, large in comparison with r) the difference between the proved result and the conjectured one is very small.

Note added in proof. (Feb. 20, 1958). The expression *term rank* has been used recently to describe the order of the largest minor of A with a non-zero term in the expansion of its determinant. In section 3 information concerning the term rank of a matrix A was obtained by embedding A in a doubly stochastic matrix. The nature and structure of such embedding has since been studied by the authors and the concept of *stochastic rank* of a matrix has been introduced as follows. An n by n matrix A with non-negative entries has stochastic rank σ if A can be embedded in a doubly stochastic matrix by the addition of $n - \sigma$ rows and columns but A cannot be embedded in a doubly stochastic matrix of smaller size. The following results concerning the stochastic rank σ and term rank ρ have been obtained.

(a) For any matrix A, $\rho \geqslant \sigma$.

(b) For a doubly stochastic matrix, or for a sub permutation matrix $\rho = \sigma$.

(c) For any matrix A, $\sigma = [S/M]$.

(d) There are n by n matrices A for which $\rho - \sigma = n - 1$.

(e) If S/M is not an integer $\rho \geqslant \sigma + 1$.

Furthermore, the conjecture stated in the concluding remarks has now been proved and extended to non-square and to infinite matrices.

These and other results will be proved in a sequel to the present paper.

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