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A MULTIVARIABLE FORM OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. Let $H(\mathbf{x})$ be a homogeneous polynomial in n indeterminates over an algebraically closed field K. A necesssary and sufficient condition is given for $H(\mathbf{x})$ to admit a factorization of the form

$$\prod_{i=1}^{k} [\alpha_i(\mathbf{a}\circ\mathbf{x})+\beta_i(\mathbf{b}\circ\mathbf{x})]^{m_i}, \quad \text{where} \quad \alpha_i, \beta_i \in K, m_i \in N, \quad \text{for } i=1,\ldots .k;$$

$\mathbf{a}, \mathbf{b} \in K^n$, and "$\circ$" is the usual inner product. This condition involves the linear derivatives of $H(\mathbf{x})$.

Let $H(\mathbf{x})$ be a homogeneous polynomial in $\mathbf{x}=(x_1, \ldots, x_n)$ of degree m over an algebraically closed field K. Denote by G_H the vector space (over K) generated by the set of all the derivatives of order $m-1$ of H.

The aim of this note is to prove the following result:

THEOREM. *$H(\mathbf{x})$ admits a factorization of the form*

$$\prod_{i=1}^{k} [\alpha_i(\mathbf{a}\circ\mathbf{x})+\beta_i(\mathbf{b}\circ\mathbf{x})]^{m_i}, \tag{1}$$

where $\mathbf{a}, \mathbf{b} \in K^n$; $\alpha_i, \beta_i \in K$, $m_i \in N$ *for* $i=1, \ldots, k$ *and* $\sum m_i = m$, *if and only if* $\dim G_H \leq 2$.

Proof. First assume $H(\mathbf{x})$ admits a factorization as shown in (1). Then $\dim G_H \leq 2$ follows from explicitly differentiating (1).

Conversely, let $\dim G_H \leq 2$. Then we can find $\mathbf{a}, \mathbf{b} \in K^n$ such that G_H is generated as a vector space over K by $y_1 = \mathbf{a}\circ\mathbf{x}$ and $y_2=\mathbf{b}\circ\mathbf{x}$. Since every polynomial can be written as a polynomial of its linear derivatives (cf. [2], Theorem 1), then $H(\mathbf{x}) = H^*(y_1, y_2)$. The Fundamental Theorem of Algebra ([1]) asserts that $H^*(y_1, y_2) = \prod_{i=1}^{k} (\alpha_i y_i + \beta_i y_2)^{m_i}$ for some $\alpha_i, \beta_i \in k$, $m_i \in \mathbb{N}$

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($i=1,\ldots,k$ and $\sum m_i = m_i$), which yields (1) by substituting (y_1, y_2) by $(\mathbf{a}\circ\mathbf{x}, \mathbf{b}\circ\mathbf{x})$. This concludes the proof.

When applied to quadratic forms this theorem yields the following well-known result.

COROLLARY. *Let* $\mathbf{x}^T A\mathbf{x} \neq 0$ *be a quadratic form over* K. *Then* $\mathbf{x}^T A\mathbf{x} = (\mathbf{a}\circ\mathbf{x})(\mathbf{b}\circ\mathbf{x})$, *where* $\mathbf{a}, \mathbf{b} \in K^n$, *if and only if rank* $A \leq 2$.

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REFERENCES

1. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, third edition, N.Y.: Macmillan, 1965.

2. P. M. Salzberg, The Minimal Generating Space of a Polynomial, *Linear Algebra and its Applications*, **36** (1981), 89–96.

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