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ON q-HYPERELLIPTIC k-BORDERED TORI

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Abstract. A compact Klein surface X is a compact surface with a dianalytic structure. Such a surface is said to be q-hyperelliptic if it admits an involution ϕ, that is an order two automorphism, such that $X/<\phi>$ has algebraic genus q. A Klein surface of genus 1 and k boundary components is a k-bordered torus.

By means of NEC groups, q-hyperelliptic k-bordered tori are studied and a geometrical description of their associated Teichmüller spaces is given.

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1. Introduction. Klein surfaces, introduced from a modern point of view by Alling and Greenleaf [**1**], are surfaces endowed with a dianalytic structure. A compact orientable Klein surface X with topological genus 1 and $k \geq 1$ boundary components is a k-bordered torus. The surface X is said to be *q-hyperelliptic* if and only if X admits an involution ϕ, that is an order two automorphism, such that $X/<\phi>$ is an orbifold with algebraic genus q. In the particular cases $q = 0, 1$, X is hyperelliptic and elliptic-hyperelliptic respectively.

Non-Euclidean crystallographic groups (NEC groups in short) where introduced by Wilkie [**16**] and Macbeath [**10**], and they are an important tool in the study of Klein surfaces since the results of Preston [**14**] and May [**13**]. Klein surfaces can be seen as quotients of the hyperbolic plane under the action of an NEC group. In particular, when X is a torus, then $X = \mathcal{D}/\Gamma$, where $\mathcal{D}$ denotes the hyperbolic plane and Γ is a surface NEC group with signature:

$$\sigma(\Gamma) = (1, +, [-], \{(-)^k\}), \qquad k \geq 1. \tag{1.1}$$

The surface X is q-hyperelliptic if and only if there exists an NEC group Γ_1 with $\Gamma \triangleleft_2 \Gamma_1$ such that $\Gamma_1/\Gamma = <\phi>$. If $k > 4q$ the group Γ_1 is unique [**2**] and Γ_1 is said to be the *q-hyperellipticity group*. In this case the automorphism ϕ is central in the group Aut(X) and it is called the *q-hyperelliptic involution*. The q-hyperelliptic surfaces have been studied in [**2**], [**3**], [**5**], and [**8**].

The aim of this work is the geometrical study of the Teichmüller space of q-hyperelliptic tori by means of fundamental regions of NEC groups. This technique was used in [**6**] and for the Moduli space of Riemann surfaces in [**7**].

In the next Section we give the necessary preliminaries about NEC groups and Klein surfaces. In Section 3 the signature of the q-hyperelliptic group Γ_1 is obtained. As a result Γ_1 may belong to one of four different classes. Afterwards we construct

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fundamental regions R with all right angles for groups belonging to the above classes. The parameters (lengths of the sides in R) can be taken as coordinates in the Teichmüller space of q-hyperelliptic tori. It is done in Section 4.

2. Preliminaries on NEC groups. An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{D}$, including reversing-orientation elements, with compact quotient $X = \mathcal{D}/\Gamma$.

Each NEC group Γ is given a signature [**10**]

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i1}, \ldots, n_{is_i}),\ i = 1, \ldots, k\}), \tag{2.1}$$

where g, m_i, n_{ij} are integers verifying $g \geq 0,\ \ m_i \geq 2,\ n_{ij} \geq 2$. g is the topological genus of X. The sign determines the orientability of X. The numbers m_i are the *proper periods* corresponding to cone points in X. The brackets $(n_{i1}, \ldots, n_{is_i})$ are the *period-cycles*. The number k of period-cycles is equal to the number of boundary components of X . Numbers n_{ij} are the periods of the period-cycle $(n_{i1}, \ldots, n_{is_i})$ also called *link-periods*, corresponding to corner points in the boundary of X. The number $p = \eta g + k - 1$, where $\eta = 1$ or 2 if the sign of $\sigma(\Gamma)$ is '$-$' or '$+$' respectively, is called the *algebraic genus* of X.

The signature determines a presentation [**10**] of Γ:

Generators

x_i $\quad i = 1, \ldots, r;$
e_i $\quad i = 1, \ldots, k;$
$c_{i,j}$ $\quad i = 1, \ldots, r;\quad j = 0, \ldots, s_i;$
a_i, b_i $\quad i = 1, \ldots, g;$ (if σ has sign '$+$');
d_i $\quad i = 1, \ldots, g.$ (if σ has sign '$-$').

Relations:

$$x_i^{m_i} = 1; \qquad i = 1, \ldots, r;$$

$$c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1; \qquad i = 1, \ldots, k;\ j = 1, \ldots, s_i;$$

$$e_i^{-1} c_{i,0} e_i c_{i,s_i} = 1; \qquad i = 1, \ldots, k;$$

$$\prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1; \qquad i = 1, \ldots, g;\ \text{(if } \sigma \text{ has sign '+')};$$

$$\prod_{i=1}^{r} x_i \prod_{i=1}^{k} e_i \prod_{i=1}^{g} d_i^2 = 1; \qquad i = 1, \ldots, g;\ \text{(if } \sigma \text{ has sign '-')};$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide reflections.

Wilkie in [**16**] found a fundamental region R_W from which he obtained the algebraic structure of NEC groups. The region R_W is called a *canonical region* or Wilkie region.

For an NEC group Γ with signature as (2.1) the region R_W is a hyperbolic polygon with sides labelled in anticlockwise order as follows

$$\varepsilon_1, \gamma_{10}, \ldots, \gamma_{1s_1}, \varepsilon_1', \ldots, \varepsilon_k, \gamma_{k0}, \ldots, \gamma_{ks_k}, \varepsilon_k', \alpha_1, \beta_1', \alpha_1', \beta_1, \ldots, \alpha_g, \beta_g', \alpha_g', \beta_g,$$

if sign '$+$', or

$$\varepsilon_1, \gamma_{10}, \ldots, \gamma_{1s_1}, \varepsilon_1^{'}, \ldots, \varepsilon_k, \gamma_{k0}, \ldots, \gamma_{ks_k}, \varepsilon_k^{'}, \delta_1, \delta_1^*, \ldots, \delta_g, \delta_g^*,$$

if sign '−', where

$$e_i(\varepsilon_i^{'}) = \varepsilon_i, \quad c_i(\gamma_i) = \gamma_i, \quad a_i(\alpha_i^{'}) = \alpha_i, \quad b_i(\beta_i^{'}) = \beta_i, \quad d_i(\delta_i^*) = \delta_i.$$

Let us denote by $< s_1, s_2 >$ the angle between two consecutive sides. In R_W we have

$$< \varepsilon_i, \gamma_i > = < \gamma_i, \varepsilon^{'} > = \pi/2,$$

and the sum of the remaining angles (accidental cycle) is 2π. Without a loss of generality we may suppose R_W is a convex polygon.

Every NEC group Γ with signature (2.1) has associated to it a fundamental region whose area $\mu(\Gamma)$, called the *area of the group* (see [**15**]), is:

$$\mu(\Gamma) = 2\pi\left(\eta g + k - 2 + \sum_{i=1}^{r} (1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i} (1 - \frac{1}{n_{i,j}})\right) \tag{2.2}$$

An NEC group with signature (2.1) actually exists if and only if the right hand side of (2.2) is greater than 0 (see [**17**]).

If Γ is a subgroup of an NEC group Γ' of finite index N, then Γ is also an NEC group and the following Riemann-Hurwitz formula holds:

$$\mu(\Gamma) = N\mu(\Gamma'). \tag{2.3}$$

Let X be a Klein surface of topological genus g with k boundary components. Then by [**14**] there exists an NEC group Γ with signature

$$\sigma(\Gamma) = (g; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\}) \tag{2.4}$$

such that $X = \mathcal{D}/\Gamma$, where the sign is "+" if X is orientable and "−" if not. An NEC group with signature (2.4) is called a *surface group*.

For each automorphism group G of a surface $X = \mathcal{D}/\Gamma$ of algebraic genus $p > 2$ there exists an NEC group Γ' such that $G = \Gamma'/\Gamma$ where $\Gamma \subset \Gamma' \subset N_{\mathcal{G}}$, and $N_{\mathcal{G}}$ denotes the normalizer of Γ in the group $\mathcal{G}$, the full group of isometries of $\mathcal{D}$ [**13**].

We give two previous results from [**2**] in Proposition 2.1 for future reference.

PROPOSITION 2.1. (a) *The Klein surface $X = \mathcal{D}/\Gamma$ is q-hyperelliptic if and only if there exists an* NEC *group Γ_1 with algebraic genus q such that $\Gamma \triangleleft_2 \Gamma_1$.*

(b) *Let X be a q-hyperelliptic Klein surface of algebraic genus $p \geq 2$ such that* $p > 4q + 1$. The*n the group Γ_1* is *unique and the automorphism ϕ, $< \phi > = \Gamma_1/\Gamma$, is central in* Aut($X$).

In our case the algebraic genus of q-hyperelliptic tori is $k + 1$ so that the inequality in Proposition 2.1 (b) becomes $k > 4q$.

3. The signature of the q-hyperellipticity group Γ_1. Let Γ be a surface NEC group with signature (1.1) and let Γ_1 be an NEC group with $\Gamma \triangleleft_2 \Gamma_1$. Then the signature of Γ_1 is [**2**]:

$$\sigma(\Gamma_1) = (g; \pm; [2^r], \{(2^{s_1}), \ldots, (2^{s_n})\}), \tag{3.1}$$

where s_i are even and $q = \eta g + n - 1$. We have denoted by $[2^r]$ the set of proper periods $[2, \overset{r}{\ldots}, 2]$, and in a similar way the link periods in the period-cycles.

Our first task is to look for the possible values for g, r, n and s_i in (3.1). This is done by means of (2.3). Let m be the number of non empty period-cycles in (3.1).

Proposition 3.1. *The actual values for* g, r *and* m *in* (3.1) *are given in the following table*:

Case	g	η	r	m
I	0	2	0	0
II	0	2	0	1
III	0	2	0	2
IV	0	2	1	0
V	0	2	1	1
VI	0	2	2	0
VII	0	2	2	1
VIII	0	2	3	0
IX	0	2	4	0
X	1	1	0	0
XI	1	1	0	1
XII	1	1	1	0
XIII	1	1	2	0
XIV	1	2	0	0
XV	2	1	0	0

Proof. From (2.3) we have

$$k = 2\left(\eta g + n - 2 + \frac{r}{2} + \frac{1}{4}\sum_{i=1}^{n} s_i\right). \tag{3.2}$$

The number of non-empty period-cycles is m and so

$$k \leq 2(n-m) + \frac{1}{2}\sum_{i=1}^{n} s_i,$$

or, equivalently,

$$-2m \geq k - 2n - \frac{1}{4}\sum_{i=1}^{n} s_i. \tag{3.3}$$

From (3.2) we obtain

$$k - 2n - \frac{1}{2}\sum_{i=1}^{n} s_i = 2\eta g - 4 + r, \tag{3.4}$$

and from (3.3) and (3.4)

$$2m \leq 4 - (2\eta g + r).$$

Giving numeric values to g and r and taking account of the sign in the signature we obtain the entries of the table. □

We are interested in the case when the group of the q-hyperellipticity is unique. As we saw in the previous section, k must be greater than $4q$; from now on we always suppose that this condition holds.

THEOREM 3.2. *Let* $X = \mathcal{D}/\Gamma$ *be a* k*-bordered torus,* $k > 4q$. *Then* X *is* q*-hyperelliptic if and only if there exists a unique* NEC *group* Γ_1 *with algebraic genus* q, *such that* $\Gamma \lhd_2 \Gamma_1$ and *the signature of* Γ_1 *is one of the following four signatures*:

(1) $\left(0, +, [-], \left\{(-)^q, (2^{2(k-2q+2)})\right\}\right)$;

(2) $\left(0, +, [-], \left\{(-)^{q-1}, (2^{s_1}), (2^{s_2})\right\}\right)$, *where* $s_1 + s_2 = 2(k - 2q + 2)$, s_1 *and* s_2 *are even*;

(3) $\left(0, +, [2^4], \{(-)\}\right)$, *where* $q = 0$, $k = 2$;

(4) $\left(1, -, [-], \left\{(-)^{q-1}, (2^{2(k-2q+2)})\right\}\right)$.

Proof. First of all let us observe from (3.2) that if $m = 0$ (every period-cycle is empty) then

$$k = 2(q - 1) + r,$$

and in this case $k > 4q$ if and only if $r > 2q + 2$. From Proposition 3.1 we have $r \leq 4$, then $m \neq 0$ except for the case IX.

Now we may discard a lot of cases in Proposition 3.1. The available ones are the cases II, III, V, VII, IX and XI. Each case gives us a possible signature for Γ_1 and for each one we must study the existence of an epimorphism

$$\theta_1 : \Gamma_1 \longrightarrow Z_2 = \{1, y\},$$

with $\ker \theta_1 = \Gamma$.

In order to construct such an epimorphism let us observe that since $\ker \theta_1$ must be an orientable surface group ([**4**, Chapter 2]),

(a) consecutive reflections in a period-cycle cannot have the same image by θ_1,

(b) non orientable words (words in the generators of $\Gamma_1 - \Gamma$) cannot belong to $\ker \theta_1$, and

(c) the image by θ_1 of every elliptic generator must have order two.

Case II: $\sigma(\Gamma_1) = \left(0, +, [-], \left\{(-)^{n-1}, (2^s)\right\}\right)$.

Since Γ_1 has algebraic genus q then $n - 1 = q$. From (3.2) we have $k = 2(n - 2 + \frac{s}{4})$; therefore $s = 2(k - 2q + 2)$.

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To define θ_1, we see (a) implies

$$\theta_1(c_{q+1,2j}) = y, \quad j = 0, \ldots, k-2q+2$$
$$\theta_1(c_{q+1,2j+1}) = 1, \quad j = 0, \ldots, k-2q+1.$$

Thus we obtain $k-2q+2$ empty period-cycles in $\ker\theta_1$. The remaining $2(q-1)$ must be obtained from the empty period-cycles of Γ_1: $C_1, \ldots, C_q$. Then $q-1$ reflections from the set $\{c_{1,0}, \ldots, c_{q,0}\}$ will be in $\ker\theta_1$, and for each one $\theta_1(e_i) = 1$. Let us define

$$\theta_1(c_{i,0}) = \theta_1(e_i) = 1,$$

for $i = 1, \ldots, q-1$, and $\theta_1(c_{q,0}) = 1$. To complete the epimorphism there still are two images to determine: $\theta_1(e_q)$ and $\theta_1(e_{q+1})$.

From the canonical relation $e_1 \cdots e_{q+1} = 1$, we have

$$\theta_1(e_q) = \theta_1(e_{q+1}),$$

and, by (b), $\theta_1(e_q) = 1$; otherwise $e_q c_{q,0}$ would be a non-orientable word in $\ker\theta_1$. Then

$$\begin{array}{lccll} \theta_1 : & \Gamma_1 & \longrightarrow & Z_2 & \\ & e_i & \longrightarrow & 1 & i = 1, \ldots, q+1, \\ & c_{i,0} & \longrightarrow & 1 & i = 1, \ldots, q-1, \\ & c_{q,0} & \longrightarrow & y & \\ & c_{q+1,2j} & \longrightarrow & y & \\ & c_{q+1,2j+1} & \longrightarrow & 1 & \end{array}$$

Furthermore, by construction, θ_1 is unique up to automorphisms of Γ_1.

Case III: $\sigma(\Gamma_1) = (0, +, [-], \{(-)^{q-1}, (2^{s_1}), (2^{s_2})\})$, with s_1 and s_2 even.

From (3.2) we have

$$k = 2\left(q - 1 + \frac{s_1 + s_2}{4}\right) = 2(q-1) + \frac{s_1 + s_2}{2},$$

and hence

$$s_1 + s_2 = 2(k - 2q + 2).$$

Reasoning as in Case II we obtain the epimorphism θ_1 (unique up to $\mathrm{Aut}(\Gamma_1)$) defined as follows:

$$\begin{array}{lccll} \theta_1 : & \Gamma_1 & \longrightarrow & Z_2 & \\ & e_i & \longrightarrow & 1 & i = 1, \ldots, q+1, \\ & c_{i,0} & \longrightarrow & 1 & i = 1, \ldots, q-1, \\ & c_{i,j} & \longrightarrow & y & i = q, q+1, \quad j \text{ even}, \\ & c_{i,j} & \longrightarrow & 1 & i = q, q+1, \quad j \text{ odd}. \end{array}$$

Case V: $\sigma(\Gamma_1) = (0, +, [2], \{(-)^q, (2^s)\})$.

From the relation (3.2) we obtain

$$k = 2(q - 1 + \frac{1}{2} + \frac{s}{2}) = 2q - 1 + \frac{s}{2},$$

and hence

$$s = 2(k - 2q + 2).$$

But the number of period-cycles in $\ker\theta_1$ is

$$2l + (k - 2q + 1),$$

where $l = \#\{c_{i,0} : \theta_1(c_{i,0}) = 1,\ i = 1, \dots, q\}$. This number never equals k. So this case must be discarded.

Case VII: $\sigma(\Gamma_1) = (0, +, [2, 2], \{(-)^q, (2^s)\})$.

Every epimorphism $\theta_1 : \Gamma_1 \longrightarrow Z_2$ such that $\ker\theta_1$ is a surface group must satisfy

$$\theta_1(x_1) = \theta_1(x_2) = \theta_1(c_{i,j}) = y,$$

for some $c_{i,j} \in \Gamma_1$. Then $\ker\theta_1$ is non-orientable, and this case must also be discarded.

Case IX: $\sigma(\Gamma_1) = (0, +, [2^4], \{(-)\})$.

In this case the epimorphism $\theta_1 : \Gamma_1 \longrightarrow Z_2$ such that $\ker\theta_1$ is a torus with two boundaries is defined by

$$\begin{array}{llcll} \theta_1 : & \Gamma_1 & \longrightarrow & Z_2 & \\ & x_i & \longrightarrow & y & i = 1, \dots, 4, \quad \text{(see(c))} \\ & e_1 & \longrightarrow & 1 & \\ & c_1 & \longrightarrow & 1 & \end{array}$$

is unique up to $\mathrm{Aut}(\Gamma_1)$.

Case XI: $\sigma(\Gamma_1) = (1, -, [-], \{(-)^{q-1}, (2^s)\})$.

From (3.2) $k = 2(q - 1) + \frac{s}{2}$. So that $s = 2(k - 2q + 2)$ and reasoning as in Case II, the epimorphism θ_1 is defined by

$$\begin{array}{llcll} \theta_1 : & \Gamma_1 & \longrightarrow & Z_2 & \\ & d_i & \longrightarrow & y & \\ & e_i & \longrightarrow & 1 & i = 1, \dots, q, \\ & c_{i,0} & \longrightarrow & 1 & i = 1, \dots, q - 1, \\ & c_{q,2j} & \longrightarrow & y & \\ & c_{q,2j+1} & \longrightarrow & 1 & \end{array}$$

4. Dimension of the Teichmüller space. In this Section we study the Teichmüller space associated to q-hyperelliptic k-bordered tori.

Let $\mathcal{G}$ be the full group of isometries of the hyperbolic plane $\mathcal{D}$. Given an NEC group Λ let us denote by $\mathcal{R}(\Lambda, \mathcal{G})$ the set of monomorphisms $r : \Lambda \longrightarrow \mathcal{G}$ such that $r(\Lambda)$ is a discrete group and the quotient $\mathcal{D}/r(\Lambda)$ is compact. Two elements $r_1, r_2 \in \mathcal{R}(\Lambda, \mathcal{G})$ are equivalent, $r_1 \sim r_2$, if and only if there exists an element $g \in \mathcal{G}$ satisfying $r_1(\lambda) = g\ \ r_2(\lambda)\ \ g^{-1}$, for every $\lambda \in \Lambda$ The quotient space $\mathcal{T}(\Lambda, \mathcal{G}) = \mathcal{R}(\Lambda, \mathcal{G})/\sim$, the Teichmüller space of Λ, is homeomorphic to a cell with dimension $d(\Lambda)$. If Λ is a Fuchsian group with (NEC) signature $(g, +, [m_1, \ldots .m_r], \{-\})$ it is well known that $d(\Lambda) = 6g + 2r - 6$. It is proved in **[15]** that if Λ is a proper NEC group then $d(\Lambda) = \frac{d(\Lambda^+)}{2}$.

The Teichmüller modular group $\mathcal{M}(\Lambda)$ of Λ is the quotient $\mathrm{Aut}(\Lambda)/\mathrm{Inn}(\Lambda)$ **[11]**, where $\mathrm{Aut}(\Lambda)$ is the full group of automorphisms of Λ and we denote by $\mathrm{Inn}(\Lambda)$ the inner automorphisms.

Now let Γ be an NEC group with signature $(1, +, [-], \{(-)^k\})$ and $X = \mathcal{D}/\Gamma$. Let ϕ an automorphism of order two such that $X/ < \phi >= X_1$ has algebraic genus q and let Γ_1 be an NEC group such that $X_1 = \mathcal{D}/\Gamma_1$. We have seen in the previous section that if $k > 4q$ the group Γ_1 has a signature of four possible types. So we divide the q-hyperelliptic k-bordered tori into four classes according to whether the quotient by the q-hyperelliptic involution is:

(1) a sphere with corner points in a single connected boundary component;
(2) a sphere with corner points in two connected boundary components;
(3) a disc with four cone points;
(4) a non-orientable surface.

Hence the Teichmüller space

$$\mathcal{T}_q = \{[r] \in \mathcal{T} :\ \mathcal{D}/r(\Gamma) \text{ is a } q\text{-hyperelliptic } k\text{-bordered torus, } k > 4q\}$$

becomes divided into four subspaces corresponding to the above classes:

$$\mathcal{T}_q = \mathcal{T}_q^1 \cup \mathcal{T}_q^2 \cup \mathcal{T}_q^3 \cup \mathcal{T}_q^4.$$

From **[9]** we have for $i = 1, 3$ and 4,

$$\mathcal{T}_q^i = \bigcup_{\overline{\alpha} \in \mathcal{M}(\Gamma)} \overline{\alpha}(\bigcup_{i_\phi \in \Phi(\Gamma, \Gamma_1, \Gamma_1/\Gamma)} i_\phi^*(\mathcal{T}(\Gamma_1, \mathcal{G}))), \tag{4.1}$$

where Γ_1 is the (unique) NEC group of the q-hyperellipticity of X, $\Phi(\Gamma, \Gamma_1, \Gamma_1/\Gamma)$ is the family of equivalence classes of surjections $\phi : \Gamma_1 \longrightarrow Z_2$ with $\ker \phi = \Gamma$ modulo the action of $\mathrm{Aut}(\Gamma_1)$ and $\mathrm{Aut}(Z_2)$ and i_ϕ^* is the induced isometry by the inclusion $i_\phi : \ker \phi \longrightarrow \Gamma_1$:

$$\begin{array}{rccc} i_\phi^* : & \mathcal{T}(\Gamma_1) & \hookrightarrow & \mathcal{T}(\Gamma) \\ & [\tau] & \longrightarrow & [\tau\, i_\phi] \end{array}$$

where $\tau \in \mathcal{R}(\Gamma, \mathcal{G})$.

In the class (2), we have a family of q-hyperellipticity groups, that will be denoted by $\Gamma_1^{s_1,s_2}$, with signature as in Theorem 3.2(2). Then, $\mathcal{T}_q^2$ is decomposed as follows:

$$\mathcal{T}_q^2 = \bigcup_{s_1+s_2=2(k-2q+2)} \mathcal{T}_q^{s_1,s_2},$$

where $\mathcal{T}_q^{s_1,s_2}$ has the same expression as in (4.1), changing Γ_1 to $\Gamma_1^{s_1,s_2}$.

In all cases, the families $\Phi(\Gamma, \Gamma_1^i, \Gamma_1^i/\Gamma)$, $i = 1, 3$ and 4, and $\Phi(\Gamma, \Gamma_1^{s_1,s_2}, \Gamma_1^{s_1,s_2}/\Gamma)$ have a single element, as was shown in the proof of Theorem 3.2. There, we constructed the unique class of epimorphisms

$$\phi : \Gamma_1 \longrightarrow Z_2, \quad \ker\phi = \Gamma, \quad \text{for all } q\text{-hyperellipticity groups } \Gamma_1$$

(see Cases II, III, IX and XI).

So the conditions of Maclachlan's method [**12**, Lemma 3] hold. Thus, we may conclude that $\mathcal{T}_q^i$ is a submanifold of $\mathcal{T}(\Gamma)$ of dimension $d(\Gamma_1) = 2k - q - 1$. We have proved the following Theorem.

THEOREM 4.1. *The subspace of the Teichmüller space associated to each class of q-hyperelliptic k-bordered tori, with $k > 4q$, is a submanifold of dimension $2k - q + 1$.*

5. Geometrical description of $\mathcal{T}$. The abstract concept of Teichmüller space $\mathcal{T}(\Gamma)$ of an NEC group Γ can be interpreted by means of fundamental regions. As we have seen two elements $r_1, r_2 \in \mathcal{R}(\Gamma)$ belong to the same class in $\mathcal{T}(\Gamma)$ if and only if there exists $g \in \mathcal{G}$ such that

$$r_1(\gamma) = g\, r_2(\gamma)\, g^{-1}, \text{ for all } \gamma \in \Gamma.$$

Equivalently, the fundamental regions of the NEC groups $r_1(\Gamma)$ and $r_2(\Gamma)$ are congruent, that is, there exists an isometry $g \in \mathcal{G}$ which transforms one of them on the other one. For this reason we can associate to each class in $\mathcal{T}(\Gamma)$ a normalized fundamental region R such that the number of parameters involved in the construction of R equals $d(\mathcal{T}(\Gamma))$, the dimension of the Teichmüller space.

Let R_1 be a fundamental region of the q-hyperellipticity group Γ_1. The canonical epimorphism $\theta_1 : \Gamma_1 \longrightarrow \Gamma_1/\Gamma$ gives us a way to obtain a fundamental region R of Γ from two copies of R_1. Our goal in this section will be the description of the necessary parameters in the construction of R_1. To do this we will transform a canonical Wilkie region R_W into a right-angled fundamental region by a cutting and pasting procedure.

Description of $\mathcal{T}_q^1$. Let Γ_1 be the q-hyperellipticity group with signature

$$(0, +, [-], \{(-)^q, (2^{2(k-2q+2)})\}),$$

and let R_W be a Wilkie region of Γ_1 (see Figure 1).

Let us consider the following geodesics in R_W: let λ_i be the common orthogonal to γ_i and $\gamma_{q+1,0}, (i = 1, \ldots, q)$. Every side γ_i is divided by λ_i in two pieces, $\gamma_i = \gamma_i^1 \cup \gamma_i^2$, and $\gamma_{q+1,0}$ is divided by the λ_i in $q + 1$ pieces:

$$\gamma_{q+1,0} = \overline{\gamma}_0 \cup \ldots \cup \overline{\gamma}_q.$$

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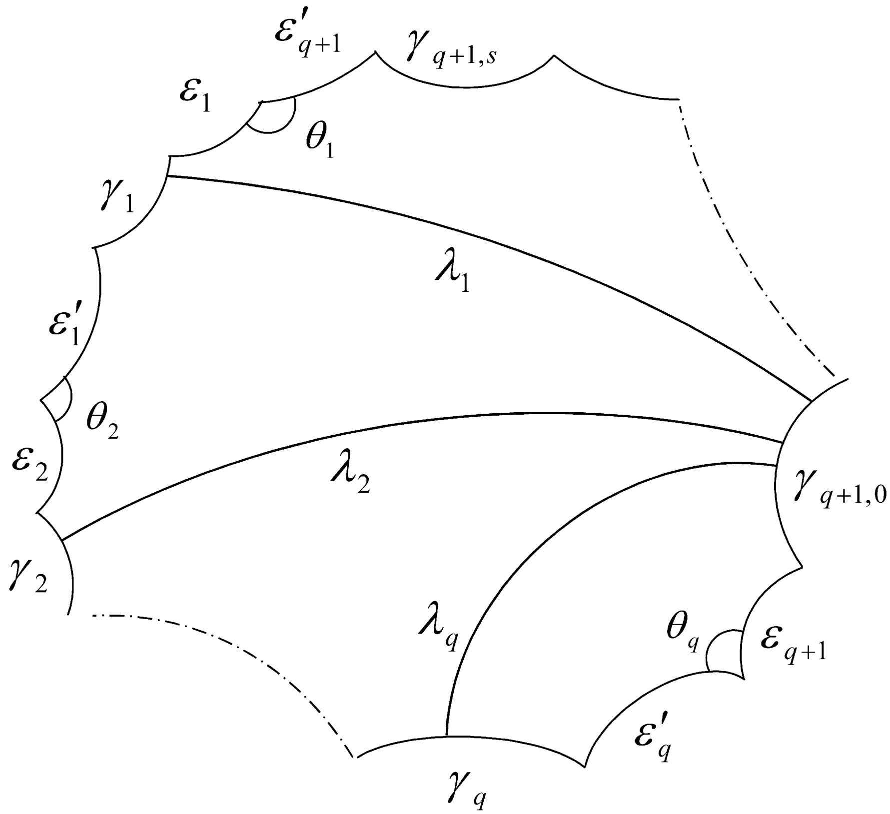

Figure 1. R_W.

Let us denote R_W by R_{q+1} and define transformations Q_{λ_i} by the following rule: cut in R_{i+1} by λ_i the polygon which contains the side ε'_i and paste this side with ε_i via e_i to obtain R_i.

Then the region

$$R^* = Q_{\lambda_1} \cdots Q_{\lambda_q}(R_W)$$

is a right-angled fundamental region of Γ_1 with $2k+4$ sides:

$$\underbrace{\ldots, f_{i-1}(\overline{\gamma}_{i-1}), f_{i-1}(\lambda_i), f_{i-1}(\gamma_i^*), f_i(\lambda_i), \ldots,}_{i=1,\ldots,q} f_q(\overline{\gamma}_q) \cup \gamma_{q+1,s}, \gamma_{q+1,s-1}, \ldots, \gamma_{q+1,1}, \qquad (5.1)$$

where $s = 2(k - 2q + 2)$ and

$$f_0 = i_d,$$
$$f_i = e_1 \cdots e_i,$$
$$\gamma_i^* = e_i(\gamma_i^2) \cup \gamma_i^1, \; i = 1, \ldots, q.$$

The pairs of identified sides in R^* are $(\lambda_i, f_i(\lambda_i))$, $i = 1, \ldots, q$.

Then we have constructed a hyperbolic right-angled polygon R^* having $2k+4$ sides and the $2k+1$ consecutive sides with the following lengths:

$$\left|\gamma_{q+1,s-3}\right|, \ldots, \left|\gamma_{q+1,1}\right|, \ldots, \underbrace{\left|\overline{\gamma}_{i-1}\right|, |\lambda_i|, \left|f_{i-1}(\gamma_i^*)\right|, \left|f_i(\lambda_i)\right|}_{i=0,\ldots,q}. \tag{5.2}$$

Since $\left|f_{i-1}(\gamma_i^*)\right| = \left|\gamma_i\right|$ and $\left|f_i(\lambda_i)\right| = |\lambda_i|$ then we have the following free lengths

$\lvert\lambda_1\rvert, \ldots, \lvert\lambda_q\rvert,$	(q orthogonal lines)
$\lvert\gamma_1\rvert, \ldots, \lvert\gamma_q\rvert,$	(q empty boundaries)
$\lvert\gamma_{q+1,1}\rvert, \ldots, \lvert\gamma_{q+1,s-3}\rvert,$	($s-3$ sides of the non-empty $(q+1)$-boundary)
$\lvert\overline{\gamma}_0\rvert, \ldots, \lvert\overline{\gamma}_{q-1}\rvert,$	(q pieces in that $\gamma_{q+1,0}$ becomes divided).

Then, there are $2k-q+1$ lengths and this number equals the dimension of $\mathcal{T}_q^1$.

Description of $\mathcal{T}_q^{s_1,s_2}$. Let Γ_1 be the q-hyperellipticity group with signature

$$(0, +, [-], \{(-)^{q-1}, (2^{s_1}), (2^{s_2})\}),$$

where s_1, s_2, are even positive integers such that $s_1 + s_2 = 2(k-2q+2)$; and let R_W be a Wilkie region of Γ_1. To convert R_W in a right-angled polygon, let us consider

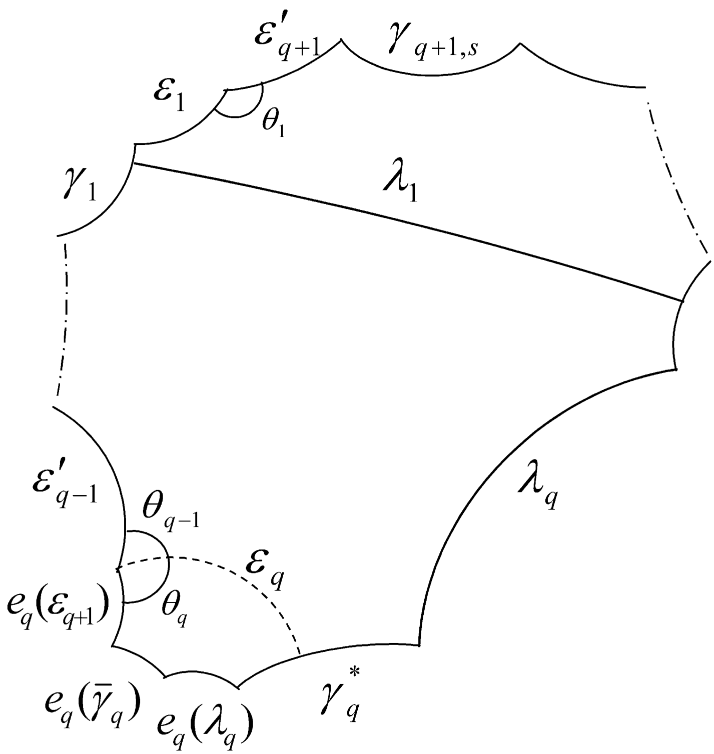

Figure 2. $Q_{\lambda_q}(R_W)$.

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the geodesics: λ_i, ($i = 1, \ldots, q-1$) as in the description of $\mathcal{T}_q^1$, and λ_q the common orthogonal to $\gamma_{q,0}$ and $\gamma_{q+1,0}$.

Then the region

$$R^* = Q_{\lambda_1} \cdots Q_{\lambda_q}(R_W)$$

is a right-angled fundamental region of Γ_1 with $2k+4$ sides. The perimeter of R^*, and the $2k+1$ lengths of consecutive sides in R^* are the same as (5.1) and (5.2), changing s to s_2, and $f_{q-1}(\gamma_q^*)$ to $f_{q-1}(\gamma_{q,0}^*) \cup f_q(\gamma_{q,1}), \ldots, f_q(\gamma_{q,s_1})$.

The involved lengths are:

$\lvert\lambda_1\rvert, \ldots, \lvert\lambda_q\rvert$,	(q orthogonal lines)
$\lvert\gamma_1\rvert, \ldots, \lvert\gamma_{q-1}\rvert$,	($q-1$ empty boundaries)
$\lvert\gamma_{q,0}\rvert, \ldots, \lvert\gamma_{q,s_1}\rvert$,	(s_1+1 sides of the non-empty q-boundary)
$\lvert\overline{\gamma}_0\rvert, \ldots, \lvert\overline{\gamma}_{q-1}\rvert$,	(q pieces in that $\gamma_{q+1,0}$ becomes divided)
$\lvert\gamma_{q+1,1}\rvert, \ldots, \lvert\gamma_{q+1,s_2-3}\rvert$,	(s_2-3 sides of the non-empty $(q+1)$-boundary)

Description of $\mathcal{T}_q^3$. Let Γ_1 be the q-hyperellipticity group with signature

$$(0, +, [2^4], \{(-)\}),$$

and let R_W be a Wilkie region of Γ_1 (see Figure 3). The side-pairings are (δ_i', δ_i) via the canonical generators x_i, $i = 1, \ldots, 4$; and $(\varepsilon_1', \varepsilon_1)$ via e_i.

To convert R_W in a right-angled polygon let us consider the orthogonal lines λ_i from the vertex X_i to γ_1, $i = 1, \ldots, 4$. These geodesic segments divide γ_1 in five pieces $\overline{\gamma}_i$, $i = 0, \ldots, 4$. Denote R_W by R_4, and define the transformations Q_{λ_i}, $i = 1, \ldots, 4$, as follows: cut in R_i the polygon which contains the side δ_i' and paste it with δ_i via x_i to obtain R_{i-1}.

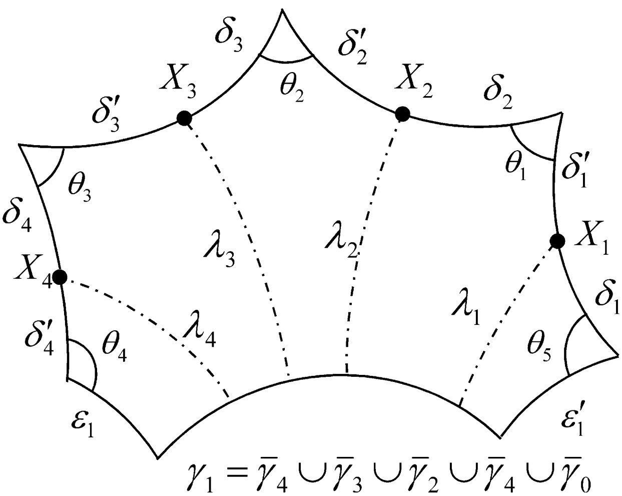

Figure 3.

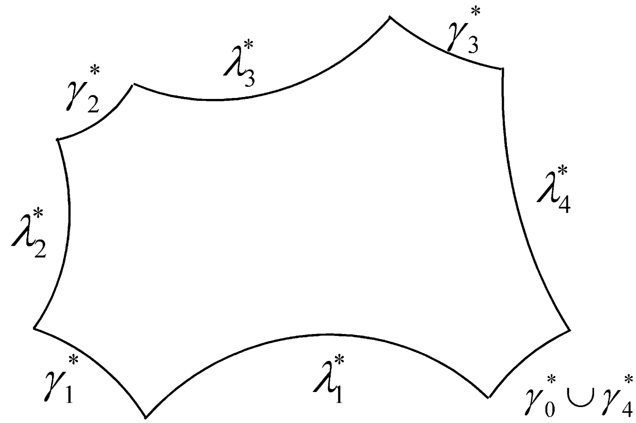

Figure 4. R^*.

Then $R^* = Q_{\lambda_1} \circ \ldots \circ Q_{\lambda_4}(R_W)$ is a right-angled octagon with the following sides:

$$\gamma_5^* \cup \gamma_1^*, \lambda_1^*, \gamma_2^*, \lambda_2^*, \gamma_3^*, \lambda_3^*, \gamma_4^*, \lambda_4^*,$$

where

$$\begin{aligned} \lambda_i^* &= f_i(\lambda_i) \cup f_{i-1}(\lambda_i), \\ \gamma_i^* &= f_i(\overline{\gamma}_i), \\ f_i &= x_1 i, \\ f_0 &= id. \end{aligned}$$

The polygon R^* is completely determined by five lengths:

$$2|\lambda_1|, \ldots, 2|\lambda_4|, |\overline{\gamma}_1|,$$

where $|\lambda_i|$ is the distance between the boundary and the cone point X_i, and $|\overline{\gamma}_1|$ is the distance between λ_1 and λ_2.

Description of $\mathcal{T}_q^4$. Let Γ_1 be the q-hyperellipticity group with signature

$$(1, -, [-], \{(-)^{q-1}, (2^{2(k-2q+2)})\}),$$

and let R_W be a Wilkie region of Γ_1 (see Figure 5).

Let d be the glide reflection which transforms δ' in δ. The axis of d is the geodesic joining the middle points of δ and δ'. Let P, Q, and S be the vertices between the pair of sides (ε_q', δ), (δ', ε_1) and (δ, δ'). Let us consider the following segments: ε (respectively ε') the orthogonal to the axis of d from P (respectively Q) (see Figure 6). Then, d^2 is a hyperbolic transformation satisfying $d^2(\zeta') = \zeta$.

We are going to convert R_W in a fundamental region of Γ_1 in which the side-pairings involves the hyperbolic transformation d^2. To do it, let us consider the geodesic m orthogonal to the axis of d which contains the vertex S, and the hyperbolic triangles T_1 and T_2 (see Figure 4). Then, the region

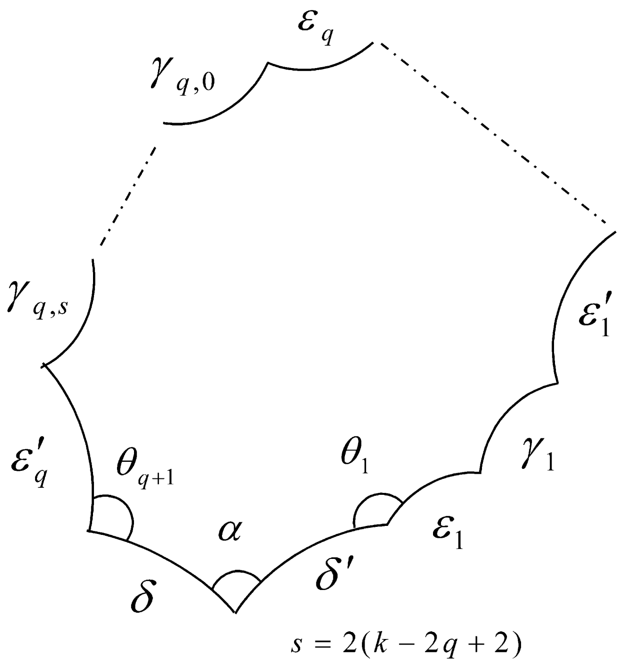

Figure 5.

$$\widehat{R} = R - (T_1 \cup T_2) \cup d(T_2) \cup d^{-1}(T_1)$$

is a fundamental region of Γ_1 that follows the pattern of identifications of R in $\mathcal{T}_q^1$ (see Figure 7).

Then, we transform $\widehat{R}$ into a right-angled region as we did for R in $\mathcal{T}_q^1$, obtaining in the same way the set of necessary lengths for its construction.

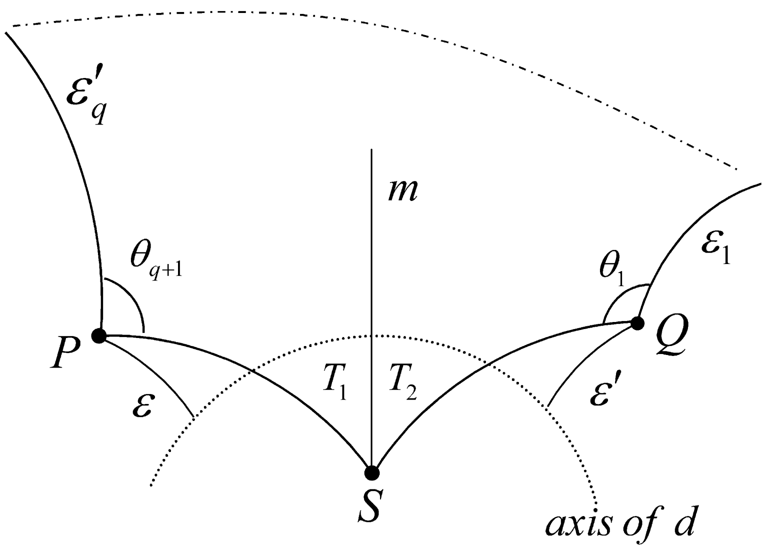

Figure 6.

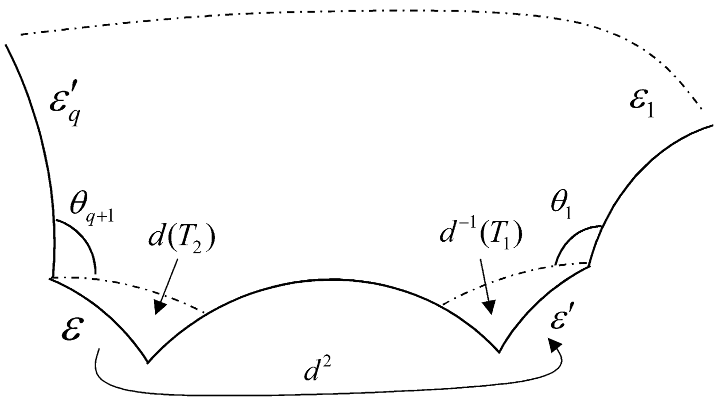

Figure 7.

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