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# GRAY IDENTITIES, CANONICAL CONNECTION AND INTEGRABILITY

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*Abstract* We characterize quasi-Kähler manifolds whose curvature tensor associated to the canonical Hermitian connection satisfies the first Bianchi identity. This condition is related to the third Gray identity and in the almost-Kähler case implies the integrability. Our main tool is the existence of generalized holomorphic frames previously introduced by the second author. By using such frames we also give a simpler and shorter proof of a theorem of Goldberg. Furthermore, we study almost-Hermitian structures having the curvature tensor associated to the canonical Hermitian connection equal to zero. We show some explicit examples of quasi-Kähler structures on the Iwasawa manifold having the Hermitian curvature vanishing and the Riemann curvature tensor satisfying the second Gray identity.

*Keywords:* almost Kähler manifolds; quasi-Kähler manifolds; integrability of almost complex structures; Gray identities

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## 1. Introduction

Quasi-Kähler and almost-Kähler manifolds are special classes of almost-Hermitian manifolds and can be considered as natural generalizations of Kähler manifolds to the context of almost-symplectic and symplectic manifolds. It is well known that if $(M,\omega)$ is an (almost-)symplectic manifold, then there always exists an almost-complex structure $J$ compatible with $\omega$. Furthermore, the choice of such an almost-complex structure is unique up to homotopy. Hence, quasi-Kähler and almost-Kähler structures can be considered as a tool to study (almost-)symplectic manifolds.

The interplay between the integrability of almost-Hermitian structures and the curvature has been largely studied in recent years (see, for example, [**2**, **11**] and references therein). One of the most important results in this topic is due to Goldberg. Indeed, Goldberg [**9**] proved that if the Riemann curvature tensor of an almost-Kähler metric $g$ satisfies the first Gray condition, i.e. if it commutes with the almost-complex structure, then $g$ is a Kähler metric. Gray's conditions were introduced in [**10**] and consist of some formulae involving the curvature tensor of an almost-Hermitian metric and the associated almost-complex structure. The Goldberg Theorem has been further generalized to the following formula:

$$s_* - s = \|\nabla\omega\|^2, \tag{1.1}$$

where $s$ and $s_*$ are the scalar curvature and the $*$-scalar curvature associated to an almost-Kähler structure $(g, J, \omega)$, respectively (see, for example, [**2**]). The classical proof of this result is based on the Weitzenböck decomposition.

Another important curvature tensor in almost-Hermitian geometry is the *Hermitian curvature tensor* $\tilde{R}$. This tensor is defined as the curvature of the unique Hermitian connection $\tilde{\nabla}$, whose torsion has $(1,1)$-part vanishing.

In [**5**] de Bartolomeis and Tomassini proved that a quasi-Kähler manifold always admits a special complex frame. This result has been improved in [**16**] by introducing generalized normal holomorphic frames. Such frames have been further taken into account in [**17**] to prove that if the holomorphic bisectional curvature associated to an almost-Kähler metric $g$ and the holomorphic bisectional curvature associated to the canonical connection coincide, then $g$ is a Kähler metric. This result is not trivial, since the Hermitian curvature tensor does not necessarily satisfy the first Bianchi identity.

As a first result of this paper we give a new proof of (1.1). Our proof is elementary and makes use not of the Weitzenböck decomposition, but only of the existence of generalized normal holomorphic frames. Sections 3 and 4 are dedicated to the study of the Hermitian curvature tensor in quasi-Kähler and almost-Kähler manifolds. We show that in the quasi-Kähler case this curvature tensor satisfies the first Bianchi identity if and only if the curvature of $g$ satisfies both the third Gray condition and another special identity involving the derivative of the Nijenhuis tensor, as follows.

**Theorem 1.1.** *Let $(M, g, J, \omega)$ be a quasi-Kähler manifold. The Hermitian curvature tensor $\tilde{R}$ satisfies the first Bianchi identity*

$$\mathop{\mathfrak{S}}_{X,Y,Z} \tilde{R}(X, Y, Z, \cdot) = 0 \quad \text{for every } X, Y, Z \in \Gamma(TM) \tag{1.2}$$

*if and only if the following conditions hold:*

(i) *the curvature tensor $R$ associated to $g$ satisfies the third Gray identity*

$$R(\bar{Z}_1, Z_2, Z_3, Z_4) = 0 \quad \text{for every } Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M);$$

(ii) *we have*

$$R(Z_1, Z_2, \bar{Z}_3, \bar{Z}_4) = \tfrac{1}{4} F(\bar{Z}_3, Z_1, Z_2, \bar{Z}_4)$$

*for every $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)$, where $F$ is the tensor*

$$F(X, Y, Z, W) := g((\nabla_X N)(Y, Z), W),$$

*$\nabla$ is the Levi-Cività connection of $g$ and $N$ denotes the Nijenhuis tensor.*

The previous theorem allows us to prove the following.

**Corollary 1.2.** *Let $(M, g, J, \omega)$ be an almost-Kähler manifold. Assume that the Hermitian curvature tensor associated to $(g, J)$ satisfies the first Bianchi identity (1.2). Then $(M, g, J, \omega)$ is a Kähler manifold.*

In § 4 we study almost-Hermitian manifolds whose Hermitian curvature tensor vanishes. By Corollary 1.2 this condition forces a four-dimensional quasi-Kähler structure to be Kähler. In higher dimensions things work differently, even in the compact case. We show that it is possible to construct examples of strictly quasi-Kähler nilmanifolds having Hermitian curvature equal to zero.

The study of the tensor $\tilde{R}$ is also related to a conjecture of Donaldson's. Indeed, $\tilde{R}$ has recently been taken into account by Tosatti *et al.* in [**14**] to study a conjecture of Donaldson's stated in [**6**]. More precisely, they proved that if $(M, \omega)$ is a symplectic manifold, $J$ is an almost-complex structure tamed by $\omega$ and $\mathcal{R}(g, J)$ denotes the tensor

$$\mathcal{R}_{i\bar{j}k\bar{l}}(g, J) := \tilde{R}^{j}_{ik\bar{l}} + 4N^{r}_{\bar{l}\bar{j}}\bar{N}^{i}_{\bar{r}\bar{k}}, \tag{1.3}$$

where $g$ is the metric associated to $(\omega, J)$ and $N$ is the Nijenhuis tensor of $J$. Then the condition $\mathcal{R}(g, J) \geqslant 0$ implies that Donaldson's conjecture holds.

It is important to observe that in the examples described in § 4 the tensor $\mathcal{R}(g, J)$ vanishes.

## Notation

Given a differential manifold $M$, $TM$ denotes its tangent bundle. If a vector bundle $F$ is fixed, then $\Gamma(F)$ denotes the vector space of the relative smooth sections. If $Z_i$ is a complex vector field on a manifold $M$, then we usually write $Z_{\bar{\imath}}$ instead of $\bar{Z}_i$. The cyclic sum is denoted by the symbol $\mathfrak{S}$.

## 2. Review

### 2.1. Almost-Hermitian manifolds

Let $M$ be a $2n$-dimensional manifold. An *almost-complex structure* on $M$ is an endomorphism $J$ of $TM$ satisfying $J^2 = -\mathrm{Id}$. An almost-complex structure $J$ is said to be *integrable* if the Nijenhuis tensor

$$N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \quad \text{for } X, Y \in \Gamma(TM)$$

vanishes everywhere. In view of the celebrated Newlander–Nirenberg Theorem [**12**], $J$ is integrable if and only if it is induced by a system of holomorphic coordinates. Any almost-complex structure on $M$ induces a natural splitting of the complexified tangent bundle into

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces to i and $-$i, respectively. Consequently, the vector bundle $\wedge^p M \otimes \mathbb{C}$ of complex $p$-forms on $M$ splits as

$$\wedge^p M \otimes \mathbb{C} = \bigoplus_{r+s=p} \wedge^{r,s} M.$$

Since

$$\mathrm{d}(\Gamma(\wedge^{r,s}M)) \subseteq \Gamma(\wedge^{r+2,s-1}M \oplus \wedge^{r+1,s}M \oplus \wedge^{r,s+1}M \oplus \wedge^{r-1,s+2}M),$$

the exterior derivative splits as

$$\mathrm{d} = A + \partial + \bar{\partial} + \bar{A}.$$

It is well known that $J$ is integrable if and only if $A = 0$. Furthermore, it can be useful to observe that the Nijenhuis tensor satisfies

$$N(Z_1, Z_2) \in \Gamma(T^{0,1}M), \qquad N(Z_1, \bar{Z}_2) = 0 \tag{2.1}$$

for every $Z_1, Z_2 \in \Gamma(T^{1,0}M)$. A Riemannian metric $g$ on $(M, J)$ is said to be *$J$-Hermitian* if it is preserved by $J$. In this case the pair $(g, J)$ is called an *almost-Hermitian structure*. Any almost-Hermitian structure $(g, J)$ induces a natural almost-symplectic structure $\omega(\cdot\,,\cdot) := g(J\cdot\,,\cdot)$.

**Definition 2.1.** The triple $(g, J, \omega)$ is called

(i) a *quasi-Kähler* structure if $\bar{\partial}\omega = (\mathrm{d}\omega)^{1,2} = 0$,

(ii) an *almost-Kähler* structure if $\mathrm{d}\omega = 0$.

On the other hand, if $\omega$ is a non-degenerate 2-form on an almost-complex manifold $(M, J)$, then we say that $J$ is *tamed* by $\omega$ if

$$\omega(X, JX) > 0 \quad \text{for all } X \neq 0.$$

In this case we can define a Riemannian metric $g$ by

$$g(X, Y) := \tfrac{1}{2}(\omega(X, JY) + \omega(Y, JX)).$$

The following lemma will be useful in the remainder of the paper (see, for example, [**13**, **16**]).

**Lemma 2.2.** *Let $(M, g, J, \omega)$ be an almost-Hermitian manifold and let $\nabla$ be the Levi-Cività connection associated to $g$. Then the following facts hold:*

(i) *the form $\omega$ is quasi-Kähler if and only if*

$$\nabla_{\bar{Z}_1} Z_2 \in \Gamma(T^{1,0}M) \quad \text{for all } Z_1, Z_2 \in \Gamma(T^{1,0}M); \tag{2.2}$$

(ii) *the form $\omega$ is almost-Kähler if and only if it is quasi-Kähler and the Nijenhuis tensor of $J$ satisfies*

$$g(\nabla_{Z_1} Z_2, Z_3) = \tfrac{1}{4} g(N(Z_2, Z_3), Z_1) \quad \text{for all } Z_1, Z_2, Z_3 \in \Gamma(T^{1,0}M). \tag{2.3}$$

**Proof.** It is well known that for an almost-Hermitian structure $(g, J, \omega)$ the following fundamental relation holds:

$$2g((\nabla_X J)Y, Z) = \mathrm{d}\omega(X, JY, JZ) - \mathrm{d}\omega(X, Y, Z) + g(N(Y, Z), JX) \tag{2.4}$$

for every $X, Y, Z \in \Gamma(TM)$. Items (i) and (ii) can be obtained just by considering the complex extension of (2.4). □

### 2.2. The canonical connection

A linear connection on an almost-Hermitian manifold $(M, g, J)$ is called *Hermitian* if it preserves $g$ and $J$. Any almost-Hermitian manifold admits a canonical Hermitian connection $\tilde{\nabla}$, which is characterized by the following properties:

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} J = 0, \quad \mathrm{Tor}(\tilde{\nabla})^{1,1} = 0,$$

where $\mathrm{Tor}(\tilde{\nabla})^{1,1}$ denotes the $(1,1)$-part of the torsion of $\tilde{\nabla}$. In the special case of a quasi-Kähler structure, $\tilde{\nabla}$ is given by

$$\tilde{\nabla} = \nabla - \tfrac{1}{2} J \nabla J,$$

where $\nabla$ is the Levi-Cività connection of $g$ (see, for example, [**8**]). We will call $\tilde{\nabla}$ simply the *canonical connection*. The connection $\tilde{\nabla}$ induces the *Hermitian curvature tensor*

$$\tilde{R}(X, Y, Z, W) = g(\tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, W).$$

Since $\tilde{\nabla}$ preserves $g$, one has

$$\tilde{R}(X, Y, Z, W) = -\tilde{R}(Y, X, Z, W) = -\tilde{R}(X, Y, W, Z).$$

Note that since $\tilde{\nabla}$ has torsion, in general $\tilde{R}$ does not satisfy the first Bianchi identity (1.2). Moreover, in general we do not have $\tilde{R}(X, Y, Z, W) = \tilde{R}(Z, W, X, Y)$.

### 2.3. The Gray conditions

In [**10**] Gray considered some special classes of almost-Hermitian manifolds characterized by some identities involving the curvature tensor.

**Definition 2.3.** Let $(M, g, J)$ be an almost-Hermitian manifold and let $R$ be the curvature tensor of $g$. Then $R$ is said to satisfy

(i) the *first Gray identity* $(\mathrm{G}_1)$ if $R(Z_1, Z_2, \cdot, \cdot) = 0$,

(ii) the *second Gray identity* $(\mathrm{G}_2)$ if $R(Z_1, Z_2, Z_3, Z_4) = R(\bar{Z}_1, Z_2, Z_3, Z_4) = 0$,

(iii) the *third Gray identity* $(\mathrm{G}_3)$ if $R(\bar{Z}_1, Z_2, Z_3, Z_4) = 0$,

for every $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)$.

Clearly, one has that

$$(\mathrm{G}_1) \Longrightarrow (\mathrm{G}_2) \Longrightarrow (\mathrm{G}_3)$$

and that the curvature tensor of a Kähler manifold satisfies $(\mathrm{G}_1)$. Furthermore, in view of a theorem of Goldberg [**9**], any almost-Kähler manifold whose curvature tensor satisfies $(\mathrm{G}_1)$ is a genuine Kähler manifold. The same cannot be claimed for the condition $(\mathrm{G}_2)$. Indeed, in $d > 6$ there exist examples of compact strictly almost-Kähler manifolds whose curvature tensor satisfies $(\mathrm{G}_2)$ [**4**]. In dimension 4 there is a different behaviour, since we have the following theorem due to Apostolov *et al.*

**Theorem 2.4 (Apostolov *et al.* [3, Theorem 2]).** *In dimension 4 there is no compact strictly almost-Kähler manifold whose curvature tensor satisfies $(G_3)$.*

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### 2.4. Generalized normal holomorphic frames

Let $(M, g, J, \omega)$ be a $2n$-dimensional almost-Hermitian manifold. Denote by $\nabla$ the Levi-Cività connection associated to the metric $g$, by $R$ the curvature tensors associated to $\nabla$ and by $N$ the Nijenhuis tensor of $J$.

**Definition 2.5.** Let $o$ be an arbitrary point in $M$. A *generalized normal holomorphic frame* (GNHF) around $o$ is a local $(1,0)$-complex frame $\{Z_1, \dots, Z_n\}$ satisfying the following properties for every $i, j, k = 1, \dots, n$:

(i) $\nabla_i Z_{\bar{j}}(o) = 0$;

(ii) $\nabla_i Z_j(o)$ is of type $(0,1)$;

(iii) $g_{i\bar{j}}(o) = \delta_{ij}$, $\mathrm{d}g_{i\bar{j}}(o) = 0$;

(iv) $\nabla_i \nabla_{\bar{j}} Z_k(o) = 0$.

We recall the following result.

**Theorem 2.6 (Vezzoni [16, Theorem 1]).** *The following facts are equivalent:*

(i) *$\omega$ is a quasi-Kähler form;*

(ii) *any point $o$ in $M$ admits a generalized normal holomorphic frame.*

The following lemma, the proof of which is similar to that of [**17**, Theorem 3.3], will be useful in the remainder of the paper.

**Lemma 2.7.** *Let $F$ be the smooth tensor on $M$ defined by*

$$F(X, Y, Z, W) := g((\nabla_X N)(Y, Z), W) \quad \text{for } X, Y, Z, W \in \Gamma(TM).$$

*Consider an arbitrary point $o$ of $M$ and let $\{Z_1, \dots, Z_n\}$ be a GNHF around $o$. Then*

$$F_{\bar{\imath}jk\bar{l}}(o) = 4g([Z_j, Z_k], \nabla_{\bar{\imath}} Z_{\bar{l}})(o)$$

*for every $i, j, k, l = 1, \dots, n$.*

The next result is a slight improvement of [**17**, Theorem 3.3] and can be viewed as a corollary of Lemma 2.7.

**Theorem 2.8.** *Let $(M, g, J, \omega)$ be a quasi-Kähler manifold and assume that the Nijenhuis tensor of $J$ satisfies*

$$\underset{X,Y,Z}{\mathfrak{S}} \nabla_X N(Y, Z) = 0 \quad \text{for all } X, Y, Z \in \Gamma(TM). \tag{2.5}$$

*Then $J$ is integrable.*

**Proof.** Let $o \in M$ and let $\{Z_1, \dots, Z_n\}$ be a GNHF around $o$. By (2.1) we have

$$N_{i\bar{k}}(o) = 0, \quad N_{ik}(o) \in T_o^{0,1}M \quad \text{for every } i, k = 1, \dots, n.$$

Furthermore, by the properties of the GNHF, we have

$$\underset{\bar{\imath},j,k}{\mathfrak{S}} (\nabla_{\bar{\imath}} N)(Z_j, Z_k)(o) = \nabla_{\bar{\imath}}(N(Z_j, Z_k))(o).$$

Hence, Equation (2.5) implies $(\nabla_{\bar{\imath}} N)_{jk} = 0$, which, in view of Lemma 2.7, is equivalent to $N = 0$. □

A direct computation gives the following.

**Proposition 2.9.** *The components of the curvature tensor with respect to a GNHF $\{Z_1, \dots, Z_n\}$ around a point $o$ can be written as*

$$\begin{aligned}
R_{i\bar{j}k\bar{l}}(o) &= -g(\nabla_{\bar{j}} \nabla_i Z_k, Z_{\bar{l}})(o),\\
R_{\bar{\imath}jkl}(o) &= g(\nabla_{\bar{\imath}} \nabla_j Z_k, Z_l)(o),\\
R_{\bar{\imath}\bar{j}kl}(o) &= -g(\nabla_{[Z_{\bar{\imath}}, Z_{\bar{j}}]} Z_k, Z_l)(o),\\
R_{ijkl}(o) &= g(\nabla_i \nabla_j Z_k, Z_l)(o) - g(\nabla_j \nabla_i Z_k, Z_l)(o).
\end{aligned}$$

### 2.5. Proof of (1.1)

The aim of this section is to give an alternative proof of (1.1) without using the Weitzenböck decomposition.

**Proof of (1.1).** Let $(M, g, J, \omega)$ be an almost-Kähler manifold. First, we recall the definition of the $*$-Ricci tensor and the $*$-scalar curvature

$$r_*(X, Y) := \sum_{i=1}^{2n} R(JX, JX_i, X_i, Y), \qquad s_* := \sum_{i=1}^{2n} r_*(X_i, X_i),$$

where $\{X_1, \dots, X_{2n}\}$ is an arbitrary orthonormal frame on $M$. It is easy to see that in complex coordinates the scalar curvature and the $*$-scalar curvature can be written as

$$s = 2 \sum_{i,j=1}^{n} \{R_{i\bar{j}j\bar{\imath}} - R_{ij\bar{\imath}\bar{j}}\}, \qquad s_* = 2 \sum_{i,j=1}^{n} \{R_{i\bar{j}j\bar{\imath}} + R_{ij\bar{\imath}\bar{j}}\},$$

where $\{Z_1, \dots, Z_n\}$ is an arbitrary unitary $(1,0)$-frame on $M$. In particular,

$$s_* - s = 4 \sum_{i,j=1}^{n} R_{ij\bar{\imath}\bar{j}}$$

and (1.1) can be rewritten as

$$\sum_{i,j=1}^{n} R_{ij\bar{\imath}\bar{j}} = \tfrac{1}{4} \|\nabla \omega\|^2.$$

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Fix an arbitrary point $o$ of $M$ and let $\{Z_1, \dots, Z_n\}$ be a GNHF around $o$. Since $\nabla_i Z_j(o) \in T^{0,1}_o M$, we have $N_{ij}(o) = -4[Z_i, Z_j](o)$; hence, at $o$ (2.3) reads as

$$g([Z_i, Z_j], Z_l)(o) = -g(\nabla_l Z_i, Z_j)(o).$$

Since $\{Z_1, \dots, Z_n\}$ is a unitary frame we have

$$[Z_i, Z_j](o) = -\sum_{l=1}^{n} \Gamma^{\bar{j}}_{li}(o) Z_{\bar{l}}(o),$$

where $\Gamma^{\bar{j}}_{li} := g(\nabla_l Z_i, Z_j)$. Furthermore, we have

$$\begin{aligned} R_{ij\bar{\imath}\bar{\jmath}}(o) &= -g(\nabla_{[Z_i,Z_j]} Z_{\bar{\imath}}, Z_{\bar{\jmath}})(o) \\ &= \sum_{l=1}^{n} \Gamma^{\bar{j}}_{li} g(\nabla_{\bar{l}} Z_{\bar{\imath}}, Z_{\bar{\jmath}})(o) \\ &= \sum_{l=1}^{n} \Gamma^{\bar{j}}_{li}(o) \Gamma^{j}_{\bar{l}\bar{\imath}}(o) \\ &= \sum_{l=1}^{n} |\Gamma^{\bar{j}}_{li}|^2(o). \end{aligned}$$

Hence,

$$\sum_{i,j=1}^{n} R_{ij\bar{\imath}\bar{\jmath}}(o) = \sum_{l,i,j=1}^{n} |\Gamma^{\bar{j}}_{li}|^2(o)$$

and the claim follows since $(\nabla_Z \omega)(X, Y) = \frac{1}{2} g(N(X, Y), JZ)$. □

Condition (1.1) is related to the subspace $\mathcal{W}_4$ described in [**15**, p. 372] (see also [**7**], where $\mathcal{W}_4 = \mathcal{C}_4$). Indeed, by using [**15**, Lemma 4.5, p. 371] it is easy to see that the projection $R^{\mathcal{W}_4}$ of $R$ to $\mathcal{W}_4$ is given by

$$R^{\mathcal{W}_4} = \frac{(s - s_*)}{16n(n-1)} = \frac{1}{4n(n-1)} \sum_{i,j=1}^{n} R_{ij\bar{\imath}\bar{\jmath}} = \frac{1}{16n(n-1)} \|\nabla\omega\|^2.$$

## 3. The first Bianchi identity for the Hermitian curvature

In this section we shall prove Theorem 1.1 and its corollary (Corollary 1.2).

Let $\tilde{\nabla}$ be the canonical connection associated to a quasi-Kähler structure $(g, J, \omega)$ on a $2n$-dimensional manifold $M$. We have the following.

**Lemma 3.1.** *Let $Z_1, Z_2$ be two arbitrary $(1,0)$-vector fields on $M$. Then*

$$\tilde{\nabla}_{Z_1} Z_2 \in \Gamma(T^{1,0}M), \qquad \tilde{\nabla}_{\bar{Z}_1} Z_2 = \nabla_{\bar{Z}_1} Z_2 \in \Gamma(T^{1,0}M).$$

**Proof.** It is sufficient to consider the definition of $\tilde{\nabla}$ and to apply Lemma 2.2. □

As a direct consequence of Lemma 3.1 we have the following.

**Proposition 3.2.** *Let $\{Z_1, \dots, Z_n\}$ be an arbitrary $(1,0)$-frame on $M$ and let $\tilde{R}$ be the Hermitian curvature tensor of $M$. Then*

(i) $\tilde{R}_{ijk\bar{l}} = R_{ijk\bar{l}}$,

(ii) $\tilde{R}_{\bar{\imath}\bar{\jmath}kl} = \tilde{R}_{ijkl} = \tilde{R}_{i\bar{\jmath}kl} = 0$.

**Lemma 3.3.** *Let $o$ be an arbitrary point of $M$ and let $\{Z_1, \dots, Z_n\}$ be a GNHF around $o$. Then*

$$\tilde{\nabla}_i Z_j(o) = 0, \quad \tilde{\nabla}_{\bar{\imath}} Z_j(o) = 0 \quad \text{for any } i, j = 1, \dots, n,$$

*i.e. the canonical connection acts on generalized normal holomorphic frames in quasi-Kähler manifolds as the Levi-Cività connection acts on normal holomorphic frames in Kähler manifolds.*

**Proof.** Let $\{Z_1, \dots, Z_n\}$ be a GNHF around $o$. Since $\nabla_i Z_j(o) \in T_o^{0,1}M$, we have

$$\begin{aligned} \tilde{\nabla}_i Z_j(o) &= \tfrac{1}{2}\{\nabla_i Z_j - J\nabla_i J Z_j\}(o) \\ &= \tfrac{1}{2}\nabla_i Z_j(o) - \mathrm{i}\,\tfrac{1}{2} J\nabla_i Z_j(o) \\ &= \tfrac{1}{2}\nabla_i Z_j(o) - \tfrac{1}{2}\nabla_i Z_j(o) \\ &= 0. \end{aligned}$$

Moreover, since $\nabla_{\bar{\imath}} Z_j(o) = 0$, we have

$$\tilde{\nabla}_{\bar{\imath}} Z_j(o) = \tfrac{1}{2}\{\nabla_{\bar{\imath}} Z_j - J\nabla_{\bar{\imath}} J Z_j\}(o) = \tfrac{1}{2}\nabla_{\bar{\imath}} Z_j(o) - \mathrm{i}\,\tfrac{1}{2} J\nabla_{\bar{\imath}} Z_j(o) = 0$$

and the claim follows. □

We have the following.

**Proposition 3.4.** *The components of the Hermitian curvature tensor $\tilde{R}$ with respect to a GNHF $\{Z_1, \dots, Z_n\}$ around a point $o$ can be written as*

(i) $\tilde{R}_{i\bar{\jmath}k\bar{l}}(o) = R_{i\bar{\jmath}k\bar{l}}(o) - g(\nabla_i Z_k, \nabla_{\bar{\jmath}} Z_{\bar{l}})(o)$,

(ii) $\tilde{R}_{ijk\bar{l}}(o) = R_{ijk\bar{l}}(o)$,

(iii) $\tilde{R}_{\bar{\imath}\bar{\jmath}kl}(o) = \tilde{R}_{ijkl}(o) = \tilde{R}_{i\bar{\jmath}kl}(o) = 0$.

**Proof.** Items (ii) and (iii) come from Proposition 3.2. The proof of the first identity can be obtained as follows.

By definition of $\tilde{R}$ and the equation $[Z_i, Z_{\bar{\jmath}}](o) = 0$ we have

$$\begin{aligned} \tilde{R}_{i\bar{\jmath}k\bar{l}}(o) &= g(\tilde{\nabla}_i \tilde{\nabla}_{\bar{\jmath}} Z_k - \tilde{\nabla}_{\bar{\jmath}} \tilde{\nabla}_i Z_k - \tilde{\nabla}_{[Z_i, Z_{\bar{\jmath}}]} Z_k, Z_{\bar{l}})(o) \\ &= g(\tilde{\nabla}_i \tilde{\nabla}_{\bar{\jmath}} Z_k - \tilde{\nabla}_{\bar{\jmath}} \tilde{\nabla}_i Z_k, Z_{\bar{l}})(o). \end{aligned}$$

Applying Lemmas 3.1 and 3.3, we get

$$\begin{aligned}\tilde R_{i\bar jk\bar l}(o) &= g(\tilde\nabla_i\tilde\nabla_{\bar j}Z_k - \tilde\nabla_{\bar j}\tilde\nabla_i Z_k, Z_{\bar l})(o)\\ &= g(\tilde\nabla_i\nabla_{\bar j}Z_k, Z_{\bar l})(o) - g(\tilde\nabla_{\bar j}\tilde\nabla_i Z_k, Z_{\bar l})(o)\\ &= Z_i g(\nabla_{\bar j}Z_k, Z_{\bar l})(o) - g(\nabla_{\bar j}Z_k, \tilde\nabla_i Z_{\bar l})(o) - Z_{\bar j} g(\tilde\nabla_i Z_k, Z_{\bar l})(o) + g(\tilde\nabla_i Z_k, \tilde\nabla_{\bar j} Z_{\bar l})(o)\\ &= Z_i g(\nabla_{\bar j}Z_k, Z_{\bar l})(o) - Z_{\bar j} g(\tilde\nabla_i Z_k, Z_{\bar l})(o).\end{aligned}$$

Finally, taking into account Lemma 2.2 and the fact that $\nabla$ and $\tilde\nabla$ preserve $g$, we obtain

$$\begin{aligned}\tilde R_{i\bar jk\bar l}(o) &= Z_i g(\nabla_{\bar j}Z_k, Z_{\bar l})(o) - Z_{\bar j} g(\tilde\nabla_i Z_k, Z_{\bar l})(o)\\ &= g(\nabla_i\nabla_{\bar j}Z_k, Z_{\bar l})(o) + g(\nabla_{\bar j}Z_k, \nabla_i Z_{\bar l})(o) - Z_{\bar j}Z_i g_{k\bar l}(o) + Z_{\bar j} g(Z_k, \tilde\nabla_i Z_{\bar l})(o)\\ &= -Z_{\bar j}Z_i g_{k\bar l}(o) + Z_{\bar j} g(Z_k, \tilde\nabla_i Z_{\bar l})(o)\\ &= -Z_{\bar j} g(\nabla_i Z_k, Z_{\bar l})(o) - Z_{\bar j} g(Z_k, \nabla_i Z_{\bar l})(o) - Z_{\bar j} g(Z_k, \nabla_i Z_{\bar l})(o)\\ &= -g(\nabla_{\bar j}\nabla_i Z_k, Z_{\bar l})(o) - g(\nabla_i Z_k, \nabla_{\bar j} Z_{\bar l})(o)\\ &\qquad\qquad - g(\nabla_{\bar j}Z_k, \nabla_i Z_{\bar l})(o) - g(Z_k, \nabla_{\bar j}\nabla_i Z_{\bar l})(o)\\ &= R_{i\bar jk\bar l}(o) - g(\nabla_i Z_k, \nabla_{\bar j} Z_{\bar l})(o),\end{aligned}$$

i.e.

$$\tilde R_{i\bar jk\bar l}(o) = R_{i\bar jk\bar l}(o) - g(\nabla_i Z_k, \nabla_{\bar j} Z_{\bar l})(o),$$

and the claim follows. □

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $o \in M$ be an arbitrary point and let $\{Z_1, \dots, Z_n\}$ be a GNHF around $o$. By Proposition 3.4 we have

$$\underset{i,j,k}{\mathfrak S}\tilde R_{ijkl}(o) = \underset{i,j,k}{\mathfrak S}\tilde R_{ijk\bar l}(o) = 0.$$

Moreover,

$$\underset{i,\bar j,k}{\mathfrak S}\tilde R_{i\bar jkl}(o) = R_{ki\bar jl}(o). \tag{3.1}$$

Furthermore,

$$\begin{aligned}\underset{i,\bar j,k}{\mathfrak S}\tilde R_{i\bar jk\bar l}(o) &= \tilde R_{i\bar jk\bar l}(o) + \tilde R_{ki\bar j\bar l}(o) + \tilde R_{\bar jki\bar l}(o)\\ &= \tilde R_{i\bar jk\bar l}(o) + \tilde R_{\bar jki\bar l}(o)\\ &= R_{i\bar jk\bar l}(o) + R_{\bar jki\bar l}(o) - g(\nabla_i Z_k, \nabla_{\bar j} Z_{\bar l})(o) + g(\nabla_k Z_i, \nabla_{\bar j} Z_{\bar l})(o)\\ &= -R_{ki\bar j\bar l}(o) - g([Z_i, Z_k], \nabla_{\bar j} Z_{\bar l})(o),\end{aligned}$$

i.e.

$$\underset{i,\bar j,k}{\mathfrak S}\tilde R_{i\bar jk\bar l}(o) = R_{ik\bar j\bar l}(o) - g([Z_i, Z_k], \nabla_{\bar j} Z_{\bar l})(o). \tag{3.2}$$

Hence, the Hermitian curvature $\tilde{R}$ satisfies the first Bianchi identity at $o$ if and only if the following equations hold:

$$R_{ki\bar{j}\bar{l}}(o) = 0, \tag{3.3}$$

$$R_{ik\bar{j}\bar{l}}(o) - g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = 0. \tag{3.4}$$

Equation (3.3) is the third Gray condition, while, in view of Lemma 2.7, Equation (3.4) is satisfied if and only if

$$R(Z_1, Z_2, \bar{Z}_3, \bar{Z}_4) = \tfrac{1}{4} g((\nabla_{\bar{Z}_3} N)(Z_1, Z_2), \bar{Z}_4)$$

for every $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)$. □

Now we can prove Corollary 1.2.

**Proof of Corollary 1.2.** Assume that $(M, g, J, \omega)$ is an almost-Kähler manifold and let $\tilde{R}$ be the Hermitian curvature of $(g, J)$. Fix an arbitrary point $o$ of $M$, consider a GNHF $\{Z_1, \dots, Z_n\}$ around $o$ and assume that $\tilde{R}$ satisfies the first Bianchi identity. Then, in view of Theorem 1.1, we have

$$0 = R_{ik\bar{j}\bar{l}}(o) - g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = -g(\nabla_{[Z_i, Z_k]} Z_{\bar{j}}, Z_{\bar{l}})(o) - g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o),$$

i.e.

$$g(\nabla_{[Z_i, Z_k]} Z_{\bar{j}}, Z_{\bar{l}})(o) = -g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o). \tag{3.5}$$

In particular,

$$g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = -g([Z_i, Z_k], \nabla_{\bar{l}} Z_{\bar{j}})(o),$$

i.e. $g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o)$ is skew-symmetric with respect to the indexes $\bar{j}$, $\bar{l}$. In view of (2.3) we have

$$\begin{aligned} g(\nabla_{[Z_i, Z_k]} Z_{\bar{j}}, Z_{\bar{l}})(o) &= \tfrac{1}{4} g(N_{\bar{j}\bar{l}}, [Z_i, Z_k])(o) \\ &= -g([Z_{\bar{j}}, Z_{\bar{l}}], [Z_i, Z_k])(o) \\ &= -2g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o). \end{aligned}$$

Hence, Equation (3.5) implies

$$g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = 0,$$

which forces $J$ to be integrable. □

## 4. The condition $\tilde{R} = 0$ in quasi-Kähler manifolds

In this section we investigate the case $\tilde{R} = 0$. We start by considering the following preliminary results.

**Lemma 4.1.** *Let $(M, g, J, \omega)$ be a quasi-Kähler manifold. Then the following are equivalent:*

(i) *the curvature tensor of the canonical connection associated to $(g, J)$ vanishes;*

(ii) *every $o \in M$ admits an open neighbourhood $U$ and a complex unitary $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ on $U$ such that*

$$\nabla_i Z_j \in \Gamma(T^{0,1}U), \quad \nabla_{\bar{\imath}} Z_j = 0, \quad i, j = 1, \dots, n.$$

**Proof.** The condition $\tilde{R} = 0$ is equivalent to require that every point $o$ of $M$ admits an open neighbourhood $U$ equipped with a complex unitary $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ such that

$$\tilde{\nabla}_i Z_j = 0, \quad \tilde{\nabla}_{\bar{\imath}} Z_j = 0, \quad i, j = 1, \dots, n. \tag{4.1}$$

Since

$$\tilde{\nabla}_i Z_j = 0 = \tfrac{1}{2}\nabla_i Z_j - \tfrac{1}{2} J \nabla_i J Z_j = \tfrac{1}{2}\nabla_i Z_j - \tfrac{1}{2}\,\mathrm{i}\, J \nabla_i Z_j$$

and

$$\tilde{\nabla}_{\bar{\imath}} Z_j = 0 = \tfrac{1}{2}\nabla_{\bar{\imath}} Z_j - \tfrac{1}{2} J \nabla_{\bar{\imath}} J Z_j = \tfrac{1}{2}\nabla_{\bar{\imath}} Z_j - \tfrac{1}{2}\,\mathrm{i}\, J \nabla_{\bar{\imath}} Z_j,$$

(4.1) is equivalent to the requirement that $\nabla_i Z_j, \nabla_{\bar{\imath}} Z_j \in \Gamma(T^{0,1}U)$ for every $i, j = 1, \dots, n$. Moreover, since $M$ is quasi-Kähler, the mixed derivatives $\nabla_{\bar{\imath}} Z_j$ are of type $(1,0)$. Hence $\nabla \bar{\imath} Z_j = 0$, as required. □

**Remark 4.2.** Note that the second item of the previous lemma in particular implies that if $g$ is an $\tilde{R}$-flat quasi-Kähler metric, then we can always find a local unitary $(1,0)$-coframe $\{\zeta_1, \dots, \zeta_n\}$ such that

$$\partial \zeta_i = \bar{\partial} \zeta_i = 0, \quad i = 1, \dots, n.$$

We recall that a four-dimensional quasi-Kähler manifold is always almost-Kähler. Hence, in view of Theorem 1.1, if a four-dimensional quasi-Kähler manifold has $\tilde{R} = 0$, then it is Kähler. In higher dimensions things work differently.

**Theorem 4.3.** *There exists a quasi-Kähler structure $(g_0, J_0, \omega_0)$ on the Iwasawa manifold with the following properties:*

(i) *the Hermitian curvature of $(g_0, J_0)$ vanishes;*

(ii) *the Riemann curvature of $g_0$ satisfies the second Gray identity $(G_2)$.*

**Proof.** Let $G$ be the *complex Heisenberg* group

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C},\ i = 1, 2, 3 \right\}$$

and let $M$ be the compact manifold $M = G/\Gamma$, where $\Gamma$ is the co-compact lattice of $G$ formed by the matrices with integral entries. Then $M$ is the *Iwasawa manifold*. It is well known that $M$ admits a global frame $\mathcal{B} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ satisfying the following structure equations:

$$[X_1, X_2] = X_3, \quad [X_4, X_5] = -X_3, \quad [X_2, X_4] = X_6, \quad [X_5, X_1] = X_6.$$

Let $J_0$ be the almost-complex structure defined on the basis $\mathcal{B}$ by

$$\begin{aligned} J_0X_1 &= X_4, & J_0X_2 &= X_5, & J_0X_3 &= X_6, \\ J_0X_4 &= -X_1, & J_0X_5 &= -X_2, & J_0X_6 &= -X_3, \end{aligned}$$

let $g_0$ be the $J_0$-almost-Hermitian metric

$$g_0 = \sum_{i=1}^{6} \alpha_i \otimes \alpha_i,$$

and let

$$\omega_0 := \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6,$$

where $\{\alpha_1, \dots, \alpha_6\}$ is the dual frame of $\mathcal{B}$. Then $(g_0, J_0, \omega_0)$ is a quasi-Kähler structure on $M$.

The almost-complex structure $J_0$ induces the $(1, 0)$-frame

$$Z_1 = X_1 - \mathrm{i}\, X_4, \qquad Z_2 = X_2 - \mathrm{i}\, X_5, \qquad Z_3 = X_3 - \mathrm{i}\, X_6.$$

Clearly,

$$[Z_1, Z_2] = 2Z_{\bar{3}}, \qquad [Z_{\bar{1}}, Z_{\bar{2}}] = 2Z_3$$

and all other brackets involving the vectors of the frame vanish. Furthermore, a direct computation gives $\nabla_{\bar{\imath}} Z_j = 0$ for $i, j = 1, 2, 3$ and

$$\begin{aligned} \nabla_1 Z_1 &= 0, & \nabla_2 Z_1 &= -Z_{\bar{3}}, & \nabla_3 Z_1 &= Z_{\bar{2}}, \\ \nabla_1 Z_2 &= Z_{\bar{3}}, & \nabla_2 Z_2 &= 0, & \nabla_3 Z_2 &= Z_{\bar{1}}, \\ \nabla_1 Z_3 &= -Z_{\bar{2}}, & \nabla_2 Z_3 &= Z_{\bar{1}}, & \nabla_3 Z_3 &= 0, \end{aligned}$$

where $\nabla$ is the Levi-Cività connection associated to $g_0$. Hence, $\nabla_i Z_j \in \Gamma(T^{0,1}M)$ and in view of Lemma 4.1 the Hermitian curvature tensor of $(g_0, J_0)$ vanishes. Furthermore, a straightforward application of our formulae yields that the curvature tensor associated to $g_0$ satisfies the second Gray identity. □

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**Remark 4.4.** The almost-Hermitian structure $J_0$ described in the proof of the above theorem corresponds to the almost-complex structure denoted by $J_3$ [**1**].

The Iwasawa manifold is (in some fashion) the unique example of a six-dimensional non-Kähler almost-complex nilmanifold admitting a quasi-Kähler $\tilde{R}$-flat metric. More precisely we have the following.

**Theorem 4.5.** *Let $(G, J)$ be a six-dimensional Lie group equipped with a left-invariant non-integrable almost-complex structure admitting a $J$-compatible quasi-Kähler metric $g$ with vanishing Hermitian curvature tensor. Then the Lie algebra of $G$ endowed with the almost-complex structure induced by $J$ is isomorphic as complex Lie algebra to the one of the complex Heisenberg group equipped with the almost-complex structure induced by $J_0$.*

**Proof.** Let $\mathfrak{g}$ be the Lie algebra of $G$. In view of Lemma 4.1 there exists a complex $(1,0)$-frame $\{Z_1, Z_2, Z_3\}$ on $\mathfrak{g}$ such that

$$[Z_i, Z_j] = \sum_{k=1}^{3} A_{ij}^{\bar{k}} Z_{\bar{k}}, \quad [Z_i, Z_{\bar{j}}] = 0, \quad i, j = 1, 2, 3.$$

Since $J$ is by hypothesis non-integrable, there exists at least a bracket different from zero. We may assume that

$$[Z_1, Z_2] \neq 0.$$

Now we observe that $A_{12}^{\bar{3}} \neq 0$. Indeed, if by contradiction $A_{12}^{\bar{3}} = 0$, then

$$[Z_1, Z_2] = A_{12}^{\bar{1}} Z_{\bar{1}} + A_{12}^{\bar{2}} Z_{\bar{2}}$$

and, by the Jacobi identity,

$$\begin{aligned} 0 &= [[Z_1, Z_2], Z_{\bar{1}}] = -A_{12}^{\bar{2}} [Z_{\bar{1}}, Z_{\bar{2}}], \\ 0 &= [[Z_1, Z_2], Z_{\bar{2}}] = -A_{12}^{\bar{1}} [Z_{\bar{1}}, Z_{\bar{2}}], \end{aligned}$$

which implies $[Z_1, Z_2] = 0$. Hence, $A_{12}^{\bar{3}}$ has to be different from zero and, consequently,

$$W_1 := Z_1, \qquad W_2 = Z_2, \qquad W_3 := \frac{1}{A_{\bar{1}\bar{2}}^{3}} (Z_3 - A_{\bar{1}\bar{2}}^{1} Z_1 - A_{\bar{1}\bar{2}}^{2} Z_2)$$

is a $(1,0)$-frame on $(\mathfrak{g}, J)$. Such a frame satisfies

$$[W_1, W_2] = W_{\bar{3}}.$$

Finally, again using the Jacobi identity, we get

$$\begin{aligned} 0 &= [[W_1, W_2], W_{\bar{1}}] = -[W_{\bar{2}}, W_{\bar{3}}], \\ 0 &= [[W_1, W_2], W_{\bar{2}}] = -[W_{\bar{1}}, W_{\bar{3}}], \end{aligned}$$

i.e.

$$[W_2, W_3] = [W_1, W_3] = 0,$$

which ends the proof. □

It is possible to find some non-equivalent quasi-Kähler structures on the Iwasawa manifold having $\tilde{R} = 0$. For instance, we have the following example.

**Example 4.6.** It easy to show that the Iwasawa manifold $M$ admits a global coframe $\{\alpha_1, \dots, \alpha_6\}$ satisfying the following structure equations:

$$\begin{aligned} \mathrm{d}\alpha_1 &= \mathrm{d}\alpha_3 = -\alpha_1 \wedge \alpha_2 + \alpha_4 \wedge \alpha_5 - \alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6, \\ \mathrm{d}\alpha_2 &= \mathrm{d}\alpha_5 = 0, \\ \mathrm{d}\alpha_4 &= \mathrm{d}\alpha_6 = -\alpha_2 \wedge \alpha_4 + \alpha_1 \wedge \alpha_5 - \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_6. \end{aligned}$$

Let $\{X_1, \dots, X_6\}$ be the frame dual to $\{\alpha_1, \dots, \alpha_6\}$ and consider the almost-complex structure $J$ on $M$ defined on $\{X_1, \dots, X_6\}$ by

$$\begin{aligned} JX_1 &= X_4, & JX_2 &= X_5, & JX_3 &= X_6, \\ JX_4 &= -X_1, & JX_5 &= -X_2, & JX_6 &= -X_3. \end{aligned}$$

Let

$$\omega := \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6;$$

a direct computation then gives that $\omega$ is a $\bar{\partial}$-closed form compatible with $J$. The basis $\{X_1, \dots X_6\}$ induces the complex $(1,0)$-frame

$$Z_1 = X_1 - \mathrm{i}\, X_4, \qquad Z_2 = X_2 - \mathrm{i}\, X_5, \qquad Z_3 = X_3 - \mathrm{i}\, X_6.$$

One easily obtains

$$[Z_1, Z_2] = 2(Z_{\bar{1}} + Z_{\bar{3}}), \quad [Z_2, Z_3] = 2(Z_{\bar{1}} + Z_{\bar{3}}), \quad [Z_1, Z_3] = 0.$$

Since $[Z_i, Z_{\bar{j}}] = 0$ and $(g, J, \omega)$ is a quasi-Kähler structure, in view of Lemma 2.2 we have

$$\nabla_{\bar{\imath}} Z_j = 0,$$

where $\nabla$ is the Levi-Cività connection associated to the metric $g$. Furthermore, a direct computation gives

$$\begin{aligned} \nabla_1 Z_1 &= -2Z_{\bar{2}}, & \nabla_2 Z_1 &= -2Z_{\bar{3}}, & \nabla_3 Z_1 &= 0, \\ \nabla_1 Z_2 &= 2Z_{\bar{1}}, & \nabla_2 Z_2 &= 0, & \nabla_3 Z_2 &= -2Z_{\bar{3}}, \\ \nabla_1 Z_3 &= 2Z_{\bar{1}}, & \nabla_2 Z_3 &= 2Z_{\bar{1}}, & \nabla_3 Z_3 &= 2Z_{\bar{2}}; \end{aligned}$$

hence,

$$\nabla_i Z_j \in \Gamma(T^{0,1}M) \quad \text{for every } i, j = 1, 2, 3.$$

By Lemma 4.1 we obtain that the Hermitian curvature tensor of $g$ vanishes. Also in this case, a straightforward computation gives that the curvature tensor of the metric $g$ satisfies the second Gray identity $(\mathrm{G}_2)$.

**Remark 4.7.** In the quasi-Kähler case, the condition $\tilde{R} = 0$ implies that the tensor $\mathcal{R}(g, J)$ described by (1.3) vanishes. Hence, it is very natural to take into account the following problem.

Does there exist a symplectic form $\omega'$ on the Iwasawa manifold taming the almost-complex structure $J_0$ and such that the pair $(\omega', J_0)$ induces an $\tilde{R}$-flat quasi-Kähler structure on $M$?

(This problem was suggested us by Valentino Tosatti.) The answer is negative. In order to show this, we fix a quasi-Kähler $\tilde{R}$-flat metric $g$ on the Iwasawa manifold $M$ compatible with $J_0$. Then we can find a global unitary $(1,0)$-coframe $\{\zeta_1, \zeta_2, \zeta_3\}$ such that

$$\mathrm{d}\zeta_1 = \mathrm{d}\zeta_2 = 0, \qquad \mathrm{d}\zeta_3 = -\zeta_{\bar{1}} \wedge \zeta_{\bar{2}}. \tag{4.2}$$

Assume that there exists a symplectic structure $\omega'$ taming $J_0$ and such that the pair $(\omega', J_0)$ induces the metric $g$. Then one necessarily has

$$\omega' = \omega + \beta + \bar{\beta},$$

where $\omega$ is the quasi-Kähler form associated to $g$ and $\beta$ is a complex form of type $(2,0)$. The equation $\mathrm{d}\omega' = 0$ can be written in terms of $\omega$ and $\beta$ as

$$\begin{aligned} A\omega + \partial\beta &= 0, \\ \bar{\partial}\beta + A\bar{\beta} &= 0. \end{aligned}$$

We can write $\beta = a\zeta_{12} + b\zeta_{23} + c\zeta_{13}$, where $a$, $b$, $c$ are smooth functions on $M$. Taking into account Equations (4.2), one has

$$\begin{aligned} \bar{\partial}\beta &= \sum_{r=1}^{3} \zeta_{\bar{r}}(a)\zeta_{12\bar{r}} + \zeta_{\bar{r}}(b)\zeta_{23\bar{r}} + \zeta_{\bar{r}}(c)\zeta_{13\bar{r}}, \\ A\bar{\beta} &= \bar{b}\zeta_{12\bar{2}} + \bar{c}\zeta_{12\bar{1}}. \end{aligned}$$

Hence, the equation $\bar{\partial}\beta + A\bar{\beta} = 0$ readily implies that $b$ and $c$ are holomorphic functions on $M$ and that the map $a$ satisfies

$$\zeta_{\bar{1}}(a) = \bar{c}, \qquad \zeta_{\bar{2}}(a) = \bar{b}, \qquad \zeta_{\bar{3}}(a) = 0.$$

Since $M$ is compact, $b$ and $c$ have to be constant. In particular, one has $\partial\bar{\partial}a = 0$ and, consequently, $a$ has to be constant. Since the components of $\beta$ are constant, one has $\partial\beta = \bar{\partial}\beta = 0$ and this condition contradicts the equation $A\omega + \partial\beta = 0$.

In view of Remark 4.2 we require that a quasi-Kähler metric $g$ locally admits a complex unitary $(1,0)$-frame $\{\zeta_1, \dots, \zeta_n\}$ satisfying

$$\partial\zeta_i = \bar{\partial}\zeta_i = 0, \quad i = 1, \dots, n.$$

This is less strict than requiring that the Hermitian curvature tensor of $g$ vanishes. Hence, it is rather natural to wonder if an almost-Kähler structure can admit such a coframe. The answer is negative, since we have the following result.

**Proposition 4.8.** *Let $(M, g, J, \omega)$ be an almost-Kähler manifold. Assume that $M$ admits a global unitary $(1,0)$-coframe $\{\zeta_1, \dots, \zeta_n\}$ satisfying*

$$\partial\zeta_i = \bar\partial\zeta_i = 0, \quad i = 1, \dots, n.$$

*Then $M$ is Kähler.*

**Proof.** Assume that such a coframe exists and let $\{Z_1, \dots, Z_n\}$ be the dual frame. Then we have

$$[Z_i, Z_{\bar j}] = 0, \quad [Z_i, Z_j] \in \Gamma(T^{0,1}M), \quad i, j = 1, \dots, n.$$

In particular, we can write

$$[Z_i, Z_j] = \sum_{k=1}^{n} A_{ij}^{\bar k} Z_{\bar k}$$

and the Nijenhuis tensor of $J$ satisfies

$$N(Z_i, Z_j) = -4 \sum_{k=1}^{n} A_{ij}^{\bar k} Z_{\bar k}.$$

Now we recall that the Nijenhuis tensor of an almost-Kähler manifold always satisfies

$$\underset{X,Y,Z}{\mathfrak{S}} g(N(X,Y), Z) = 0.$$

This formula in our case reads

$$A_{ij}^{\bar k} + A_{ki}^{\bar j} + A_{jk}^{\bar \imath} = 0, \quad 1 \leqslant i, j, k \leqslant n. \tag{4.3}$$

Since the brackets of the form $[Z_i, Z_{\bar j}]$ vanish, the Jacobi identity in terms of $Z_i$s reads

$$[[Z_i, Z_j], Z_{\bar r}] = 0, \quad 1 \leqslant i, j, r \leqslant n,$$

i.e.

$$0 = [[Z_i, Z_j], Z_{\bar r}] = \sum_{k=1}^{n} [A_{ij}^{\bar k} Z_{\bar k}, Z_{\bar r}] = -\sum_{k=1}^{n} Z_{\bar r}(A_{ij}^{\bar k}) Z_{\bar k} + \sum_{k,s=1}^{n} A_{ij}^{\bar k} \bar A_{kr}^{\bar s} Z_s.$$

In particular, one has

$$\sum_{k=1}^{n} A_{ij}^{\bar k} \bar A_{kr}^{\bar s} = 0, \quad 1 \leqslant i, j, s, r \leqslant n. \tag{4.4}$$

Using Equations (4.3) and (4.4), we get

$$0 = \sum_{k=1}^{n} A_{ij}^{\bar k} \bar A_{ki}^{\bar j} = -\sum_{k=1}^{n} \{A_{ij}^{\bar k} \bar A_{ij}^{\bar k} - A_{ij}^{\bar k} \bar A_{kj}^{\bar \imath}\} = -\sum_{k=1}^{n} |A_{ij}^{\bar k}|^2$$

which forces $(M, g, J, \omega)$ to be a Kähler manifold. □

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