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ON SPANNING AND DOMINATING CIRCUITS IN GRAPHS

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ABSTRACT. A set E of edges of a graph G is said to be a dominating set of edges if every edge of G either belongs to E or is adjacent to an edge of E. If the subgraph $\langle E\rangle$ induced by E is a trail T, then T is called a dominating trail of G. Dominating circuits are defined analogously. A sufficient condition is given for a graph to possess a spanning (and thus dominating) circuit and a sufficient condition is given for a graph to possess a spanning (and thus dominating) trail between each pair of distinct vertices. The line graph $L(G)$ of a graph G is defined to be that graph whose vertex set can be put in one-to-one correspondence with the edge set of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The existence of dominating trails and circuits is employed to present results on line graphs and second iterated line graphs, respectively.

Introduction. A set U of vertices in a graph G is said to *dominate* the vertex set of G if every vertex of G either belongs to U or is adjacent to a vertex of U. Such a set U will be referred to as a *dominating set of vertices.* In a like manner, we define a set E of edges of G to be a *dominating set of edges* if every edge of G either belongs to E or is adjacent to an edge of E.

The *line graph* $L(G)$ of a graph G is that graph whose vertex set can be put in one-to-one correspondence with the edge set of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. It follows readily that under the above correspondence, a dominating set of edges in G corresponds to a dominating set of vertices in $L(G)$.

It is convenient to give a few definitions at this point. Definitions of basic graph theory terms not given here are consistent with [1]. If u and v are (not necessarily distinct) vertices of a graph G, then a $u-v$ *walk* in G is an alternating sequence of vertices and edges of G beginning with u, ending with v, and such that each edge is incident with the two vertices immediately preceding and succeeding it. A $u-v$ walk is *open* if $u\neq v$ and *closed* if $u=v$. A $u-v$ *trail* is a $u-v$ walk in which no edge is repeated, and a $u-v$ *path* is a

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$u-v$ walk in which no vertex is repeated. A non-trivial closed trail of G is called a *circuit* of G, and a *cycle* is a circuit in which no vertices are repeated (except the first and last). A *spanning walk* in a graph G is a walk which contains all vertices of G. A spanning path (cycle) in a graph G is often referred to as a *hamiltonian path* (*hamiltonian cycle*) of G. A graph possessing a hamiltonian cycle is a *hamiltonian graph.*

If E is a dominating set of edges of a graph G such that the subgraph $\langle E\rangle$ induced by E is a trail T, then T is called a *dominating trail* of G. A circuit which is a dominating trail of G is called a *dominating circuit* of G.

It was shown by Harary and Nash–Williams [3] that the line graph $L(G)$ of a connected graph G is hamiltonian if and only if either G has a dominating circuit or G is isomorphic to a complete bipartite graph $K(1, n)$, for some $n \geq 3$. A slight modification in the proof of this result yields the following analogous result for line graphs which possess hamiltonian paths.

THEOREM 1. *The line graph $L(G)$ of a connected graph G contains a hamiltonian path if and only if G has a dominating trail.*

Thus, the line graph $L(G)$ of a connected graph G (which is not isomorphic to $K(1, n)$) containing a hamiltonian cycle (path) is equivalent to G containing a dominating circuit (trail). We now consider a special type of dominating circuit, namely a spanning circuit.

In [4] Ore proved that if G is a graph of order $p \geq 3$ such that $\deg_G u + \deg_G v \geq p$ for every pair u, v of non-adjacent vertices, then G contains a spanning cycle. In particular, such a graph contains a spanning (and thus dominating) circuit. In Theorem 2 we present an analogue to Ore's result for a graph to possess a spanning circuit. The following observation will be useful. If G is a graph of order $p \geq 2$ such that $\deg_G u + \deg_G v \geq p-1$ for every pair u, v of non-adjacent vertices, then the graph G^* obtained from G by adding a new vertex w which is adjacent to every vertex of G satisfies the hypothesis of Ore's result. Clearly, then, this implies that G contains a spanning path.

We use $\delta(G)$ to denote the minimum degree among the vertices of a graph G.

THEOREM 2. *If G is a graph of order $p \geq 6$ with $\delta(G) \geq 2$ such that $\deg_G u + \deg_G v \geq p-1$ for every pair u, v of non-adjacent vertices, then G contains a spanning circuit.*

Proof. If G is hamiltonian, then G contains a spanning circuit. We therefore assume that G is not hamiltonian. By the observation above, G contains a spanning path P which we denote by

$$P: u_1, u_2, \ldots, u_p,$$

where $u_i \in V(G)$, $1 \leq i \leq p$, and $u_iu_{i+1} \in E(G)$, $1 \leq i \leq p-1$. Since G is not

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hamiltonian, $u_1u_p \notin E(G)$. Hence, $\deg_G u_1 + \deg_G u_p \geq p-1$. Now, if $u_1u_i \in E(G)$, $2 \leq i \leq p$, then $u_{i-1}u_p \notin E(G)$; for otherwise, G contains the hamiltonian cycle $u_1, u_2, \ldots, u_{i-1}, u_p, u_{p-1}, \ldots, u_i, u_1$. Thus $\deg_G u_p \leq (p-1) - \deg_G u_1$ or, equivalently, $\deg_G u_1 + \deg_G u_p \leq p-1$. Therefore $\deg_G u_1 + \deg_G u_p = p-1$ and, since $u_1u_p \notin E(G)$, there is a vertex w in the set $\{u_2, u_3, \ldots, u_{p-1}\}$ which is adjacent to both u_1 and u_p. It suffices to show there exists such a vertex in the set $\{u_3, \ldots, u_{p-2}\}$ for then $u_1, u_2, \ldots, u_p, w, u_1$ is a spanning circuit of G.

Suppose this is not the case. We consider the following two possible (and exhaustive) cases.

(1) Precisely one of u_1u_{p-1}, u_pu_2 is an edge of G.

(2) Both u_1u_{p-1} and u_pu_2 are edges of G.

Case 1. Suppose $u_pu_2 \in E(G)$ and $u_1u_{p-1} \notin E(G)$. Since G is not hamiltonian, $u_1u_3 \notin E(G)$. By assumption, $\deg_G u_1 \geq 2$. Let k be the minimum integer ≥ 4 such that $u_1u_k \in E(G)$. Then $k < p-1$ and $u_1u_{k-1} \notin E(G)$. Since G is not hamiltonian, $u_pu_{k-1} \notin E(G)$. Hence if $U = \{u_i : 3 \leq i \leq p-1$ and $u_1u_i \in E(G)\} \cup \{u_{k-1}\}$, $\deg_G u_1 = |U|$ and u_p is adjacent to no vertex in U. Therefore u_p is adjacent to at most $(p-3) - |U|$ vertices in the set $\{u_3, u_4, \ldots, u_{p-1}\}$ which implies that $\deg_G u_p \leq (p-2) - |U|$. But then $\deg_G u_1 + \deg_G u_p \leq |U| + (p-2-|U|) = p-2$ which is a contradiction. In an analogous manner, if $u_1u_{p-1} \in E(G)$ and $u_pu_2 \notin E(G)$ we are led to the same contradiction. Therefore the first case cannot occur.

Case 2. Suppose u_1u_{p-1}, $u_pu_2 \in E(G)$. Let $S = \{u_i : 3 \leq i \leq p-2$ and $u_1u_i \in E(G)\}$ and $T = \{u_i : 3 \leq i \leq p-2$ and $u_pu_i \in E(G)\}$. Then $|S| + |T| = p-5$, $S \cap T = \varnothing$, and there is exactly one vertex u_j in $\{u_3, \ldots, u_{p-2}\}$ that is in neither S nor T. Also, since G is not hamiltonian, there is no value of $i \in \{2, \ldots, p-2\}$ such that $u_pu_i \in E(G)$ and $u_1u_{i+1} \in E(G)$. It follows easily that

(i) $j = 3$, $S = \{u_4, \ldots, u_{p-2}\}$ and $T = \varnothing$

(ii) $j = p-2$, $S = \varnothing$ and $T = \{u_3, \ldots, p-3\}$ or

(iii) $3 < j < p-2$, $S = \{u_{j+1}, \ldots, u_{p-2}\}$ and $T = \{u_3, \ldots, u_{j-1}\}$. In case (i), G has the hamiltonian cycle $u_1, u_{p-1}, u_p, u_2, u_3, \ldots, u_{p-2}, u_1$, which is impossible. Similarly, case (ii) cannot occur. Thus $3 < j < p-2$ and $p \geq 7$.

Since $\deg_G u_1 + \deg_G u_p = p-1$, either $\deg_G u_1 \leq (p-1)/2$ or $\deg_G u_p \leq (p-1)/2$. Therefore, because u_j is adjacent to neither u_1 or u_p, $\deg_G u_j \geq (p-1)/2 \geq 3$. So u_j must be adjacent to some u_k, where $2 \leq k \leq j-2$ or $j+2 \leq k \leq p-1$. This, however, gives rise to at least one of the two following hamiltonian cycles of $G: u_j, u_k, u_{k-1}, \ldots, u_1, u_{j+1}, u_{j+2}, \ldots, u_p, u_{k+1}, u_{k+2}, \ldots, u_j$ or $u_j, u_k, u_{k+1}, \ldots, u_p, u_{j-1}, u_{j-2}, \ldots, u_1, u_{k-1}, u_{k-2}, \ldots, u_j$. Therefore, the second case cannot occur.

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The graph G of order $p \geq 6$ composed of two disjoint complete subgraphs K_n and K_{p-n}, $3 \leq n \leq p-3$, and an edge joining vertices of the two subgraphs has no spanning circuit. Thus we cannot replace the condition "$\deg_G u + \deg_G v \geq p-1$" by "$\deg_G u + \deg_G v \geq p-2$" in Theorem 2, even in the case of connected graphs.

As observed previously, if the sum of the degrees of each pair of non-adjacent vertices of a graph having order p is at least $p-1$, then there is a spanning path in the graph. It can be verified that if G is a connected graph of order $p \geq 5$ such that $\deg_G u + \deg_G v \geq p-2$ for every pair u, v of non-adjacent vertices, then G contains a spanning trail. However, no example is known to show that this result cannot be improved.

A graph which satisfies Ore's condition for a spanning cycle also contains numerous spanning trails, as we now verify.

THEOREM 3. *Let G be a graph of order $p \geq 5$ such that $\deg_G u + \deg_G v \geq p$ for every pair u, v of non-adjacent vertices. Then each pair of distinct vertices is joined by a spanning trail.*

Proof. By Ore's result, G is hamiltonian. Let $C: v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle of G and let $v_i, v_j \in V(G)$ where $i < j$. If $v_iv_j \in E(C)$, then v_i and v_j are clearly joined by a spanning trail in G. If $v_iv_j \in E(G) - E(C)$, then $v_i, v_{i+1}, \ldots, v_i, v_j$ is a spanning $v_i - v_j$ trail in G.

Assume that $v_iv_j \notin E(G)$. Thus $\deg_G v_i + \deg_G v_j \geq p$ and there exist distinct vertices $v_k, v_\ell \in V(G) - \{v_i, v_j\}$ with v_iv_k, v_iv_ℓ, v_jv_k, $v_jv_\ell \in E(G)$. If either of v_k or v_ℓ is not one of v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}, then C together with the path v_i, v_k, v_j, or C together with the path v_i, v_ℓ, v_j produces a spanning $v_i - v_j$ trail in G. If $\{v_k, v_\ell\} \subseteq \{v_{i-1}, v_{i+1}, v_{j-1}, v_{j+1}\}$, we note that since $p \geq 5$ we cannot have $i+1 = j-1$ and $j+1 = i-1$ (modulo p).

Thus we must have one of the following:

(a) $v_iv_{j-1} \in E(G)$ and $v_iv_{j-1} \notin E(C)$,
(b) $v_iv_{j+1} \in E(G)$ and $v_iv_{j+1} \notin E(C)$,
(c) $v_jv_{i-1} \in E(G)$ and $v_jv_{i-1} \notin E(C)$,
(d) $v_jv_{i+1} \in E(G)$ and $v_jv_{i+1} \notin E(C)$,

yielding one of the following spanning $v_i - v_j$ trails in G:

(a) $v_i, v_{j-1}, v_{j-2}, \ldots, v_j$
(b) $v_i, v_{j+1}, v_{j+2}, \ldots, v_j$
(c) $v_i, v_{i+1}, v_{i+2}, \ldots, v_{i-1}, v_j$
(d) $v_i, v_{i-1}, v_{i-2}, \ldots, v_{i+1}, v_j$.

This completes the proof.

The bound given in Theorem 3 is sharp, even among graphs with minimum degree at least two. For example, let G be the graph of order $p \geq 5$ consisting of the complete graph K_{p-2}, two additional adjacent vertices w and z, and the edges xw and yz, where x and y are distinct vertices of K_{p-2}. In G, we have $\deg_G u + \deg_G v \geq p-1$ for every pair u, v of non-adjacent vertices. However, G contains no spanning $x-z$ trail.

We have already noted the strong relationship which exists between dominating trails and circuits in a graph G and the hamiltonian properties of $L(G)$. This relationship allows one to investigate the hamiltonian properties of the iterated line graph $L(L(G))$ of a graph G. For example, the existence of dominating circuits in the line graph of a graph G was used by Chartrand and Wall in [2] to prove that if G is a connected graph for which $\delta(G) \geq 3$, then $L(L(G))$ is hamiltonian. Now, suppose G is a connected graph with at least four edges in which every vertex of degree two is adjacent to an end vertex. If we let S be the set of all end vertices of G which are adjacent to some vertex of degree two and let G' denote the graph G—S, then G' has at least three edges and $\deg_{G'} w \neq 2$, for each $w \in V(G')$. Using an almost identical technique to that employed in the proof of the aforementioned result of Chartrand and Wall, we can show that $L(G')$ contains a spanning circuit C. Since C is also a dominating circuit of $L(G)$, we conclude that $L(L(G))$ is hamiltonian. This result is stated below.

THEOREM 4. *Let G be a connected graph with at least four edges. If every vertex of degree two is adjacent to an end vertex, then $L(G)$ contains a dominating circuit and thus $L(L(G))$ is hamiltonian.*

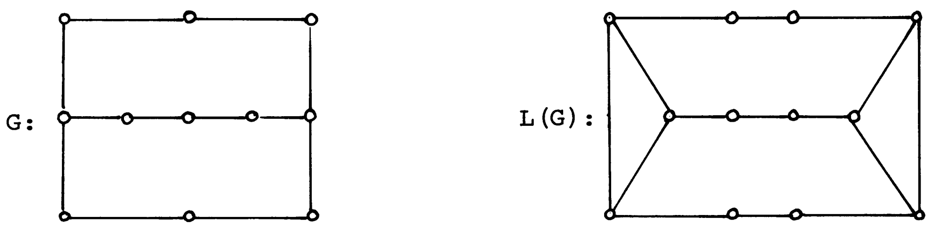

Figure 1

The graph G of Figure 1 illustrates that if a graph fails to satisfy the condition of Theorem 4 involving vertices of degree two, then the second iterated line graph need not be hamiltonian. We observe that $L(G)$ contains no dominating circuit and so, by the aforementioned result of Harary and Nash–Williams, $L(L(G))$ is not hamiltonian.

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