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COLOUR CLASSES FOR r-GRAPHS

BY
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1. **Introduction.** By an r-graph G we mean a finite set $V(G)$ of elements called vertices and a set $E(G)$ of some of the r-subsets of $V(G)$ called edges. This paper defines certain colour classes of r-graphs which connect the material of a variety of recent graph theoretic literature in that many existing results may be reformulated as structural properties of the classes for some special cases of r-graphs. It is shown that the concepts of Ramsey Numbers, chromatic number and index may be defined in terms of these classes. These concepts and some of their properties are generalized. The final subsection compares two existing bounds for the chromatic number of a graph.

We shall use the following notation. For any r-graph G, the subgraph $\langle S\rangle$ induced by the subset S of $V(G)$ is the largest subgraph of G with vertex set S and the subgraph $\langle F\rangle$ generated by the subset F of $E(G)$ is that graph for which $V(\langle F\rangle)=\bigcup_{f\in F}\{v: v\in f\}$ and $E(\langle F\rangle)=F$. If the r-graph B contains a subgraph isomorphic to the r-graph A we write $A<B$ or $B>A$. K_p $(p\geq r)$ will denote the complete r-graph with p vertices (i.e. with $\binom{p}{r}$ edges) and G-v will mean the r-graph obtained by deleting from G, the vertex v and all edges incident with v.

2. **The colour classes.**

DEFINITION Let P_i $(i=1,\ldots,t)$ be any t properties associated with r-graphs. A vertex $(P_1, P_2,\ldots, P_t)$-colouring of an r-graph G is a partition of $V(G)$ into t subsets $S_1, S_2,\ldots, S_t$ such that for each $i=1,\ldots,t$, $\langle S_i\rangle$ has property P_i. An edge $(P_1, P_2,\ldots, P_t)$-colouring of G is similarly defined as a partition of $E(G)$ into $F_1, F_2,\ldots, F_t$ such that for each $i=1,\ldots,t$, $\langle F_i\rangle$ has property P_i. $\mathscr{V}(P_1, P_2,\ldots, P_t)$ and $\mathscr{E}(P_1, P_2,\ldots, P_t)$ are those classes which contain all r-graphs having vertex $(P_1, P_2,\ldots, P_t)$-colourings and edge $(P_1, P_2,\ldots, P_t)$-colourings respectively.

We now give some additional notation. If $\mathscr{R}$ denotes the class of all r-graphs, then $\overline{\mathscr{V}}(P_1, P_2,\ldots, P_t)=\mathscr{R}-\mathscr{V}(P_1, P_2,\ldots, P_t)$ and $\overline{\mathscr{E}}(P_1, P_2,\ldots, P_t)=\mathscr{R}-\mathscr{E}(P_t, P_2,\ldots, P_t)$. If $P_i=P$ for all $i=1,\ldots,t$ we use the terms vertex and edge P_t-colourings, $\mathscr{V}(P^t)$, $\overline{\mathscr{V}}(P^t)$, $\mathscr{E}(P^t)$ and $\overline{\mathscr{E}}(P^t)$ as abbreviations for $(P_1, P_2,\ldots, P_t)$-colourings, etc. Finally we note the trivial fact that all these quantities are invariant under permutations of the subscripts $1,\ldots,t$.

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3 **Ramsey-type properties of r-graphs.** Throughout this section $G_1, \ldots, G_t$ will denote r-graphs and for each $i=1, \ldots, t$ a graph G has property P_i if and only if $G \not> G_i$.

THEOREM 1. *$\overline{\mathscr{V}}(P_1, P_2, \ldots, P_t)$ and $\overline{\mathscr{E}}(P_1, P_2, \ldots, P_t)$ are nonempty.*

Proof. Suppose that G_i has p_i vertices ($i=1, \ldots, t$) and consider the r-graph K_λ where $\lambda=\sum_{i=1}^{t}(p_i-1)+1$. Then in any vertex partition of K_λ into $S_1, \ldots, S_t$ some S_i contains at least p_i vertices and $\langle S_i\rangle > K_{p_i} > G_i$. Hence $K_\lambda \in \overline{\mathscr{V}}(P_1, P_2, \ldots, P_t)$.

Secondly suppose μ is greater than or equal to the Ramsey Number $N(p_1, p_2, \ldots, p_t; r)$ [1]. Then by Ramsey's theorem [2] if the edges of K_μ are partitioned arbitrarily into $F_1, \ldots, F_t$, for at least one i in $\{1, \ldots, t\}$ $\langle F_i\rangle > K_{p_i} > G_i$ and $K_\mu \in \overline{\mathscr{E}}(P_1, P_2, \ldots, P_t)$. This proof as well as some simple properties of $\overline{\mathscr{E}}(P_1, P_2, \ldots, P_t)$ appeared in [3]. The properties are repeated below for completeness.

DEFINITION. The Ramsey edge number $N(G_1, G_2, \ldots, G_t)$ is the smallest integer n such that $K_n \in \overline{\mathscr{E}}(P_1, P_2, \ldots, P_t)$.

We note that if $G_i=K_{p_i}$ then the Ramsey edge number $N(G_1, G_2, \ldots, G_t)$ is the standard Ramsey number $N(p_1, p_2, \ldots, p_t; r)$. Some properties of the classes and Ramsey edge numbers follow:

(i) If $t=1$, $\mathscr{V}(P_1)=\mathscr{E}(P_1)=\{G: G \not> G_1\}$.

(ii) $G \in \mathscr{E}(P_1, P_2, \ldots, P_t)$ and $F < G \Rightarrow F \in \mathscr{E}(P_1, P_2, \ldots, P_t)$, (and similarly for $\mathscr{V}(P_1, P_2, \ldots, P_t)$).

(iii) For each $i=1, \ldots, t$ let an r-graph G have property Q_i if and only if $G \not> H_i$ and suppose that $G_i > H_i$. Then $\mathscr{V}(Q_1, Q_2, \ldots, Q_t) \subseteq \mathscr{V}(P_1, P_2, \ldots, P_t)$, $\mathscr{E}(Q_1, Q_2, \ldots, Q_t) \subseteq \mathscr{E}(P_1, P_2, \ldots, P_t)$ and $N(H_1, H_2, \ldots, H_t) \leq N(G_1, G_2, \ldots, G_t)$.

(iv) Let G' be the r-graph obtained from G by removing a vertex of maximum degree.

THEOREM 2. *$N(G_1, G_2, \ldots, G_t) \leq N(s_1, s_2, \ldots, s_t; r-1) + 1$ where*

$$s_1 = N(G_1', G_2, \ldots, G_t)$$
$$s_2 = N(G_1, G_2', \ldots, G_t) \tag{1}$$
$$\ldots$$
$$s_t = N(G_1, G_2, \ldots, G_t')$$

and $N(s_1, s_2, \ldots, s_t; r-1)$ is the standard Ramsey number.

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Proof. This is a straightforward generalization of the proof of the recurrence inequality for Ramsey numbers [1, p. 41]. Let x be an element of the n-set S where $n \geq N(s_1, s_2, \ldots, s_t; r-1)+1$ and let $F_1, F_2, \ldots, F_t$ be an arbitrary partition of the edges of the complete r-graph with vertex set S. This partition defines a partition $E_1, \ldots, E_t$ of the edges of Y, the complete $(r-1)$-graph whose vertex set is $T=S-\{x\}$ as follows. An edge e of Y is in E_i if and only if $e \cup \{x\}$ is in F_i. Now $|T| \geq N(s_1, s_2, \ldots, s_t; r-1)$ hence by Ramsey's theorem for some j in $\{1, \ldots, t\}$, $\langle E_j \rangle$ contains a subgraph W which is isomorphic to the complete $(r-1)$-graph on s_j vertices. Without losing generality let $j=1$. Next consider the complete r-graph on $V(W)$. Its edges are partitioned among $F_1, \ldots, F_t$ and since $s_1 = N(G_1', G_2, \ldots, G_t)$, either for some k in $\{2, \ldots, t\}$, $\langle F_k \rangle > G_k$ or $\langle F_1 \rangle > G_1'$. If the latter possibility occurs, the r-graph obtained by adjoining to G_1' the r-edges formed by uniting each $(r-1)$-edge of W with $\{x\}$, has a subgraph isomorphic to G_1 and by construction each of the adjoined r-edges is in F_1. Hence the augmented graph is a subgraph of $\langle F_1 \rangle$, showing that $\langle F_1 \rangle > G_1$. Thus in all cases for some i in $\{1, \ldots, t\}$, $\langle F_i \rangle > G_i$ and the theorem is proved.

When $r=2$, $N(s_1, s_2, \ldots, s_t; r-1) = \sum_i^t s_i - t + 1$ and hence Theorem 2 specializes to

$$(2) \quad \begin{aligned} N(G_1, G_2, \ldots, G_t) \leq {} & N(G_1', G_2, \ldots, G_t) \\ & + N(G_1, G_2', \ldots, G_t) + \cdots + N(G_1, G_2, \ldots, G_t') - t + 2 \end{aligned}$$

The proof techniques of [4, Theorem 3] enable one to show that if $\sum_i^t s_i - t$ is even and at least one s_i is even, then the inequality (2) is strict.

Setting $G_i = G$ for each $i=1, \ldots, t$ we obtain

$$N(G^t) \leq tN(G^{t-1}, G') - t + 2$$

and this inequality is strict if both t and $N(G^{t-1}, G')$ are even.

4. Generalized chromatic numbers.

DEFINITION. The vertex (edge) P-chromatic number of an r-graph G, denoted by $\chi_P(G)$ $(\chi_P'(G))$, is the least integer t such that G has a vertex (edge) P^t-colouring.

Equivalently $\chi_P(G)$ is the smallest integer t such that $G \in \mathscr{V}(P^t)$. If $r=2$ and P means totally disconnected, then $\chi_P(G)$ is the usual chromatic number and if a 2-graph has property P if and only if it has no subgraph isomorphic to the graph with 3 vertices and 2 edges, then $\chi_P'(G)$ is the chromatic index or line chromatic number of G. Several papers on particular P-chromatic numbers of 2-graphs have already appeared in the literature. (See [5] and [6].)

4.1. In this section we establish an upper bound for $\chi_P(G)$ which generalizes a result of Szekeres and Wilf [7]. Let G be an r-graph, P be a hereditary property

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(see [8, p. 96]) and for $v \in V(G)$ let $\mathscr{S}_P(v)$ be a family of subgraphs of G with the following properties:

(a) $H \in \mathscr{S}_P(v) \Rightarrow v \in V(H)$.

(b) $H \in \mathscr{S}_P(v) \Rightarrow H$ does not have property P but for all $u \in V(H)$, $H-u$ has property P.

(c) $H_1, H_2 \in \mathscr{S}_P(v) \Rightarrow H_1, H_2$ have no common vertex except for v, i.e. $V(H_1) \cap V(H_2)=\{v\}$.

We define $d_P(v)$ to be the largest cardinality of all such classes $\mathscr{S}_P(v)$ and $\delta_P(G)=\min_{v\in V(G)} d_P(v)$.

THEOREM 3. *If P is a hereditary property then*

$$\chi_P(G) \leq 1+\max_{G'<G} \delta_P(G'). \tag{3}$$

We note that if $r=2$ and P means no edge, then $d_P(v)$ is merely the degree of $v \in G$ and (3) reduces to the bound of Szekeres and Wilf:

$$\chi(G) \leq 1+\max_{G'<G} \min_{v\in V(G')} d(v) \tag{4}$$

($d(v)$ here refers to the degree of v in G').

Proof. By removing successive vertices from G, if necessary, we can form a subgraph G^c such that $\chi_P(G^c)=\chi_P(G)$ but for any $v \in V(G^c)$, $\chi_P(G^c-v)=\chi_P(G)-1$. Then for each $v \in V(G^c)$,

$$d_P(v) \geq \chi_P(G)-1. \tag{5}$$

For suppose the contrary, i.e. $\exists\ u \in V(G^c)$ s.t. $d_P(u)\leq\chi_P(G)-2$. Let $t=\chi_P(G)$ and $V_1, V_2, \ldots, V_{t-1}$ be a vertex P^{t-1}-colouring of G^c-u. By definition each $\langle V_i\rangle$ has property P. Therefore if each of the $t-1$ subgraphs of G^c, $\langle V_i \cup \{u\}\rangle$ $i=1,\ldots,t-1$, were without property P, then for each $i=1,\ldots,t-1$, $\langle V_i \cup \{u\}\rangle > W_i$ where $u \in V(W_i)$, W_i does not have property P but the removal of any vertex from W_i restores the property P. Then the family $\{W_i\colon i=1,\ldots,t-1\}$ satisfies (a), (b) and (c) above and hence $d_P(u)\geq t-1$ contrary to hypothesis. We may therefore conclude that for some j in $1,\ldots,t-1$, $\langle V_j \cup \{u\}\rangle$ has property P. But this implies that $V_1, V_2, \ldots, V_{j-1}, V_j \cup \{u\}, V_{j+1}, \ldots, V_{t-1}$ is a vertex P^{t-1}-colouring of G^c contrary to the definition of G^c. Hence (5) is true, i.e. for all $v \in G^c$.

$$\chi_P(G) \leq 1+d_P(v).$$

Therefore $\chi_P(G)\leq 1+\delta_P(G^c)\leq 1+\max_{G'<G} \delta_P(G')$.

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4.2. Throughout this section, property P will mean no subgraph isomorphic to the r-graph H and we shall use the more convenient notation $\chi_H(G)$ rather than $\chi_P(G)$. The following properties are easily established.

(i) Let H_1, H_2, G be r-graphs and $H_1 < H_2$. Then $\chi_{H_2}(G) \le \chi_{H_1}(G)$.

(ii) Let H, G_1, G_2 be r-graphs and $G_1 < G_2$. Then $\chi_H(G_1) \le \chi_H(G_2)$.

(iii) Let $\{x\}$ be the smallest integer greater than or equal to x. Then for any r, $\chi_{K_p}(K_q) = \left\{\frac{q}{p-1}\right\}$.

For the r-graph M, we define

$$\lambda(M) = \max\{n : M > K_n\}$$

and

$$\beta_H(M) = \max\{|S| : S \subseteq V(M) \text{ and } \langle S\rangle \not> H\}.$$

THEOREM 4. *If G has p vertices then*

$$p/\beta_H(G) \le \chi_H(G) \le \left\{\frac{p-\beta_H(G)}{\lambda(H)-1}\right\}+1. \tag{6}$$

If $r=2$ and $H=K_2$, then $\beta_H(G)$ is the point independence number β_0 of G and $\lambda(H)=2$. Thus (6) reduces to:

$$p/\beta_0 \le \chi(G) \le p-\beta_0+1 \tag{7}$$

which are well known inequalities. (See [8, p. 128]).

Proof. If $\chi_H(G)=t$, there is a partition $V_1, V_2, \ldots, V_t$ of $V(G)$ such that no $V_i > H$. Then $|V_i| \le \beta_H(G)$ for each $i=1, \ldots, t$ and $p=\sum_{i=1}^t |V_i| \le t\beta_H(G)$. Therefore $\chi_H(G) \ge p/\beta_H(G)$.

Let F be an r-graph with q vertices. Using properties (i), (ii) and (iii) of this section:

$$\chi_H(F) \le \chi_{K_{\lambda(H)}}(F) \le \chi_{K_{\lambda(H)}}(K_q) = \left\{\frac{q}{\lambda(H)-1}\right\}. \tag{8}$$

Finally let S be a maximal subset of $V(G)$ such that $\langle S\rangle \not> H$, i.e. $|S| = \beta_H(G)$. Denote by G-S the r-graph formed by deleting S and all edges incident with S from G. Then

$$\chi_H(G-S) \ge \chi_H(G)-1. \tag{9}$$

But $|V(G-S)| = p-\beta_H(G)$ and so applying (8) with $F=G-S$ we obtain

$$\chi_H(G-S) \le \left\{\frac{p-\beta_H(G)}{\lambda(H)-1}\right\}$$

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and then by (9) the required result

$$\chi_H(G) \leq 1+\left\{\frac{p-\beta_H(G)}{\lambda(H)-1}\right\}.$$

4.3. In the preceding sections two known upper bounds for the standard chromatic number of a 2-graph have been mentioned. They are

$$\chi(G) \leq 1+\max_{G'>G}\min_{v\in V(G')} d(v)$$

and

$$\chi(G) \leq 1+p-\beta_0(G).$$

The following result shows that the first of these is the sharper bound.

THEOREM 5. *For any graph G*

$$\max_{G'<G}\min_{v\in V(G')} d(v) \leq p-\beta_0(G).$$

Proof. Let S be a set of $\beta_0(G)$ independent points of G and let G' be any subgraph of G. Either G' contains no vertex of S in which case $G'<K_{p-\beta_0(G)}$ and

$$\min_{v\in V(G')} d(v) \leq p-\beta_0(G)-1$$

or G' contains a vertex in S which has degree $\leq p-\beta_0(G)$ and therefore

$$\min_{v\in V(G')} d(v) \leq p-\beta_0(G).$$

Thus for all subgraphs G' of G, $\min_{v\in V(G')} d(v) \leq p-\beta_0(G)$. Hence

$$\max_{G'<G}\min_{v\in V(G')} d(v) \leq p-\beta_0(G)$$

as required.

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REFERENCES

1. H. J. Ryser, *Combinatorial mathematics*, Carus Math. Monograph, Math. Assoc. America, **14** (1965), 38–43.
2. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), 264–286.
3. E. J. Cockayne, *An application of Ramsey's theorem*, Canad. Math. Bull. (1) **13** (1970), 145–6.
4. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955), 1–7.
5. G. Chartrand, D. Geller and S. Hedetniemi, *A generalization of the chromatic number*, Proc. Cambridge Philos. Soc. **64** (1968), 265–271.
6. S. Hedetniemi, *Disconnected colourings of graphs*, *Combinatorial Structures and their applications*. Gordon & Breach, New York (1970), 163–168.
7. G. Szekeres and H. S. Wilf, *An inequality for the chromatic number of a graph*, J. Comb. Theory (1) **4** (1968), 1–3.
8. F. Harary, *Graph theory*, Addison-Wesley, Reading, Mass., 1969.

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