Proceedings of the Edinburgh Mathematical Society (1998) **41**, 133–139 ©

THE SPECTRA OF TOEPLITZ OPERATORS WITH UNIMODULAR SYMBOLS

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(Received 9th January 1996)

The spectrum $\sigma(T_\phi)$ of a Toeplitz operator T_ϕ on the open unit disc D for a unimodular symbol ϕ is studied and many sufficient conditions for $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$ are given. In particular if ϕ is a unimodular function in $H^\infty + C$, then $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$.

1991 *Mathematics subject classification:* Primary 47B35.

1. Introduction

Let L^p be the Lebesgue space on the unit circle ∂D and let H^p be the corresponding Hardy space for $0 < p \leq \infty$. The Toeplitz operator T_ϕ with symbol ϕ in L^∞ is the operator on H^2 defined by $T_\phi x = P(\phi x)$ for x in H^2, where P is the orthogonal projection of L^2 onto H^2.

In this paper we study the spectrum $\sigma(T_\phi)$ of a Toeplitz operator T_ϕ. It is known that $\sigma(T_\phi)$ is always connected. This is a hard and deep result due to H. Widom (cf. [**2**, Corollary 7.46]). If ϕ is a continuous function on ∂D, $\sigma(T_\phi)$ consists of the range of ϕ together with those points not in the range of ϕ that have a nonzero index with respect to ϕ (cf. [**2**, Corollary 7.28]). If ϕ is a real-valued function in L^∞, $\sigma(T_\phi) =$ [ess inf ϕ, ess sup ϕ] (cf. [**2**, Theorem 7.20]) and if ϕ is a function in H^∞, $\sigma(T_\phi) =$ the closure of $\phi(D)$ (cf. [**2**, Theorem 7.21]). In particular, we are interested in the spectrum $\sigma(T_\phi)$ of a Toeplitz operator T_ϕ when ϕ is a unimodular function in L^∞. M. Lee and D. Sarason [**6**], and R. G. Douglas and D. Sarason [**3**] have considered $\sigma(T_\phi)$ when ϕ is a quotient of two inner functions. Under some conditions, they showed that $\sigma(T_\phi) = \bar{D}$ [**6**]. In this paper, we consider such a problem when ϕ is an arbitrary unimodular function. Theorem 1 in [**6**] is a corollary of (2) of Theorem 2 in this paper. For a real-valued function s in L^∞, $\tilde{s}$ denotes the harmonic conjugate with $\tilde{s}(0) = 0$. Our main tool is the following theorem [**1**].

Widom and Devinatz's Theorem. *Let ϕ be a unimodular function in L^∞. Then the following* (1)–(3) *are equivalent.*

* This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

(1) T_ϕ *is invertible.*

(2) ϕ *has the form:* $\phi = e^{it}$ *where* t *is a real-valued function in* L^1 *such that* $\inf\{\|t - \tilde{s} - a\|_\infty;\ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$.

(3) ϕ *has the form:* $\phi = g_1 g_2/|g_1 g_2|$ *where both* g_1 *and* g_1^{-1} *are in* H^∞, *and both* g_2 *and* g_2^{-1} *are in* $\bigcup_{p>1} H^p$ *with Re* g_2 *bounded away from* 0 *on* ∂D.

In this paper, we give sufficient conditions for $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$, using $\inf \|t - \tilde{s} - a\|_\infty$ in Section 1 and $g/|g|$ in Section 2. Throughout this paper, for a function space X on ∂D, we let $X_R = \{Ref;\ f \in X\}$, where Ref is a real part of f. C denotes a set of continuous functions on ∂D and so C_R is a set of all real valued continuous functions on ∂D.

2. Sufficient conditions using $\inf \|t - \tilde{s} - a\|_\infty$

Lemma 1. *Let* ϕ *be unimodular in* L^∞ *and* $\lambda = a + ib$ *in* D. *Then* $\lambda \notin \sigma(T_\phi)$ *if and only if* ϕ *has the form* $\phi = e^{it}$ *where* t *is a real-valued function in* L^1 *such that*

$$\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty\ ;\ s \in L^\infty_R \text{ and } a \in R\} < \pi/2$$

and $v_\lambda = \arctan\{(a \sin t - b \cos t)/(1 - (a \cos t + b \sin t))\}$.

Proof. We will first show the "if" part. There exists a function s_λ in L^∞ such that $(1 - \lambda\bar{\phi})/|1 - \lambda\bar{\phi}| = e^{is_\lambda}$ and $\|s_\lambda\|_\infty < \pi/2$ because $|\lambda| < 1$. Then

$$\frac{1 - (a \cos t + b \sin t)}{|1 - \lambda\bar{\phi}|} + i\frac{a \sin t - b \cos t}{|1 - \lambda\bar{\phi}|} = \cos s_\lambda + i \sin s_\lambda.$$

Since $|a \cos t + b \sin t| \le |\lambda| < 1$, $\|v_\lambda\|_\infty < \pi/2$. Hence $\|v_\lambda - s_\lambda\|_\infty < \pi$ and $\tan v_\lambda = \tan s_\lambda$ *a.e.* and so $v_\lambda = s_\lambda$ *a.e.*. Therefore

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \phi\frac{1 - \lambda\bar{\phi}}{|1 - \lambda\bar{\phi}|} = e^{it}e^{iv_\lambda}$$

and by Widom and Devinatz's Theorem in the Introduction $T_{\phi-\lambda}$ is invertible because $\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty\ ;\ s \in L^\infty_R$ and $a \in R\} < \pi/2$. Conversely if $\lambda \notin \sigma(T_\phi)$, by Widom and Devinatz's Theorem there exists a real-valued function t_λ such that $(\phi - \lambda)/|\phi - \lambda| = e^{it_\lambda}$ and $\inf\{\|t_\lambda - \tilde{s} - a\|_\infty\ ;\ s \in L^\infty_R$ and $a \in R\} < \pi/2$. As in the proof of the "if" part, there exists s_λ such that $(1 - \lambda\bar{\phi})/|1 - \lambda\bar{\phi}| = e^{is_\lambda}$. Moreover $\phi = e^{it}$ and $s_\lambda = v_\lambda$ if $t = t_\lambda - s_\lambda$. This implies the "only if" part.

Theorem 1. *Let* ϕ *be a unimodular function in* L^∞.

(1) *If* $\phi = e^{it}$ *and* t *is a real-valued function in* L^1 *such that* $\inf\{\|t - \tilde{s} - a\|_\infty;\ s \in L_R^\infty$ *and* $a \in R\} = 0$, *then* $\sigma(T_\phi) \subseteq \partial D$.

(2) *If* $\inf\{\|t - \tilde{s} - a\|_\infty\ ;\ s \in L_R^\infty$ *and* $a \in R\} \geq \pi$ *for any* $t \in L_R^1$ *with* $\phi = e^{it}$, *then* $\sigma(T_\phi) = \bar{D}$.

(3) *If* $\sigma(T_\phi) = \bar{D}$, *then* $\inf\{\|t - a\|_\infty\ ;\ a \in R\} \geq \pi$ *for any* $t \in L_R^1$ *with* $\phi = e^{it}$.

Proof. (1) If $\lambda = a + ib \in D$ and $v_\lambda = \arctan\{(a\sin t - b\cos t)/1 - (a\cos t + b\sin t)\}$, then $\|v_\lambda\|_\infty < \pi/2$ and hence $\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty\ ;\ s \in L_R^\infty$ and $a \in R\} < \pi/2$ because $\inf\{\|t - \tilde{s} - a\|_\infty\ ;\ s \in L_R^\infty$ and $a \in R\} = 0$. By Lemma 1, $\lambda \notin \sigma(T_\phi)$ and hence $\sigma(T_\phi) \subseteq \partial D$.

(2) If $\lambda \in D$ and $\lambda \notin \sigma(T_\phi)$, then by Lemma 1 $\inf\{\|t + v_\lambda - \tilde{s} - a\|_\infty\ ;\ s \in L_R^\infty$ and $a \in R\} < \pi/2$. Since $\|v_\lambda\|_\infty < \pi/2$, $\inf\{\|t - \tilde{s} - a\|_\infty\ ;\ s \in L_R^\infty$ and $a \in R\} < \pi$. This implies (2).

(3) (3) is a result of a theorem of A. Brown and P. R. Halmos (cf. [**2**, Corollary 7.19]).

Corollary 1. *Suppose* $\phi = e^{it}$ *and* t *is a real-valued function which satisfies one of the following* (i)–(iii), *then* $\sigma(T_\phi) \subseteq \partial D$.

(i) $t = \tilde{u} + v$ *where* $u \in L_R^\infty$ *and* $v \in C_R$.

(ii) $t = \tilde{u} + v$ *where* $u \in L_R^\infty$ *and* v *is in the norm closure of* H_R^∞.

(iii) $t = \tilde{u} + v$ *where* $u \in L_R^\infty$ *and* $v = s \circ q$ *for* $s \in C_R$ *and an inner function* q.

Proof. If $v \in C_R$, then v is in the norm closure of H_R^∞ and so (i) is a result of (ii). If $v \in H_R^\infty$, then $v = \tilde{s} + a$ for $s \in H_R^\infty$ and $a \in R$, and hence a simple computation implies (ii). If s is a real-valued polynomial of z and $\bar{z}$, then $v = s \circ q$ belongs to H_R^∞ for an inner function q. Thus (iii) is a result of (ii).

Corollary 2. *Let* Q_j *be a non-constant inner function,* $a_j \in D$ *and* $b_j \in D$ *for* $1 \leq j \leq \max(n, m)$. *Suppose* $\phi = \bar{q}_1 q_2$ *where* $q_1 = \prod_{j=1}^n (Q_j - a_j)/(1 - \bar{a}_j Q_j)$ *and* $q_2 = \prod_{j=1}^m (Q_j - b_j)/(1 - \bar{b}_j Q_j)$. *Then* $\sigma(T_\phi) \subseteq \partial D$ *if and only if* $n = m$.

Proof. If $n = m$, put $u = 2\sum_{j=1}^n \log|(1 - \bar{a}_j Q_j)/(1 - \bar{b}_j Q_j)|$, then $u \in L_R^\infty$ and $\phi = \bar{q}_1 q_2 = \alpha e^{i\tilde{u}}$ for some constant α. (1) of Theorem 1 implies the corollary. Suppose $\sigma(T_\phi) \subseteq \partial D$. If $n > m$, then $\phi = \bar{q}_1 q_2 = \phi_1 \phi_2$ where $\phi_1 = \prod_{j=m+1}^n (1 - \bar{a}_j Q_j / Q_j - a_j)$, $\phi_2 = \alpha e^{i\tilde{u}}$, α is a constant and $u = 2\sum_{j=1}^m \log|(1 - \bar{a}_j Q_j)/(1 - \bar{b}_j Q_j)|$. Therefore $T_\phi = T_{\phi_1} T_{\phi_2}$, and both T_ϕ and T_{ϕ_2} are invertible. This contradicts the fact that T_{ϕ_1} is not invertible.

3. Sufficient conditions using $g/|g|$ for g in H^p

Theorem 2. *Let ϕ be a unimodular function in L^∞.*

(1) *If $\phi = g/|g|$ where both g and g^{-1} are in H^∞, then $\sigma(T_\phi) \subseteq \partial D$.*

(2) *If $\phi \neq g/|g|$ for any g in $\bigcup_{p>1/2} H^p$ whose inverse is in $\bigcup_{p>1/2} H^p$, then $\sigma(T_\phi) = \bar{D}$.*

Proof. (1) This is a corollary of (1) of Corollary 1. But we will give another proof. If $\phi = g/|g|$ where both g and g^{-1} are in H^∞, put $h = g^{1/2}$, then $\phi = h/\bar{h}$ and both h and h^{-1} are in H^∞. For any $\lambda \in D$, $\phi - \lambda = (1/\bar{h})(1 - \lambda\bar{h}/h)h$ and hence

$$T_{\phi-\lambda} = T_{(1/\bar{h})} T_{(1-\lambda\bar{h}/h)} T_h.$$

This implies that $T_{\phi-\lambda}$ is invertible by Widom and Devinatz's Theorem.

(2) For any $\lambda \in D$, $1 - \lambda\bar{\phi} = \phi_0 \ell$ where $|\phi_0| = 1$ *a.e.*, and both ℓ and ℓ^{-1} are in H^∞. Hence

$$\phi - \lambda = \phi(1 - \lambda\bar{\phi}) = \phi\phi_0\ell \text{ and } \bar{\phi}_0 - \ell = \lambda\bar{\phi}_0\bar{\phi}.$$

Since $\|\bar{\phi}_0 - \ell\|_\infty = |\lambda| < 1$, by Widom and Devinatz's Theorem $T_{\bar{\phi}_0}$ is invertible and $\bar{\phi}_0 = h/|h|$ for some $h \in H^a$ and $a > 1$. If $T_{\phi-\lambda}$ is invertible, then $T_{\phi\phi_0}$ is invertible and hence $\phi\phi_0 = k/|k|$ for some $k \in H^b$ and $b > 1$. Therefore $\phi = \bar{\phi}_0\phi\phi_0 = hk/|hk|$ and both hk and $(hk)^{-1}$ belong to H^p for some $p > 1/2$. This implies (2).

Corollary 3. *If $\phi = g/|g|$ where $g \in \bigcap_{p<\infty} H^p$ and $g^{-1} \notin \bigcap_{p>1/2} H^p$, then $\sigma(T_\phi) = \bar{D}$.*

Proof. If $\phi = h/|h|$ for some h in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$, then $\phi = |k|/k$ with $k = 1/h$. Hence kg is non-negative *a.e.* on ∂D and $kg \in H^{1/2}$. By [7], $g = ch$ for some positive constant c and $g^{-1} \in \bigcap_{p>1/2} H^p$. Now (2) of Theorem 2 implies the corollary.

Corollary 4. *Let Q_j be a non-constant inner function, $a_j \in D$ and $b_j \in D$ for $1 \leq j \leq \max(n, m)$. Suppose $\phi = \bar{q}_1 q_2$ where $q_1 = \prod_{j=1}^{n}(Q_j - a_j)/(1 - \bar{a}_j Q_j)$ and $q_2 = \prod_{j=1}^{m}(Q_j - b_j)/(1 - \bar{b}_j Q_j)$. Then $\sigma(T_\phi) = \bar{D}$ if and only if $n \neq m$.*

Proof. By Corollary 2, it is enough to show the "if" part. If $n > m$, then by the proof of Corollary 2 $\phi = \phi_2\phi_2$ and so $\phi = \phi_1(g/|g|)$ where both g and g^{-1} are in H^∞, and ϕ_1 is a non-constant inner function. If $\phi = h/|h|$ for some h in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$, $\phi_1 g h^{-1}$ is a non-negative function in $H^{1/2}$. By [7], this contradicts that ϕ_1 is non-constant. Thus (2) of Theorem 2 implies that $\sigma(T_\phi) = \bar{D}$. When $n < m$, by a similar method we can show that $\sigma(T_\phi) = \bar{D}$.

Now using (2) of Theorem 2, we will give a proof of Theorem 1 in [6]. For each inner function q, sing q denotes the subset of ∂D on which q can not be analytically extended.

Corollary 5 ([6]). *If $\phi = \bar{q}_1 q_2$ where q_1 and q_2 are inner functions with* sing $q_1 \neq$ sing q_2, *then $\sigma(T_\phi) = \bar{D}$.*

Proof. By (2) of Theorem 2, it is enough to show that $\phi = \bar{q}_1 q_2 \neq g/|g|$ for any g in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$. We may assume that sing $q_1 \not\ni z_0 \in$ sing q_2. There exists a constant $\lambda \in D$ such that $q = (q_2 - \lambda)/(1 - \bar{\lambda} q_2)$ is a Blaschke product with sing $q =$ sing q_2 by [5, p. 176]. Then $\bar{q}_1 q_2 = \bar{q}_1 q k/|k|$ where $k = (1 - \bar{\lambda} q_2)^2$. Since both k and k^{-1} are in H^∞, we may assume that q_2 is a Blaschke product. If $\bar{q}_1 q_2 = f/|f|$ $q_1 \bar{q}_2 = g/|g|$ where $fg = 1$ *a.e.*, $f \in H^{1/2}$ and $g \in H^{1/2}$, then $\bar{q}_1 q_2 g \geq 0$ *a.e.* and $\bar{q}_2 q_1 f \geq 0$ *a.e.*. Since $\bar{q}_1 q_2 g \geq 0$ *a.e.*, $g \in H^{1/2}$ and $z_0 \notin$ sing q_1, by [4] there exists an open arc J such that $z_0 \in J$ and $q_2 g$ can be continued analytically from D across J. The zeros of q_2 cannot cluster at any point of J. This contradicts that $z_0 \in$ sing q_2. Thus $\bar{q}_1 q_2$ satisfies the condition of (2) of Theorem 2, and hence $\sigma(T_\phi) = \bar{D}$.

Corollary 6. *Let q_1 and q_2 be inner functions, and χ_E be a characteristic function of a measurable set E in ∂D. If $\phi = \bar{q}_1 q_2 (2\chi_E - 1)$ and there exists an open arc J in E such that* (sing $q_2) \cap J \neq \emptyset$ *and* (sing $q_1) \cap J = \emptyset$, *or* (sing $q_1) \cap J \neq \emptyset$ *and* (sing $q_2) \cap J = \emptyset$, *then $\sigma(T_\phi) = \bar{D}$.*

Proof. As in Corollary 5, we may assume that q_2 is a Blaschke product. If $\phi = \bar{q}_1 q_2 (2\chi_E - 1) = f/|f| = |g|/g$ where $fg = 1$ *a.e.*, $f \in H^{1/2}$ and $g \in H^{1/2}$, then $\bar{q}_1 q_2 (2\chi_E - 1) g \geq 0$ *a.e.* and $\bar{q}_2 q_1 (2\chi_E - 1) f \geq 0$ *a.e.*. If there exists an open arc J in E such that (sing $q_2) \cap J \neq \emptyset$ and (sing $q_1) \cap J = \emptyset$, then

$$\bar{q}_1 q_2 (2\chi_E - 1) g = \bar{q}_1 q_2 g \geq 0 \textit{ a.e.} \text{ on } J.$$

Now as in Corollary 5, we can get a contradiction and hence $\sigma(T_\phi) = \bar{D}$.

Let $q_a = \exp\{-a(1+z)/(1-z)\}$ for $a > 0$ and suppose b is a Blaschke product with sing $b = \{1\}$. Put $\phi_a = \bar{q}_a b$. Theorem 4 in [6] shows that if ϕ_a belongs to $H^\infty + C$ for all $a > 0$, then $\sigma(T_{\phi_a}) = \bar{D}$. This is a corollary of Corollary 7.

Corollary 7. *If ϕ_a belongs to $H^\infty + C$ for some $a > 0$, then $\sigma(T_{\phi_c}) = \bar{D}$ for $0 < c < a$. If T_{ϕ_a} is invertible or $\sigma(T_{\phi_a}) \subseteq \partial D$, then $\sigma(T_{\phi_c}) = \bar{D}$ for arbitrary $c > 0$ with $c \neq a$.*

Proof. By Theorem 2 in [8], $\phi_a = q e^{i(u+\tilde{v})}$ where q is inner, and u and v are in C_R. For $0 < c < a$, $\phi_c = q_{a-c} q e^{i(u+\tilde{v})}$ and so by (2) of Theorem 2 $\sigma(T_{\phi_c}) = \bar{D}$. For if $q_{a-c} q e^{i(u+\tilde{v})} = g/|g|$ for some g in $\bigcup_{p>1/2} H^p$ with $h = g^{-1} \in \bigcup_{p>1/2} H^p$, then $h q_{a-c} q e^{i(u+\tilde{v})} \geq 0$ *a.e.* and so $hkq_{a-c} q \geq 0$ *a.e.* where $k = e^{-\tilde{u}+v+i(u+\tilde{v})}$. Since both k and k^{-1} belong to $\bigcap_{p<\infty} H^p$, $hkq_{a-c} q$ is a non-negative function in $H^{1/2}$ and so by [7], $hkq_{a-c} q$ is constant.

This contradicts the fact that $q_{a-c}q$ is not constant. Therefore (2) of Theorem 2 shows that $\sigma(T_{\phi_c}) = \bar{D}$ for $0 < c < a$. If T_{ϕ_a} is invertible, it is known that q is constant. In fact, we can show it as in the above proof. If q is constant, then for $c > 0$ with $c \neq a$

$$\phi_c = \bar{q}_c b = q_{a-c}\bar{q}_a b = q_{a-c}e^{i(u+\bar{v})}.$$

By the first part of this theorem, we may assume that $c > a$. However, in this case we can show it as in case $0 < c < a$.

4. Remark

If $\sigma(T_\phi) \subseteq \partial D$, then $\sigma(T_\phi) = J$ for some closed arc J in ∂D because $\sigma(T_\phi)$ is connected by a theorem of H. Widom (cf. [**2**, Corollary 7.46]). Then, if the essential range $R(\phi)$ of ϕ is disconnected, by a theorem of A. Brown and P. R. Halmos (cf. [**2**, Corollary 7.19]), then $\sigma(T_\phi) \not\subseteq \partial D$. Hence if $\sigma(T_\phi) \subseteq \partial D$, $R(\phi)$ is connected and so $R(\phi) = J = \sigma(T_\phi)$ by the theorem of A. Brown and P. R. Halmos. If $\phi = \alpha e^{it}$, $\inf\{\|t - \bar{s}\|_\infty ;\ s \in L^\infty_R\} = 0$ and $R(\phi) = \partial D$, then $\sigma(T_\phi) = \partial D$ by (1) of Theorem 1. For a unimodular function ϕ in C, by Theorem 1 it is easy to see that $\sigma(T_\phi) \subseteq \partial D$ if and only if $\phi = e^{iv}$ for some $v \in C_R$. For a unimodular function ϕ in $H^\infty + C$, by [**8**, Theorem 2] and Theorem 1 it is easy to see that $\sigma(T_\phi) \subseteq \partial D$ if and only if $\phi = e^{i(\tilde{u}+v)}$ for some $u, v \in C_R$. In fact, by [**8**, Theorem 2], Theorems 1 and 2, $\sigma(T_\phi) \subseteq \partial D$ or $\sigma(T_\phi) = \bar{D}$ for a unimodular function ϕ in $H^\infty + C$.

In Corollary 3, we can not change the condition: $g^{-1} \notin \bigcup_{p>1/2} H^p$ to $g^{-1} \notin \bigcup_{p>1} H^p$ even if $g \in H^\infty$. For example, put $g = 1 + z$ then $\sigma(T_\phi) \neq \bar{D}$. If $\phi = (1+q)^\alpha/|1+q|^\alpha$ where q is a non-constant inner function and $2 \leq \alpha < \infty$, then by Corollary 3 $\sigma(T_\phi) = \bar{D}$ because $(1+q)^\alpha \in H^\infty$ and $(1+q)^{-\alpha} \notin \bigcup_{p>1/2} H^p$. We can show a more general theorem than Corollary 6, that is, for a symbol $\phi = \bar{q}_1 q_2 \phi_0$ where ϕ_0 is a unimodular step function. Let ϕ be an arbitrary unimodular function in L^∞, then by [**8**] $\phi = \bar{q}_1 q_2 e^{i(u+\bar{v})}$ where both q_1 and q_2 are Blaschke products and $u, v \in C_R$. If sing $q_1 \neq$ sing q_2, then by the proof of Corollary 5 it is easy to see that $\phi \neq g/|g|$ for any g in $\bigcap_{p>1/2} H^p$ whose inverse is in $\bigcap_{p>1/2} H^p$. Thus by Theorem 2 $\sigma(T_\phi) = \bar{D}$.

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