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APPROXIMATION OF ENTIRE FUNCTIONS OVER CARATHÉODORY DOMAINS

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Let D be a domain bounded by a Jordan curve. For $1 \le p \le \infty$, let $L^p(D)$ be the class of all functions f holomorphic in D such that $\|f\|_{D,p} = \left((1/A) \iint_D |f(z)|^p dxdy\right)^{1/p} < \infty$, where A is the area of D. For $f \in L^p(D)$, set

(*) $$E_n^p(f) = \inf_{g \in \pi_n} \|f-g\|_{D,p} ;$$

π_n consists of all polynomials of degree at most n. Recently, Andre Giroux (*J. Approx. Theory* 28 (1980), 45-53) has obtained necessary and sufficient conditions, in terms of the rate of decrease of the approximation error $E_n^p(f)$, such that $f \in L^p(D)$, $2 \le p \le \infty$, has an analytic continuation as an entire function having finite order and finite type. In the present paper we have considered the approximation error (*) on a Carathéodory domain and have extended the results of Giroux for the case $1 \le p < 2$.

1. Introduction

Let B denote a Carathéodory domain, that is, a bounded simply

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connected domain such that the boundary of B coincides with the boundary of the domain lying in the complement of the closure of B and containing the point ∞. In particular, a domain bounded by a Jordan curve is a Carathéodory domain. Let $L^p(B)$, $1 \le p \le \infty$, be the class of all functions f holomorphic on B and satisfying

$$\|f\|_{B,p} = \left(\iint_B |f(z)|^p dxdy\right)^{1/p} < \infty ,$$

where the last inequality is understood to be $\sup_{z \in B} |f(z)| < \infty$ for $p = \infty$.

Then $\|\cdot\|_{B,p}$ is called the L^p-norm on $L^p(B)$. For $f \in L^p(B)$, we define $E_n^p(f)$, the error in approximating the function f by polynomials of degree at most n in L^p-norm, as

$$E_n^p(f) \equiv E_n^p(f, B) = \inf_{g \in \pi_n} \|f-g\|_{B,p} , \quad n = 0, 1, 2, \ldots ,$$

where π_n consists of all polynomials of degree at most n.

We prove

THEOREM 1. *Let $f \in L^p(B)$, $1 \le p \le \infty$. Then f is the restriction to B of an entire function, if and only if,*

$$\lim_{n\to\infty} \left(E_n^p(f)\right)^{1/n} = 0 . \tag{1.1}$$

For the case $p = \infty$, it is sufficient to assume that f is continuous on B.

THEOREM 2. *Let $f \in L^p(B)$, $1 \le p \le \infty$. Then f is the restriction to B of an entire function of finite order ρ, if and only if,*

$$\limsup_{n\to\infty} \left[(n \log n)/\left(-\log E_n^p(f)\right)\right] = \rho \tag{1.2}$$

and, if $\rho > 0$, of nonzero finite type T, if and only if,

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(1.3) $$\limsup_{n\to\infty} n\left[E_n^p(f)\right]^{\rho/n} = e\rho d^{\rho}T$$

where d is the transfinite diameter of the closure of B. For the case $p = \infty$, it is enough to assume that f is continuous on B.

REMARKS. (i) Results of the nature of Theorems 1 and 2, in L^{∞}-norm, have been extensively studied by various workers (for example, Bernstein [1, p. 113], [5, pp. 76-78], Varga [10], Shah [8], Kapoor and Nautiyal [4], Winiarski [11]).

(ii) For $p = 2$ and $B = \{z : |z| < 1\}$, Theorems 1 and 2 are due to Reddy [7].

(iii) Theorems 1 and 2 extend and generalize the results of Giroux [3], obtained for the case $2 \le p \le \infty$ with B as a domain bounded by a Jordan curve.

2. Proofs of the theorems

Let B^* be the component of the complement of $\overline{B}$, the closure of B, that contains the point ∞. Set $B_r = \{z : |\varphi(z)| = r\}$, $r > 1$, where the function $w = \varphi(z)$ maps B^* conformally onto $|w| > 1$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$.

We need the following lemmas.

LEMMA 1 ([11, Lemma 3.1]). *The order ρ of an entire function $f(z)$ is given by*

$$\rho = \limsup_{r\to\infty} \left(\log\log \overline{M}(r, f)\right)/(\log r)$$

and, if $0 < \rho < \infty$, the type T of $f(z)$ is given by

$$Td^{\rho} = \limsup_{r\to\infty} \left(\log \overline{M}(r, f)\right)/r^{\rho}$$

where d is the transfinite diameter of $\overline{B}$ and

$$M(r, f) = \max_{z\in B_r} |f(z)| .$$

LEMMA 2. *Let $f \in L^p(B)$, $1 \le p < \infty$, be the restriction to B of an entire function and let r' (> 1) be given. Then, for all $r > 2r'$*

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and all sufficiently large values of n *, we have*

$$E_n^p(f) \le K\overline{M}(r, f)(r'/r)^n ,$$

where K *is a constant independent of* n *and* r *.*

Proof. Since $f(z)$ is entire, there exists a sequence of polynomials $\{Q_n\}$, Q_n being of degree at most n , such that

$$(2.1) \qquad |f(z)-Q_n(z)| \le (3/2)\overline{M}(r, f) \frac{(r'/r)^{n+1}}{1-(r'/r)} , \quad z \in \overline{B} ,$$

for all $r > r'$ and all sufficiently large values of n ([6, p. 114]). It follows from the definition of $E_n^p(f)$, since $Q_n \in \pi_n$, that

$$(2.2) \qquad E_n^p(f) \le \|f-Q_n\|_{B,p} \le A^{1/p} \max_{z \in \overline{B}} |f(z)-Q_n(z)| ,$$

where A is the area of B . The lemma now follows from (2.1) and (2.2).

Proof of Theorem 1. If $f \in L^p(B)$, $1 \le p < \infty$, is entire, it follows from Lemma 2 that $\limsup_{n\to\infty} \left(E_n^p(f)\right)^{1/n} \le r'/r$ for all $r > 2r'$. Letting $r \to \infty$, we get

$$\lim_{n\to\infty} \left(E_n^p(f)\right)^{1/n} = 0 .$$

This proves the necessity part of the theorem for $1 \le p < \infty$.

Now, let $z_0 \in B$ and let $R > 0$ be such that $D_R = \{z : |z-z_0| \le R\}$ is contained in B . If $f \in L^p(B)$, $1 \le p < \infty$, then $f(z)$ is holomorphic on D_R and has the following Taylor expansion

$$(2.3) \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

in D_R , where the a_n's are given by

$$\frac{\pi R^{2(n+1)}}{n+1} a_n = \iint_{D_R} f(z)\overline{(z-z_0)^n}\,dxdy .$$

Thus, for any $g \in \pi_{n-1}$, we have

$$\frac{\pi R^{2(n+1)}}{n+1} |a_n| = \left| \iint_{D_R} (f(z)-g(z)) \overline{(z-z_0)^n} dxdy \right|$$

$$\leq R^n \|f-g\|_{B,1} .$$

Now, using Hölder's inequality, the above relation gives that

$$\frac{\pi R^{n+2}}{n+1} |a_n| \leq A^q \|f-g\|_{B,p} ,$$

where A is the area of B and $q = 1 - 1/p$. Since the above relation holds for any $g \in \pi_{n-1}$, we have

$$\frac{\pi R^{n+2}}{n+1} |a_n| \leq A^q E^p_{n-1}(f) . \tag{2.4}$$

If, for $f \in L^p(B)$, $1 \leq p < \infty$, equation (1.1) holds, then it follows, from (2.4), that

$$\lim_{n\to\infty} |a_n|^{1/n} = 0$$

and so, (2.3) gives that $f(z)$ is an entire function. This proves the sufficiency part of the theorem for $1 \leq p < \infty$.

The theorem is thus proved for $1 \leq p < \infty$. For the case $p = \infty$, the theorem is essentially due to Winiarski [11].

Proof of Theorem 2. (i) First, let $f(z)$ be an entire function. Then ([9, p. 273]), for all finite z ,

$$f(z) = \sum_{n=0}^{\infty} b_n p_n(z) \tag{2.5}$$

where $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials, p_n being of degree n , such that

$$\iint_B p_n(z)\overline{p_m(z)}dxdy = \delta^n_m , \quad b_n = \iint_B f(z)\overline{p_n(z)}dxdy , \tag{2.6}$$

$\delta^n_m = 1$ for $m = n$ and $\delta^n_m = 0$ otherwise. It is also known [9, p. 272]

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that, given $r_* > 1$, we have

$$\max_{z\in\overline{B}} |p_n(z)| \le Cr_*^n , \quad n = 1, 2, \ldots , \tag{2.7}$$

where C is a constant independent of n .

From (2.6) and (2.7), for any $g \in \pi_{n-1}$, $n \ge 1$, we obtain

$$|b_n| = \left|\iint_B (f(z)-g(z))\overline{p_n(z)}dxdy\right| \le Cr_*^n\|f-g\|_{B,1} . \tag{2.8}$$

On applying Holder's inequality, (2.8) gives

$$|b_n|/r_*^n \le CA^q\|f-g\|_{B,q} , \quad 1 \le p < \infty ,$$

where A is the area of B and $q = 1 - 1/p$. Since the above relation holds for any $g \in \pi_{n-1}$, we have

$$|b_n|/r_*^n \le CA^qE_{n-1}^p(f) , \quad 1 \le p < \infty . \tag{2.9}$$

Now, using (2.5) and (2.7) and applying Bernstein's inequality [6, p. 112] to each term of the series $\sum_{n=0}^{\infty} b_np_n(z)$, we obtain

$$|f(z)| \le |b_0| + C\sum_{n=1}^{\infty} |b_n|(rr_*)^n , \quad z \in B_r .$$

The above relation, in view of (2.9), gives that

$$\overline{M}(r, f) \le |b_0| + C^2A^q\sum_{n=1}^{\infty} E_{n-1}^p(f)\left(rr_*^2\right)^n , \quad 1 \le p < \infty . \tag{2.10}$$

Set $f_p(z) = \sum_{n=0}^{\infty} E_n^p(f)z^n$, $1 \le p < \infty$. By Theorem 1, $f_p(z)$ is an entire function. Further, (2.10) gives that

$$\overline{M}(r, f) \le |b_0| + C^2A^qrr_*^2M\left(rr_*^2, f_p\right) . \tag{2.11}$$

In view of Lemma 1, from (2.11) we obtain

$$\rho \le \rho_p \tag{2.12}$$

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where ρ is the order of $f(z)$ and ρ_p is the order of $f_p(z)$.

On the other hand, by Lemma 2, we get

$$M(r/r', f_p) \le P(r) + K\overline{M}(r+1, f) \sum_{n=0}^{\infty} (r/(r+1))^n = P(r) + K(r+1)\overline{M}(r+1, f) \tag{2.13}$$

where $P(r)$ is a polynomial. From (2.13) and Lemma 1, we have

$$\rho_p \le \rho . \tag{2.14}$$

Combining (2.12) and (2.14) we get $\rho_p = \rho$. Thus, applying the formula expressing the order of an entire function in terms of its Taylor coefficients [2, p. 9] to the function $f_p(z)$, it follows that the order ρ of $f(z)$ is given by (1.2).

(ii) If, for $f \in L^p(B)$, $1 \le p < \infty$, the limit superior on the left hand side of (1.2) is finite, it follows that $\lim_{n\to\infty} \left(E_n^p(f)\right)^{1/n} = 0$. Hence, by Theorem 1, $f(z)$ is entire. From (i) we now get that the order ρ of $f(z)$ is given by (1.2).

(iii) Let $f(z)$ be an entire function of order ρ , $0 < \rho < \infty$, and type T . Then, using (2.11), (2.13) and Lemma 1, we get

$$Td^\rho \le r_*^{2\rho} T_p , \quad T_p/(r')^\rho \le Td^\rho ,$$

where T_p is the type of the entire function $f_p(z)$. Since $r_* > 1$ and $r' > 1$ are arbitrary, we get $T_p = T$. Thus, applying the formula expressing the type of an entire function in terms of its Taylor coefficients [2, p. 11] to the function $f_p(z)$, it follows that the type T of $f(z)$ is given by (1.3).

(iv) If, for $f \in L^p(B)$, $1 \le p < \infty$, the limit superior on the left hand side of (1.3) is nonzero finite, it follows that

$$\limsup_{n\to\infty} \left[(n \log n)/\left(-\log E_n^p(f)\right)\right] = \rho .$$

Hence, by part (ii), $f(z)$ is an entire function of order ρ . From part

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(iii), we now get that the type T of $f(z)$ is given by (1.3).

For $1 \le p < \infty$, the theorem now follows from parts (i) to (iv) above. For $p = \infty$, the theorem is essentially due to Winiarski [11].

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