Canad. Math. Bull. Vol. **55** (3), 2012 pp. 611–622
http://dx.doi.org/10.4153/CMB-2011-108-1

Chen Inequalities for Submanifolds of Real Space Forms with a Semi-Symmetric Non-Metric Connection

Cihan Özgür and Adela Mihai

Abstract. In this paper we prove Chen inequalities for submanifolds of real space forms endowed with a semi-symmetric non-metric connection, *i.e.,* relations between the mean curvature associated with a semi-symmetric non-metric connection, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space. The equality cases are considered.

1 Introduction

H. A. Hayden introduced the notion of a semi-symmetric metric connection on a Riemannian manifold [10]. K. Yano studied a Riemannian manifold endowed with a semi-symmetric metric connection [20]. Some properties of a Riemannian manifold and a hypersurface of a Riemannian manifold with a semi-symmetric metric connection were studied by T. Imai [11, 12]. Z. Nakao [18] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections. N. S. Agashe and M. R. Chafle introduced the notion of a semisymmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection [1, 2].

On the other hand, one of the basic problems in submanifold theory is to find simple relationships between the extrinsic and intrinsic invariants of a submanifold. B. Y. Chen [6, 7, 9] established inequalities in this respect, called *Chen inequalities*. Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces; see, for example, [3–5, 13, 14, 19].

Recently, the present authors studied Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection and Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections [15, 16].

In the present paper, we study Chen inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection. The paper is organized as follows. In Section 2, we give a brief introduction about a semi-symmetric non-metric connection, Chen lemma and Ricci curvature. In Section 3, for submanifolds of

Received by the editors July 17, 2009.
Published electronically May 30, 2011.
The second author's work was supported in part by the strategic grant *POSDRU*/89/1.5/*S*/58852, Project *Postdoctoral programme for training scientific researchers*, cofinanced by the European Social Fund within the Sectoral Operational Program Human Resources Development 2007-2013.
AMS subject classification: **53C40**, 53B05, 53B15.
Keywords: real space form, semi-symmetric non-metric connection, Ricci curvature.

real space forms endowed with a semi-symmetric non-metric connection we establish a Chen first inequality. Section 4 gives a relation between the Ricci curvature in the direction of a unit tangent vector and the mean curvature. In Section 5, we state a relationship between the sectional curvature of a submanifold M^n of a real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$ and the associated squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n and the squared mean curvature $\|H\|^2$.

2 Preliminaries

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold and $\widetilde{\nabla}$ a linear connection on N^{n+p}. If the torsion tensor $\widetilde{T}$ of $\widetilde{\nabla}$, defined by

$$\widetilde{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}],$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, satisfies $\widetilde{T}(\widetilde{X},\widetilde{Y}) = \phi(\widetilde{Y})\widetilde{X} - \phi(\widetilde{X})\widetilde{Y}$ for a 1-form ϕ, then the connection $\widetilde{\nabla}$ is called a *semi-symmetric connection*.

Let g be a Riemannian metric on N^{n+p}. If $\widetilde{\nabla}g = 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric metric connection* on N^{n+p}. If $\widetilde{\nabla}g \neq 0$, then $\widetilde{\nabla}$ is called a *semi-symmetric non-metric connection* on N^{n+p}.

Following [1], a semi-symmetric non-metric connection $\widetilde{\nabla}$ on N^{n+p} is given by

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \mathring{\widetilde{\nabla}}_{\widetilde{X}}\widetilde{Y} + \phi(\widetilde{Y})\widetilde{X},$$

for any vector fields $\widetilde{X}$ and $\widetilde{Y}$ on N^{n+p}, where $\mathring{\widetilde{\nabla}}$ denotes the Levi–Civita connection with respect to the Riemannian metric g and ϕ is a 1-form. Denote by $P = \phi^\sharp$, *i.e.,* the vector field P is defined by $g(P,\widetilde{X}) = \phi(\widetilde{X})$, for any vector field $\widetilde{X}$ on N^{n+p}.

We will consider a Riemannian manifold N^{n+p} endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$ and the Levi–Civita connection denoted by $\mathring{\widetilde{\nabla}}$.

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N^{n+p}. On the submanifold M^n we consider the induced semi-symmetric non-metric connection denoted by ∇ and the induced Levi–Civita connection denoted by $\mathring{\nabla}$.

Let $\widetilde{R}$ be the curvature tensor of N^{n+p} with respect to $\widetilde{\nabla}$ and $\mathring{\widetilde{R}}$ the curvature tensor of N^{n+p} with respect to $\mathring{\widetilde{\nabla}}$. We also denote by R and $\mathring{R}$ the curvature tensors of ∇ and $\mathring{\nabla}$, respectively, on M^n.

The Gauss formulas with respect to ∇, respectively $\mathring{\nabla}$ can be written as:

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X,Y \in \chi(M^n),$$

$$\mathring{\widetilde{\nabla}}_X Y = \mathring{\nabla}_X Y + \mathring{h}(X,Y), \quad X,Y \in \chi(M^n),$$

where $\mathring{h}$ is the second fundamental form of M^n in N^{n+p} and h is a $(0,2)$-tensor on M^n. According to the formula (3.4) in [2],

$$h = \mathring{h}. \tag{2.1}$$

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

One denotes by H the mean curvature vector of M^n in N^{n+p}.

Let $N^{n+p}(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$.

The curvature tensor $\mathring{\widetilde{R}}$ with respect to the Levi–Civita connection $\mathring{\widetilde{\nabla}}$ on $N^{n+p}(c)$ is expressed by

$$\mathring{\widetilde{R}}(X,Y,Z,W) = c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}. \tag{2.2}$$

Then the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ on $N^{n+p}(c)$ can be written as [1]

$$\widetilde{R}(X,Y,Z,W) = \mathring{\widetilde{R}}(X,Y,Z,W) + s(X,Z)g(Y,W) - s(Y,Z)g(X,W), \tag{2.3}$$

for any vector fields $X, Y, Z, W \in \chi(M^n)$, where s is a $(0,2)$-tensor field defined by

$$s(X,Y) = (\mathring{\widetilde{\nabla}}_X \phi)Y - \phi(X)\phi(Y), \quad \forall X, Y \in \chi(M^n).$$

From (2.2) and (2.3) it follows that the curvature tensor $\widetilde{R}$ can be expressed as

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = c\{&g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\}\\ &+ s(X,Z)g(Y,W) - s(Y,Z)g(X,W).\end{aligned} \tag{2.4}$$

Denote by λ the trace of s. Using (2.1), the Gauss equation for the submanifold M^n into the real space form $N^{n+p}(c)$ is

$$\mathring{\widetilde{R}}(X,Y,Z,W) = \mathring{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)).$$

Decomposing the vector field P on M uniquely into its tangent and normal components P^T and $P^\perp$, respectively, we have $P = P^T + P^\perp$.

Let $\pi \subset T_xM^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M^n with respect to the induced semi-symmetric non-metric connection ∇. For any orthonormal basis $\{e_1, \dots, e_m\}$ of the tangent space T_xM^n, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

We recall the following algebraic lemma.

Lemma 2.1 ([6]) *Let $a_1, a_2, \dots, a_n, b$ be $(n+1)$ $(n \ge 2)$ real numbers such that*

$$\Big(\sum_{i=1}^n a_i\Big)^2 = (n-1)\Big(\sum_{i=1}^n a_i^2 + b\Big).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

Let M^n be an n-dimensional Riemannian manifold, L a k-plane section of T_xM^n, $x \in M^n$, and X a unit vector in L. We choose an orthonormal basis $\{e_1, \dots, e_k\}$ of L such that $e_1 = X$. One defines [8] the *Ricci curvature* (or *k-Ricci curvature*) of L at X by $\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}$, where K_{ij} denotes, as usual, the sectional curvature of the 2-plane section spanned by e_i, e_j. For each integer k, $2 \le k \le n$, the Riemannian invariant Θ_k on M^n is defined by

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X), \quad x \in M^n,$$

where L runs over all k-plane sections in T_xM^n and X runs over all unit vectors in L.

3 Chen First Inequality

Recall that the *Chen first invariant* is given by

$$\delta_{M^n}(x) = \tau(x) - \inf\{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\},$$

(see for example [9]), where M^n is a Riemannian manifold, $K(\pi)$ is the sectional curvature of M^n associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and τ is the scalar curvature at x.

Denote by

$$\Omega(X) = s(X,X) + g(P^{\perp}, h(X,X)), \tag{3.1}$$

for a unit vector X tangent to M^n at a point x. We remark that Ω does not depend on X. Detailed explanations will be given in the proof of Theorem 3.1.

For submanifolds of real space forms endowed with a semi-symmetric non-metric connection we establish the following optimal inequality, which we will call the *Chen first inequality*.

Theorem 3.1 *Let M^n, $n \ge 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c, endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. We have*

$$\delta_{M^n}(x) \le \Omega + (n-2)\Big[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c}{2}\Big] - \frac{1}{2}(n-1)\lambda - \frac{1}{2}(n^2-n)\phi(H),$$

where π is a 2-plane section of $T_xM^n, x \in M^n$. Equality holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_xM^n and an orthonormal basis $\{e_{n+1}, \dots, e_{n+p}\}$ of $T_x^{\perp}M^n$ such that the shape operators of M^n in $N^{n+p}(c)$ at x

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a+b=\mu,$$

$$A_{e_{n+i}} = \begin{pmatrix} h_{11}^{n+i} & h_{12}^{n+i} & 0 & \cdots & 0 \\ h_{12}^{n+i} & -h_{11}^{n+i} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad 2 \le i \le p,$$

where we define $h_{ij}^r = g(h(e_i, e_j), e_r)$ *for* $1 \le i, j \le n$ *and* $n+1 \le r \le n+p$*.*

Proof From [2], the Gauss equation with respect to the semi-symmetric non-metric connection is

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(h(X,Z),h(Y,W)) \\ &- g(h(Y,Z),h(X,W)) + g(P^{\perp},h(Y,Z))g(X,W) \\ &- g(P^{\perp},h(X,Z))g(Y,W). \end{aligned} \tag{3.2}$$

Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_{n+p}\}$ be orthonormal bases of T_xM^n and $T_x^{\perp}M^n$, respectively. For $X = W = e_i, Y = Z = e_j$, $i \ne j$, from the equation (2.4) it follows that

$$\tilde{R}(e_i, e_j, e_j, e_i) = c - s(e_j, e_j). \tag{3.3}$$

From (3.2) and (3.3) we get

$$\begin{aligned} c - s(e_j, e_j) = R(e_i, e_j, e_j, e_i) &+ g(h(e_i, e_j), h(e_i, e_j)) \\ &- g(h(e_i, e_i), h(e_j, e_j)) + \phi(h(e_j, e_j)). \end{aligned}$$

By summation after $1 \le i, j \le n$, it follows from the previous relation that

$$(n^2-n)c - (n-1)\lambda = 2\tau + \|h\|^2 - n^2\|H\|^2 + (n^2-n)\phi(H), \tag{3.4}$$

where we recall that λ is the trace of s and denote by

$$\|h\|^2 = \sum_{i,j=1}^{n} g\big(h(e_i,e_j), h(e_i,e_j)\big), \quad H = \frac{1}{n}\,\mathrm{trace}\,h,$$

$$\phi(H) = \frac{1}{n}\sum_{j=1}^{n} \phi(h(e_j,e_j)) = g(P^{\perp}, H).$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 + (n-1)\lambda - (n^2-n)c + (n^2-n)\phi(H). \tag{3.5}$$

Then from (3.4) and (3.5) we get

$$n^2\|H\|^2 = (n-1)(\|h\|^2 + \varepsilon). \tag{3.6}$$

Let $x \in M^n$, $\pi \subset T_xM^n$, $\dim \pi = 2$, $\pi = sp\{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$, and from the relation (3.6) we obtain

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big(\sum_{i,j=1}^{n}\sum_{r=n+1}^{n+p}(h_{ij}^r)^2 + \varepsilon\Big),$$

or equivalently,

$$\Big(\sum_{i=1}^{n} h_{ii}^{n+1}\Big)^2 = (n-1)\Big\{\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon\Big\}. \tag{3.7}$$

By using Lemma 2.1, we have from (3.7)

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon.$$

The Gauss equation for $X = W = e_1, Y = Z = e_2$ gives

$$\begin{aligned}
K(\pi) = R(e_1,e_2,e_2,e_1) &= c - s(e_2,e_2) - g(P^{\perp}, h(e_2,e_2)) + \sum_{r=n+1}^{p}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \\
&\geq c - s(e_2,e_2) - \phi(h(e_2,e_2)) + \frac{1}{2}\Big[\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \varepsilon\Big] \\
&\quad + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p}(h_{12}^r)^2 = c - s(e_2,e_2) - \phi(h(e_2,e_2)) \\
&\quad + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{n+p}(h_{ij}^r)^2 + \frac{1}{2}\varepsilon + \sum_{r=n+2}^{n+p} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+p}(h_{12}^r)^2 \\
&= c - s(e_2,e_2) - g(P^{\perp}, h(e_2,e_2)) + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{n+p}\sum_{i,j>2}(h_{ij}^r)^2 \\
&\quad + \frac{1}{2}\sum_{r=n+2}^{n+p}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{1}{2}\varepsilon \\
&\geq c - s(e_2,e_2) - g(P^{\perp}, h(e_2,e_2)) + \frac{\varepsilon}{2},
\end{aligned}$$

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

which implies $K(\pi) \geq c - s(e_2, e_2) - g(P^{\perp}, h(e_2, e_2)) + \varepsilon/2$. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of T_xM^n. If we take $\pi = sp\{e_1, e_2\}$, the formula (3.1) implies that $\Omega(e_1) = \Omega(e_2)$. Analogously, for $\pi' = sp\{e_1, e_3\}$, we have $\Omega(e_1) = \Omega(e_3)$. Therefore, $\Omega(e_1) = \Omega(e_2) = \cdots = \Omega(e_n)$. Thus $\Omega(X)$ does not depend on X and denote it simply by Ω. By using (3.5) we get

$$K(\pi) \geq \tau - \Omega - (n-2)\left[\frac{n^2}{2(n-1)}\|H\|^2 + (n+1)\frac{c}{2}\right] + \frac{1}{2}(n-1)\lambda + \frac{1}{2}(n^2-n)\phi(H),$$

which represents the inequality.

The equality case holds at a point $x \in M^n$ if and only if it achieves the equality in all the previous inequalities and we have the equality in the lemma.

$$\begin{aligned}
h_{ij}^{n+1} &= 0,\ \forall i \neq j, i, j > 2,\\
h_{ij}^{r} &= 0,\ \forall i \neq j, i, j > 2, r = n+1, \dots, n+p,\\
h_{11}^{r} + h_{22}^{r} &= 0,\ \forall r = n+2, \dots, n+p,\\
h_{1j}^{n+1} &= h_{2j}^{n+1} = 0,\ \forall j > 2,\\
h_{11}^{n+1} + h_{22}^{n+1} &= h_{33}^{n+1} = \cdots = h_{nn}^{n+1}.
\end{aligned}$$

We may chose $\{e_1, e_2\}$ such that $h_{12}^{n+1} = 0$ and we denote by $a = h_{11}^{r}$, $b = h_{22}^{r}$, $\mu = h_{33}^{n+1} = \cdots = h_{nn}^{n+1}$. It follows that the shape operators take the desired forms. ∎

4 Ricci Curvature in the Direction of a Unit Tangent Vector

In this section, we establish a sharp relation between the Ricci curvature in the direction of a unit tangent vector X and the mean curvature H with respect to the semi-symmetric non-metric connection $\tilde{\tilde{\nabla}}$.

Denote by $N(x) = \{X \in T_xM^n \mid h(X, Y) = 0,\ \forall Y \in T_xM^n\}$.

Theorem 4.1 *Let M^n, $n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric non-metric connection $\tilde{\tilde{\nabla}}$.*

(i) *For each unit vector X in T_xM we have*

$$\|H\|^2 \geq \frac{4}{n^2}\Big[\mathrm{Ric}(X) - (n-1)c + \frac{n-1}{2}\lambda - \frac{(n-2)(n-1)}{2}s(X,X) + \frac{1}{2}(n^2-n)\phi(H)\Big]. \tag{4.1}$$

(ii) *If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of* (4.1) *if and only if $X \in N(x)$.*

(iii) *The equality case of inequality* (4.1) *holds identically for all unit tangent vectors at x if and only if either x is a totally geodesic point, or $n = 2$ and x is a totally umbilical point.*

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

Proof (i) Let $X \in T_xM$ be a unit tangent vector at x. We choose an orthonormal basis $e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ such that $e_1, e_2, \dots, e_n$ are tangent to M at x, with $e_1 = X$.

From (3.4) we obtain $n^2\|H\|^2 = 2\tau + \|h\|^2 + (n-1)\lambda - (n^2-n)c + (n^2-n)\phi(H)$. From Gauss equation (3.2) and the formula (3.3), for $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, we get

$$\begin{aligned} K_{ij} &= \widetilde{R}(e_i, e_j, e_j, e_i) + g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_i, e_j)) \\ &= c - s(e_j, e_j) + \sum_{r=n+1}^{n+p} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2]. \end{aligned}$$

By summation and by using formula (3.4), it follows that

$$\begin{aligned} \sum_{2 \le i < j \le n} K_{ij} &= \sum_{r=n+1}^{n+p} \sum_{2 \le i < j \le n} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2] + \sum_{2 \le i < j \le n} [c - s(e_j, e_j)] \\ &= \sum_{r=n+1}^{n+p} \sum_{2 \le i < j \le n} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2] \\ &\qquad + \frac{(n-2)(n-1)}{2} c - \frac{(n-2)(n-1)}{2} s(e_1, e_1). \end{aligned}$$

On the other hand,

$$\begin{aligned} n^2\|H\|^2 &= 2\tau + \frac{1}{2} n^2\|H\|^2 + \frac{1}{2} \sum_{r=n+1}^{n+p} (h^r_{11} - h^r_{22} - \cdots - h^r_{nn})^2 \\ &\quad - 2 \sum_{r=n+1}^{n+p} \sum_{2 \le i < j \le n} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2] + (n-1)\lambda - n(n-1)c + (n^2-n)\phi(H). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \frac{1}{2} n^2\|H\|^2 &= 2\operatorname{Ric}(e_1) + 2 \sum_{2 \le i < j \le n} K_{ij} + \frac{1}{2} \sum_{r=n+1}^{n+p} (h^r_{11} - h^r_{22} - \cdots - h^r_{nn})^2 \\ &\qquad + (n-1)\lambda - n(n-1)c + n^2\phi(H) - 2 \sum_{r=n+1}^{n+p} \sum_{2 \le i < j \le n} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2] \\ &= 2\operatorname{Ric}(e_1) + (n-2)(n-1)c - (n-2)(n-1)s(e_1, e_1) \\ &\qquad + \frac{1}{2} \sum_{r=n+1}^{n+p} (h^r_{11} - h^r_{22} - \cdots - h^r_{nn})^2 \\ &\qquad + (n-1)\lambda - n(n-1)c + (n^2-n)\phi(H) \end{aligned}$$

$$\geq 2\,\mathrm{Ric}(e_1) - 2(n-1)c + (n-1)\lambda - (n-2)(n-1)s(e_1,e_1) + (n^2-n)\phi(H).$$

Finally,

$$\mathrm{Ric}(e_1) \leq \frac{1}{4}n^2\|H\|^2 + (n-1)c - \frac{n-1}{2}\lambda + \frac{(n-2)(n-1)}{2}s(e_1,e_1) - \frac{1}{2}(n^2-n)\phi(H),$$

or, equivalently,

$$\|H\|^2 \geq \frac{4}{n^2}\Big[\mathrm{Ric}(X) - (n-1)c + \frac{n-1}{2}\lambda - \frac{(n-2)(n-1)}{2}s(X,X) + \frac{1}{2}(n^2-n)\phi(H)\Big],$$

for every unit vector $X \in T_xM$, which represents to inequality to prove.

(ii) Assume $H(x) = 0$. Equality holds in (4.1) if and only if

$$h^r_{12} = \cdots = h^r_{1n} = 0, \quad h^r_{11} = h^r_{22} + \cdots + h^r_{nn},\ r \in \{n+1,\dots,n+p\}.$$

Then $h^r_{1j} = 0$ for all $j \in \{1,\dots,n\}$, $r \in \{n+1,\dots,n+p\}$, *i.e.*, $X \in N(x)$.

(iii) The equality case of (4.1) holds for all unit tangent vectors at x if and only if

$$h^r_{ij} = 0, \quad i \neq j,\ r \in \{n+1,\dots,n+p\},$$
$$h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1,\dots,n\},\ r \in \{n+1,\dots,n+p\}.$$

We distinguish two cases:

- $n \neq 2$, then x is a totally geodesic point;
- $n = 2$, it follows that x is a totally umbilical point.

The converse is trivial. ■

5 k-Ricci Curvature

We first state a relationship between the sectional curvature of a submanifold M^n of a real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$ and the associated squared mean curvature $\|H\|^2$. Using this inequality, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the squared mean curvature $\|H\|^2$ (extrinsic invariant), as another answer of the basic problem in submanifold theory which we mentioned in the introduction.

Theorem 5.1 *Let M^n, $n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c + \frac{1}{n}\lambda + \phi(H). \tag{5.1}$$

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

Proof Let $x \in M^n$ and $\{e_1, e_2, \dots, e_n\}$ an orthonormal basis of $T_x M^n$. The relation (3.4) is equivalent to

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 + (n-1)\lambda - n(n-1)c + (n^2 - n)\phi(H). \tag{5.2}$$

We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+p}\}$ at x such that e_{n+1} is parallel to the mean curvature vector $H(x)$ and $e_1, \dots, e_n$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},$$

$$A_{e_r} = (h^r_{ij}), i, j = 1, \dots, n, \quad r = n+2, \dots, n+p, \quad \operatorname{trace} A_r = 0.$$

From (5.2), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+p} \sum_{i,j=1}^{n} (h^r_{ij})^2 + (n-1)\lambda - n(n-1)c + (n^2 - n)\phi(H). \tag{5.3}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j,$$

we obtain

$$n^2 \|H\|^2 = \Big(\sum_{i=1}^{n} a_i \Big)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2.$$

We have from (5.3)

$$n^2 \|H\|^2 \geq 2\tau + n\|H\|^2 + (n-1)\lambda - n(n-1)c + (n^2 - n)\phi(H)$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c + \frac{1}{n}\lambda + \phi(H). \qquad \blacksquare$$

Using Theorem 5.1, we obtain the following.

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press

Theorem 5.2 *Let M^n, $n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant sectional curvature c endowed with a semi-symmetric non-metric connection $\widetilde{\nabla}$. Then for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - c + \frac{1}{n}\lambda + \phi(H). \tag{5.4}$$

Proof Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_xM. Denote by $L_{i_1\cdots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1\cdots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\ldots,i_k\}} \mathrm{Ric}_{L_{i_1\cdots i_k}}(e_i),$$

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<\cdots<i_k\leq n} \tau(L_{i_1\ldots i_k}).$$

From (5.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2}\Theta_k(p),$$

which implies (5.4). ∎

References

[1] N. S. Agashe and M. R. Chafle, *A semi-symmetric non-metric connection on a Riemannian manifold*. Indian J. Pure Appl. Math. **23**(1992), no. 6, 399–409.

[2] ———, *On submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection* Tensor **55** (1994), no. 2, 120-130.

[3] K. Arslan, R. Ezentaş, I. Mihai, C. Murathan, and C. Özgür, *B. Y. Chen inequalities for submanifolds in locally conformal almost cosymplectic manifolds*. Bull. Inst. Math., Acad. Sinica **29**(2001), no. 3, 231–242.

[4] ———, *Certain inequalities for submanifolds in $(k\mu)$-contact space forms*. Bull. Aust. Math. Soc. **64**(2001), no. 2, 201–212. http://dx.doi.org/10.1017/S0004972700039873

[5] ———, *Ricci curvature of submanifolds in locally conformal almost cosymplectic manifolds*. Math. J. Toyama Univ. **26** (2003), 13–24. Zbl 1076.53030

[6] B-Y. Chen, *Some pinching and classification theorems for minimal submanifolds* Arch. Math. (Basel) **60** (1993), no. 6, 568–578. http://dx.doi.org/10.1007/BF01236084

[7] ———, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*. In: The Third Pacific Rim Geometry Conference. Monogr. Geom. Topology 25. Int. Press, Cambridge, MA, 1998, pp. 7–60.

[8] ———, *Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions*. Glasg. Math. J. **41**(1999), no. 1, 33–41. http://dx.doi.org/10.1017/S0017089599970271

[9] ———, *δ-invariants, inequalities of submanifolds and their applications.* In: Topics in Differential Geometry. Ed. Acad. Române, Bucharest, 2008, pp. 29–155.

[10] H. A. Hayden, *Subspaces of a space with torsion*. Proc. London Math. Soc. **34** (1932), 27–50.

[11] T. Imai, *Hypersurfaces of a Riemannian manifold with semi-symmetric metric connection*. Tensor **23** (1972), 300–306.

[12] ———, *Notes on semi-symmetric metric connections*. Tensor **24**(1972), 293–296.

[13] K. Matsumoto, I. Mihai, and A. Oiagă, *Ricci curvature of submanifolds in complex space forms*. Rev. Roumaine Math. Pures Appl. **46**(2001), no. 6, 775–782.

[14] A. Mihai, *Modern Topics in Submanifold Theory*. Editura Universitătii din Bucuresti, Bucharest, 2006.

[15] A. Mihai and C. Özgür, *Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection.* Taiwanese J. Math. **14**(2010), no. 4, 1465–1477.

[16] ———, *Chen inequalities for submanifolds of complex space forms and Sasakian space forms endowed with semi-symmetric metric connections*, to appear Rocky Mountain J. Math.

[17] I. Mihai, *Slant submanifolds in complex space forms.* In: *Topics in Differential Geometry*. Editura Academiei Române, Bucharest, 2008, pp. 157–187.

[18] Z. Nakao, *Submanifolds of a Riemannian manifold with semisymmetric metric connections*. Proc. Amer. Math. Soc. **54**(1976), 261–266. http://dx.doi.org/10.1090/S0002-9939-1976-0445416-9

[19] A. Oiaga and I. Mihai, *B. Y. Chen inequalities for slant submanifolds in complex space forms*. Demonstratio Math. **32** (1999), no. 4, 835–846.

[20] K. Yano, *On semi-symmetric metric connection*. Rev. Roumaine Math. Pures Appl. **15**(1970), 1579–1586.

Balıkesir University, Department of Mathematics, 10145, Çağış, Balıkesir, Turkey
e-mail: cozgur@balikesir.edu.tr

University of Bucharest, Faculty of Mathematics and Computer Science, Department of Mathematics, Academiei 14, 010014 Bucharest, Romania
e-mail: adela_mihai@fmi.unibuc.ro

https://doi.org/10.4153/CMB-2011-108-1 Published online by Cambridge University Press