AN APPLICATION OF NEVANLINNA-PÓLYA THEOREM TO A COSINE FUNCTIONAL EQUATION

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1. Introduction

We consider the cosine functional equation (see [1, 2, 3])

$$f(0)(f(x+y)+f(x-y)) = 2f(x)f(y), \tag{1}$$

where $f(z)$ is an entire function of a complex variable z and x, y are complex variables.

It is clear that the only entire solution of (1) is $a \cos bz$ where a, b are arbitrary complex constants.

In Section 2 we shall prove the following

THEOREM 1. *If $f(z)$ is an entire function of a complex variable z, then* (1) *implies the following functional equation* (2) *with some g*:

$$|f(0)|^2(|f(x+y)|^2+|f(x-y)|^2) = 2|f(x)|^2|f(y)|^2 + 2|g(x)|^2|g(y)|^2, \tag{2}$$

where $g(z)$ is an entire function of a complex variable z.

In Section 3 we shall prove a converse of Theorem 1, i.e., the following

THEOREM 2. *If $f(z)$, $g(z)$ are entire functions of a complex variable z, then* (2) *implies* (1).

To this end we shall use the following

THEOREM A. *If $f(z)$, $g(z)$, $h(z)$, $k(z)$ are entire functions of a complex variable z and satisfy $|f(z)|^2+|g(z)|^2 = |h(z)|^2+|k(z)|^2$ in $|z| < +\infty$, then there exists a unitary matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ such that*

$$h(z) = \alpha f(z)+\beta g(z)$$
$$k(z) = \gamma f(z)+\delta g(z)$$

in $|z| < +\infty$, where α, β, γ, δ are complex constants.

PROOF. See [4, 5].

COROLLARY OF THEOREM A. *If $f(z)$, $g(z)$, $h(z)$, $k(z)$ are entire functions of z and satisfy $|f'(z)|^2+|g'(z)|^2 = |h'(z)|^2+|k'(z)|^2$ in $|z| < +\infty$ and if $f(0) = g(0) = h(0) = k(0) = 0$, then $|f(z)|^2+|g(z)|^2 = |h(z)|^2+|k(z)|^2$ in $|z| < +\infty$.*

PROOF. By the hypothesis and by Theorem A there exists a unitary matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ such that

$$h'(z) = \alpha f'(z)+\beta g'(z) \tag{3}$$

$$k'(z) = \gamma f'(z)+\delta g'(z) \tag{4}$$

in $|z| < +\infty$, where α, β, γ, δ are complex constants.

By (3), (4) and by $f(0) = g(0) = h(0) = k(0) = 0$ we have

$$h(z) = \alpha f(z)+\beta g(z) \tag{5}$$

$$k(z) = \gamma f(z)+\delta g(z). \tag{6}$$

Since $(\alpha\ \ \beta/\gamma\ \ \delta)$ is a unitary matrix, by (5), (6) the Corollary is proved.

By Theorems 1,2 we have the following

THEOREM 3. *The only system of entire solutions of (2) is $f(z) = a \cos bz$, $g(z) = a \exp(i\theta) \sin bz$ where a, b are arbitrary complex constants and θ is an arbitrary real constant.*

PROOF. It is clear from Theorems 1, 2.

2. Proof of Theorem 1

We may assume that $f(z) \not\equiv$ const. Otherwise the proof is clear.

Differentiating both sides of (1) twice with respect to y and putting $y = 0$, we have

$$f(0)f''(x) = f''(0)f(x). \tag{7}$$

Differentiating both sides of (1) with respect to x and then with respect to y, we have

$$f(0)(f''(x+y)-f''(x-y)) = 2f'(x)f'(y). \tag{8}$$

We can deduce that $f''(0) \neq 0$. Otherwise, by (1), (7) $f(z)$ is a complex constant, contradicting the assumption that $f(z) \not\equiv$ const.

By (7), (8) we have

$$f(0)(f(x+y)-f(x-y)) = 2\frac{f(0)}{f''(0)}f'(x)f'(y). \tag{9}$$

(1), (9) and the parallelogram identity $|a+b|^2+|a-b|^2 = 2|a|^2+2|b|^2$ (a, b complex) yield that

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$$(10)\qquad |f(0)|^2(|f(x+y)|^2+|f(x-y)|^2) = 2|f(x)|^2|f(y)|^2+2\left|\frac{f(0)}{f''(0)}\right|^2|f'(x)|^2|f'(y)|^2.$$

By (10) we see that (1) implies (2) with $g(z) = \sqrt{f(0)/f''(0)}\, f'(z)$ which is an entire function of a complex variable z. Q.E.D.

3. Proof of Theorem 2

Upon putting $x = y = 0$ in (2), we see that

$$(11)\qquad g(0) = 0.$$

We next take Laplacians $\partial^2/\partial s^2+\partial^2/\partial t^2$ of both sides of (2) with respect to $y = s+it$ (s, t real) and obtain

$$|f(0)|^2(4|f'(x+y)|^2+4|f'(x-y)|^2) = 8|f(x)|^2|f'(y)|^2+8|g(x)|^2|g'(y)|^2,$$

or

$$(12)\qquad |f(0)|^2(|f'(x+y)|^2+|f'(x-y)|^2) = 2\,|f(x)|^2|f'(y)|^2+2|g(x)|^2|g'(y)|^2,$$

since, by [6], $\Delta|f|^2 = 4|f'|^2$.

When x is arbitrarily fixed, $f(0)(f(x+y)-f(x))$, $f(0)(f(x-y)-f(x))$, $\sqrt{2}f(x)(f(y)-f(0))$, $\sqrt{2}g(x)g(y)$ are entire functions with

$$(f(0)(f(x+y)-f(x)))_{y=0} = (f(0)(f(x-y)-f(x)))_{y=0} = (\sqrt{2}f(x)(f(y)-f(0)))_{y=0} = (\sqrt{2}g(x)g(y))_{y=0} = 0$$

(by (11)). Moreover, by (12) we have in $|y| < +\infty$

$$\left|\frac{\partial}{\partial y}(f(0)(f(x+y)-f(x)))\right|^2+\left|\frac{\partial}{\partial y}(f(0)(f(x-y)-f(x)))\right|^2 = \left|\frac{\partial}{\partial y}(\sqrt{2}f(x)(f(y)-f(0)))\right|^2+\left|\frac{\partial}{\partial y}(\sqrt{2}g(x)g(y))\right|^2$$

Hence, by Corollary of Theorem A we have in $|y| < +\infty$

$$(13)\qquad |f(0)|^2(|f(x+y)-f(x)|^2+|f(x-y)-f(x)|^2) = 2|f(x)|^2|f(y)-f(0)|^2+2|g(x)|^2|g(y)|^2.$$

Subtracting (13) from (2) and using the identity $|a-b|^2 = |a|^2+|b|^2-2\mathrm{Re}\,(a\bar{b})$, we see that

$$(14)\qquad |f(0)|^2\,\mathrm{Re}\,((f(x+y)+f(x-y))\overline{f(x)}) = 2|f(x)|^2\,\mathrm{Re}\,(f(y)\overline{f(0)}).$$

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We may assume that $f(0) \neq 0$. Otherwise the proof is clear. Hence, by the continuity of f there exists a neighborhood V of the origin where $f(x) \neq 0$.

So, by (14) we have in V and for every complex y

$$\operatorname{Re}\left(\frac{1}{f(x)}(f(x+y)+f(x-y))-2\frac{1}{f(0)}f(y)\right)=0. \tag{15}$$

Since $f(z)$ is an entire function of a complex variable z, by (15) we have in V and for every complex y

$$\frac{1}{f(x)}(f(x+y)+f(x-y))-2\frac{1}{f(0)}f(y)=C, \tag{16}$$

where C is a complex constant.

Upon putting $y = 0$ in (16), we see that

$$C = 0. \tag{17}$$

By (16), (17) and by the Identity Theorem we have (1). Q.E.D.

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